Exercise 0.1. Define

$$g : \mathbf{R}^3 \to \mathbf{R} \qquad \text{by} \qquad g(x) = x_1^2 + x_2^2 - (1 - x_3)x_3^2 \qquad \text{and} \qquad V = \{ x \in \mathbf{R}^3 \mid g(x) = 0 \}.$$

(i) Verify that $V \subset \{ x \in \mathbf{R}^3 \mid x_3 \leq 1 \}$.

Given an arbitrary point $x \in V$, the result in (i) suggests to write $x_3 = 1 - s^2$ for some $s \in \mathbf{R}$.

(ii) Conclude

$$V \subset \text{im}(\phi) \qquad \text{with} \qquad \phi : \mathbf{R}^2 \to \mathbf{R}^3 \qquad \text{given by} \qquad \phi(s, t) = (1 - s^2)(s \cos t, s \sin t, 1).$$

Show that actually one has $V = \text{im}(\phi)$.

(iii) Demonstrate that ϕ is an immersion at all points of $\mathbf{R}^2$ with the exception of the points $(\pm 1, t)$ and $(0, t)$, for arbitrary $t \in \mathbf{R}$. More precisely, prove

$$\dim \ker \big(D\phi(\pm 1, t) \big) = \dim \ker \big(D\phi(0, t) \big) = 1.$$

Verify $\phi(\pm 1, t) = 0$ and $\phi(0, t) = (0, 0, 1) = n$, for all $t \in \mathbf{R}$.

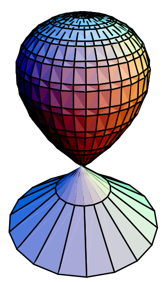

Illustration for Exercise 0.1

In the illustration we note nothing of particular interest at the point $n \in \mathbf{R}^3$ (on the contrary, we do so at $0 \in \mathbf{R}^3$).

(iv) Prove that V is a C^∞ submanifold in $\mathbf{R}^3$ of dimension 2 at every point belonging to $V \setminus \{0\}$.

Background. The fact of ϕ not being an immersion at the points $(0, t)$, therefore, is peculiar to ϕ; in this case, it does not imply singular behavior of $\text{im}(\phi)$ itself near n. Finally, without proof, we mention that V is not a submanifold in $\mathbf{R}^3$ of dimension 2 at 0.

Solution of Exercise 0.1. (i) $x \in V$ implies $0 \leq x_1^2 + x_2^2 = (1 - x_3)x_3^2$, therefore $0 \leq 1 - x_3$, that is, $x_3 \leq 1$.

(ii) If $x_3 = 1 - s^2$, then $1 - x_3 = s^2$. Accordingly, for $x \in V$,

$$x_1^2 + x_2^2 = (1 - x_3)x_3^2 = \big(s(1 - s^2)\big)^2, \qquad \text{so} \qquad (x_1, x_2) = s(1 - s^2)(\cos t, \sin t),$$

for suitable $t \in \mathbf{R}$, on account of the parametrization of a circle by trigonometric functions. Thus we obtain $V \subset \text{im}(\phi)$. Conversely, for every $x \in \text{im}(\phi)$,

$$x_1^2 + x_2^2 = \big(s(1 - s^2)\big)^2 \qquad \text{and} \qquad (1 - x_3)x_3^2 = s^2(1 - s^2)^2, \qquad \text{that is} \qquad g(x) = 0.$$

(iii) Suppose, for $h \in \mathbf{R}^2$,

$$D\phi(s,t)h = \begin{pmatrix} (1-3s^2)\cos t & -s(1-s^2)\sin t \\ (1-3s^2)\sin t & s(1-s^2)\cos t \\ -2s & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \star \\ \star \\ -2sh_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $s \neq 0$, it follows that $h_1 = 0$. The two top equations above then give

$$h_2 s(1-s^2)\sin t = h_2 s(1-s^2)\cos t = 0, \qquad \text{so} \qquad s(1-s^2)h_2 = 0.$$

Accordingly, if $s \notin \{-1, 0, 1\}$, then $h_2 = 0$ too; and therefore ϕ is immersive in this case. On the other hand,

$$D\phi(\pm 1, t) = -2 \begin{pmatrix} \cos t & 0 \\ \sin t & 0 \\ \pm 1 & 0 \end{pmatrix}, \qquad D\phi(0,t) = \begin{pmatrix} \cos t & 0 \\ \sin t & 0 \\ 0 & 0 \end{pmatrix},$$

which shows that all three of these mappings in $\mathrm{Lin}(\mathbf{R}^2, \mathbf{R}^3)$ have a one-dimensional kernel. It is direct from the definition that $\phi(\pm 1, t) = 0$ and $\phi(0, t) = n$, for all $t \in \mathbf{R}$.

(iv) We have, for $x \in V$,

$$Dg(x) = (2x_1, 2x_2, -2x_3 + 3x_3^2) \in \mathrm{Lin}(\mathbf{R}^3, \mathbf{R}).$$

This mapping fails to be surjective only if all its entries equal 0, which is the case only if $x = 0$ (the solution with $x_3 = \frac{2}{3}$ does not belong to V). Hence, g is submersive at all points of $V \setminus \{0\}$; and on the strength of the Submersion Theorem 4.5.2 we now obtain that V is a C^∞ manifold in $\mathbf{R}^3$ of dimension 2 at all of its points, with the possible exception of the point 0.