Exercise 0.1 (Green's first identity by means of Gauss' Divergence Theorem). Consider $B^2 = \{ x \in \mathbf{R}^2 \mid \|x\| < 1 \}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ given by $g(x) = x_1^2 - x_2^2$.

(i) Prove
$$\int_{B^2} \| \operatorname{grad} g(x) \|^2 \, dx = 2\pi.$$

(ii) Recall that $\frac{\partial g}{\partial \nu} = \langle \operatorname{grad} g, \nu \rangle$, the derivative in the direction of the outer normal ν to ∂B^2, and compute
$$\int_{\partial B^2} \Big(g \, \frac{\partial g}{\partial \nu} \Big)(y) \, d_1 y.$$

Hint: Use $2(\cos^2 \alpha - \sin^2 \alpha)^2 = 2 \cos^2 2\alpha = 1 + \cos 4\alpha$.

The equality of the two integrals above is no accident, as we will presently show. To this end, suppose $h : \mathbf{R}^2 \to \mathbf{R}$ to be an arbitrary C^2 function. Note that $h \operatorname{grad} h : \mathbf{R}^2 \to \mathbf{R}^2$ is a C^1 vector field and recall the identity $\operatorname{div} \operatorname{grad} = \Delta$.

(iii) Prove $\operatorname{div}(h \operatorname{grad} h) = \| \operatorname{grad} h \|^2 + h \, \Delta h$.

(iv) Suppose $\Omega \subset \mathbf{R}^2$ satisfies the conditions of Gauss' Divergence Theorem. Apply this theorem to verify
$$(\star) \qquad \int_\Omega (h \, \Delta h)(x) \, dx + \int_\Omega \| \operatorname{grad} h(x) \|^2 \, dx = \int_{\partial \Omega} \Big(h \, \frac{\partial h}{\partial \nu} \Big)(y) \, d_1 y.$$

(v) Derive $(\star)$ in part (iv) directly from Green's first identity.

(vi) Show that the equality of the integrals in parts (i) and (ii) follows from $(\star)$ in part (iv).

Solution of Exercise 0.1

(i) We have $\operatorname{grad} g(x) = 2(x_1, -x_2)$ and so $\| \operatorname{grad} g(x) \|^2 = 4 \|x\|^2$. Introducing polar coordinates (r, α) in $\mathbf{R}^2 \setminus \{ (x_1, 0) \in \mathbf{R}^2 \mid x_1 \leq 0 \}$, which leads to a C^1 change of coordinates, we find
$$\int_{B^2} \| \operatorname{grad} g(x) \|^2 \, dx = \int_{-\pi}^{\pi} \int_0^1 4 r^3 \, dr \, d\alpha = 2\pi [r^4]_0^1 = 2\pi.$$

(ii) $\partial B^2 = S^1$, which implies $\nu(y) = y$. Therefore
$$\Big(g \, \frac{\partial g}{\partial \nu} \Big)(y) = g(y) \langle \, 2(y_1, -y_2), \, (y_1, y_2) \, \rangle = 2 g(y)^2.$$

Note $S^1 = \operatorname{im}(\phi)$ with $\phi(\alpha) = (\cos \alpha, \, \sin \alpha)$. Hence $\omega_\phi(\alpha) = \|(-\sin \alpha, \, \cos \alpha)\| = 1$ and so
$$\int_{\partial B^2} \Big(g \, \frac{\partial g}{\partial \nu} \Big)(y) \, d_1 y = \int_{-\pi}^{\pi} 2(\cos^2 \alpha - \sin^2 \alpha)^2 \, d\alpha = \int_{-\pi}^{\pi} (1 + \cos 4\alpha) \, d\alpha = 2\pi.$$

(iii) We have
$$\operatorname{div}(g \operatorname{grad} g) = \sum_{1 \leq j \leq 2} D_j (g \, D_j g) = \sum_{1 \leq j \leq 2} ((D_j g)^2 + g \, D_j^2 g) = \| \operatorname{grad} g \|^2 + g \, \Delta g.$$

(iv) The assertion follows from application of Gauss' Divergence Theorem 7.8.5 to the vector field $g\ \mathrm{grad}\, g$; indeed,

$$\begin{aligned}\int_\Omega \mathrm{div}(g\ \mathrm{grad}\, g)(x)\, dx &= \int_{\partial\Omega} \langle\, g(y)\ \mathrm{grad}\, g(y),\, \nu(y)\,\rangle\, d_1 y = \int_{\partial\Omega} g(y)\,\langle\, \mathrm{grad}\, g, \nu\,\rangle(y)\, d_1 y \\ &= \int_{\partial\Omega} \Big(g\, \frac{\partial g}{\partial \nu} \Big)(y)\, d_1 y.\end{aligned}$$

(v) Set $f = g$ in Green's first identity

$$\int_\Omega (g\,\Delta f)(x)\, dx = \int_{\partial\Omega} \Big(g\, \frac{\partial f}{\partial \nu} \Big)(y)\, d_{n-1} y - \int_\Omega \langle\, \mathrm{grad}\, f,\ \mathrm{grad}\, g\,\rangle(x)\, dx.$$

(vi) This follows from $\Delta g = 2 - 2 = 0$.