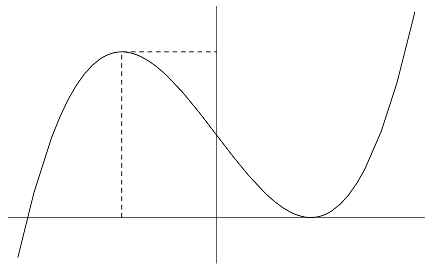
Exercise 0.1 (Family of cubic curves). Define the monic cubic polynomial function

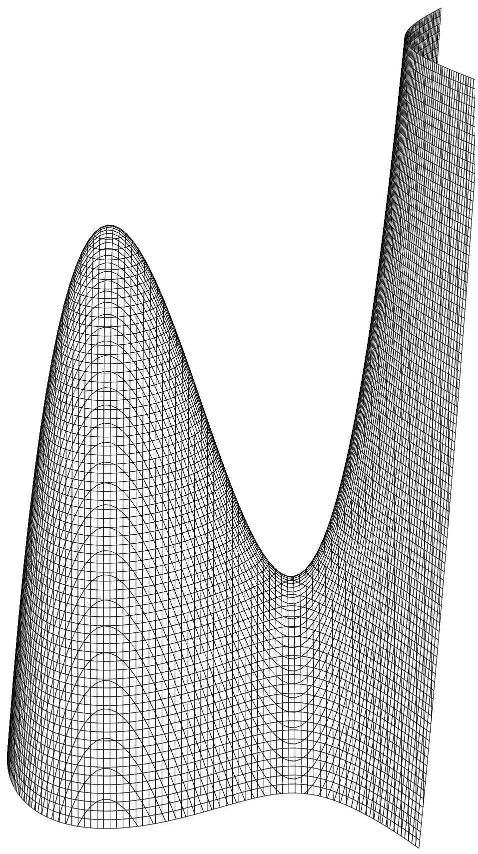
$$p : \mathbf{R} \to \mathbf{R} \qquad \text{by} \qquad p(x) = x^3 - 3x + 2.$$

(i) Prove that the extrema of p are a local maximum of value 4 occurring at -1 and a local minimum 0 at 1. Determine the zeros of p and decompose p into a product of linear factors.



Next introduce the cubic polynomial function

$$g : \mathbf{R}^3 \to \mathbf{R} \qquad \text{by} \qquad g(x) = p(x_1) - x_2^2 - x_3 \qquad \text{and the set} \qquad V = \{\, x \in \mathbf{R}^3 \mid g(x) = 0 \,\}.$$

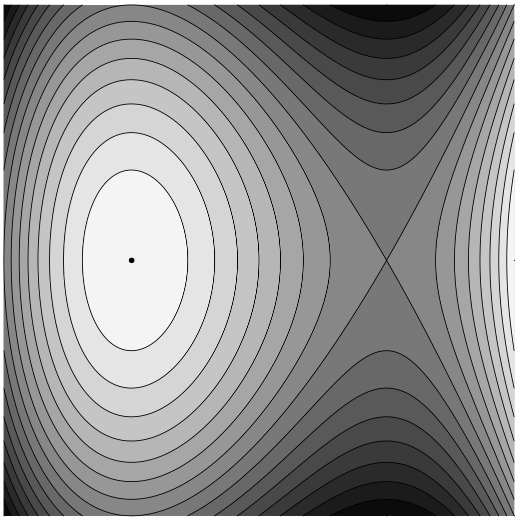


(ii) Show that V is a C^∞ submanifold in $\mathbf{R}^3$ of dimension 2 by representing it as the graph of a C^∞ function.

(iii) Verify again the claim about V as in part (ii), but now by considering $Dg(x)$, for all $x \in V$. Further, prove that $(-1, 0, 4)$ and $(1, 0, 0)$ are the only points of V at which the tangent plane of V is given by the linear subspace $\mathbf{R}^2 \times \{0\}$ of $\mathbf{R}^3$.

For every $c \in \mathbf{R}$, define the function

$$g_c : \mathbf{R}^2 \to \mathbf{R} \qquad \text{by} \qquad g_c(x_1, x_2) = g(x_1, x_2, c) \qquad \text{and the set} \qquad V_c = \{\, x \in \mathbf{R}^2 \mid g_c(x) = 0 \,\}.$$



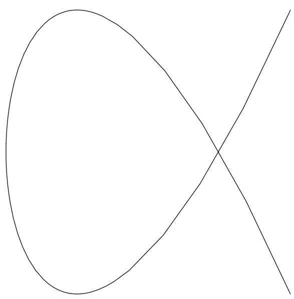
(iv) For every $c \in \mathbf{R} \setminus \{0, 4\}$, demonstrate that V_c is a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1. Prove that V_0 is a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1 in all of its points with the possible exception of $(1, 0)$. Furthermore, using part (i) show that V_4 is the disjoint union of a point (which?) and a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1.

(v) Set $I = [\,-2, \infty\,[\,\subset \mathbf{R}$ and prove by means of part (i) that $V_0 \subset I \times \mathbf{R}$. Next, use this fact to write V_0 as the union of the graphs G_+ and G_- of two distinct functions defined on I that are C^∞ on the interior of I. Furthermore, derive that $(1, 0) \in V_0$ is a point where G_+ and G_- intersect and that $\frac{\pi}{3}$ is the smallest angle between the tangent lines at $(1, 0)$ of G_+ and G_-, respectively.

(vi) From the previous part it follows that every $x \in V_0$ satisfies $x_1 \geq -2$; in this case, therefore, one may write $x_1 = t^2 - 2$ with $t \in \mathbf{R}$. Deduce $V_0 = \operatorname{im} \phi$, where

$$\phi : \mathbf{R} \to \mathbf{R}^2 \qquad \text{is given by} \qquad \phi(t) = (t^2 - 2, t^3 - 3t).$$

Verify that ϕ is an embedding on $\mathbf{R} \setminus \{\pm\sqrt{3}\}$.



Finally, suppose that $p : \mathbf{R} \to \mathbf{R}$ is an arbitrary monic cubic polynomial with real coefficients and consider $C = \{\, x \in \mathbf{R}^2 \mid p(x_1) = x_2^2 \,\}$.

(vii) Show that C possesses a singular point only if p has a root at least of multiplicity two. Describe the geometry of C if p has a root of multiplicity three.

Background. Families of curves in $\mathbf{R}^2$ of the type studied above occur in *number theory* and in the *theory of differential equations*.

Solution of Exercise 0.1

(i) $p'(x) = 3(x^2 - 1) = 0$ implies $x = \pm 1$; with corresponding values $p(-1) = 4$ and $p''(-1) = -6$, hence a local maximum; and $p(1) = 0$ and $p''(1) = 6$, hence a local minimum. Since $\lim_{x\to\pm\infty} p(x) = \pm\infty$, the extrema are not absolute. In view of $p(1) = p'(1) = 0$, one may write $p(x) = (x-1)^2(x-a) = x^3 + \cdots - a$ (see Application 3.6.A), which implies $a = -2$; hence the factorization is $p(x) = (x-1)^2(x+2)$.

(ii) $g(x) = 0$ implies $x_3 = p(x_1) - x_2^2$. This leads to $V = \{\, (x_1, x_2, p(x_1) - x_2^2) \in \mathbf{R}^3 \mid (x_1, x_2) \in \mathbf{R}^2 \,\}$, displaying V as the graph of a C^∞ function on $\mathbf{R}^2$.

(iii) $Dg(x) = (p'(x_1), -2x_2, -1)$, and this element in $\mathrm{Mat}(1\times 3, \mathbf{R})$ is of rank 1, for all $x \in \mathbf{R}^3$; therefore g is submersive on all of $\mathbf{R}^3$. The assertion about V now follows from the Submersion Theorem 4.5.2. Furthermore, $\operatorname{grad} g(x)$ is perpendicular to T_xV, for any $x \in V$ (see Example 5.3.5); hence $T_xV = \mathbf{R}^2 \times \{0\}$ if and only if $p'(x_1) = 0$, $x_2 = 0$ and $g(x) = 0$. But this implies $x_1 = \pm 1$, $x_2 = 0$ and $x_3 = p(\pm 1)$.

(iv) According to the Submersion Theorem 4.5.2, the set V_c is a a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1 in $x \in V_c$ if $Dg_c(x) = (p'(x_1), -2x_2) \neq (0,0)$ and $c = p(x_1) - x_2^2$. That is, V_c possibly does not possess the desired properties at x if

$$x_1 = \pm 1, \qquad x_2 = 0 \qquad \text{and} \qquad c \in \{\, p(\pm 1) \,\} = \{0, 4\}.$$

If $c = 0$, and $c = 4$, only the point $(1,0) \in V_0$, and $(-1,0) \in V_4$, respectively, satisfies all these conditions. Actually, the point $(-1,0)$ is an isolated point of V_4. Indeed, on the basis of part (i) one finds for $x \in V_4$ sufficiently close to $(-1,0)$ that $4 = p(-1) \geq p(x_1) = x_2^2 + 4$. But this implies $x_2 = 0$ and so $x_1 = -1$.

(v) For $x \in V_0$ one has $0 \leq x_2^2 = p(x_1)$, but then part (i) implies $x_1 \geq -2$. Under the latter assumption, the condition $x_2^2 = p(x_1) = (x_1 - 1)^2(x_1 + 2)$ on x is equivalent to

$$x_2 = \pm(x_1 - 1)\sqrt{x_1 + 2} =: f_\pm(x_1),$$

where $f_\pm : I \to \mathbf{R}$ is a C^∞ function on the interior of I. Now set $G_\pm = \operatorname{graph} f_\pm$. Since $f_\pm(1) = 0$, one sees $(1,0) \in \bigcap_\pm G_\pm$, while $f_\pm$ is C^∞ near 1. Furthermore,

$$Df_\pm(x_1) = \pm(\sqrt{x_1 + 2} + (x_1 - 1)\cdots), \qquad \text{in particular} \qquad \operatorname{graph} Df_\pm(1) = \mathbf{R}(1, \pm\sqrt{3}).$$

Noting that the norms of the two preceding generators of the tangent spaces of G_+ and G_- at (1,0) are equal to 2 and writing α for the angle between these, one gets

$$\cos\alpha = \frac{\langle\, (1,\sqrt{3}), (1,-\sqrt{3}) \,\rangle}{\|(1,\sqrt{3})\|\,\|(1,-\sqrt{3})\|} = \frac{1-3}{2\cdot 2} = -\frac{1}{2}, \qquad \text{that is} \qquad \alpha = \frac{2\pi}{3}.$$

It follows that the smallest angle between the tangent lines equals $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

(vi) Writing $x_1 = t^2 - 2$ for $x \in V_0$, one finds on the basis of part (i)

$$x_2^2 = p(x_1) = (x_1 - 1)^2(x_1 + 2) = (t^2 - 3)^2\, t^2 = (t^3 - 3t)^2.$$

This implies $V_0 \subset \operatorname{im}\phi$, whereas the reverse implication is a straightforward calculation. $D\phi(t) = (2t, 3(t^2 - 1))$ is of rank 1, for all $t \in \mathbf{R}$; hence ϕ is an immersion on $\mathbf{R}$. Further, $\phi(t) = \phi(t')$, for t and $t' \in \mathbf{R}$, leads to $t = \pm t'$, hence $t(t^2 - 3) = 0$; therefore $t = \pm\sqrt{3}$ and $t' = \mp\sqrt{3}$. If $t \neq \pm\sqrt{3}$ and $x = \phi(t)$, then $x_1 - 1 \neq 0$, which implies that $\phi(t) = x \mapsto \frac{x_2}{x_1 - 1} = t$ defines a continuous mapping. This demonstrates that ϕ is an embedding on $\mathbf{R} \setminus \{\pm\sqrt{3}\}$.

(vii) If $x \in C$ is a singular point of C, then $p(x_1) = x_2^2$ and $(p'(x_1), -2x_2) = (0, 0)$ imply $x_2 = 0$ and $p(x_1) = p'(x_1) = 0$; in other words, p must possess a root of multiplicity at least two. Suppose $p(x_1) = (x_1 - c)^3$, for some $c \in \mathbf{R}$, then the points of C satisfy the equation $(x_1 - c)^3 = x_2^2$, which is an ordinary cusp as in Example 5.3.8.