Exercise 0.1 (Application of Implicit Function Theorem). Suppose that $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is a C^∞ function and that there exists a C^∞ function $g : \mathbf{R} \to \mathbf{R}$ satisfying

$$g(0) \neq 0 \qquad \text{and} \qquad f(x; 0) = x\, g(x) \qquad (x \in \mathbf{R}).$$

Consider the equation $f(x; y) = t$, where x and $t \in \mathbf{R}$, while $y \in \mathbf{R}^n$.

(i) Prove the existence of an open neighborhood V of 0 in $\mathbf{R}^n \times \mathbf{R}$ and of a unique C^∞ function $\psi : V \to \mathbf{R}$ such that, for all $(y, t) \in V$

$$\psi(0) = 0 \qquad \text{and} \qquad f(\psi(y, t);\, y) = t.$$

(ii) Establish the following formulae, where D_1 and D_2 denote differentiation with respect to the variables in $\mathbf{R}^n$ and $\mathbf{R}$, respectively:

$$D_1\psi(0) = -\frac{1}{g(0)} D_1 f(0; 0) \qquad \text{and} \qquad D_2\psi(0) = \frac{1}{g(0)}.$$

Solution of Exercise 0.1

(i) Define $F : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by $F(x; y, t) = f(x; y) - t$. Then F is a C^∞ function satisfying

$$F(0; 0, 0) = f(0; 0) = 0 \qquad \text{and} \qquad D_1 F(0; 0, 0) = \frac{d}{dx}\Big|_{x=0} (x\, g(x)) = g(0) \neq 0.$$

The desired conclusion now follows from the Implicit Function Theorem 3.5.1.

(ii) Furthermore on account of the aforementioned theorem we obtain

$$D\psi(y, t) = -D_x F(\psi(y, t); y, t)^{-1} \circ D_{(y,t)} F(\psi(y, t); y, t).$$

In particular, this is valid for $(\psi(y, t); y, t) = (0; 0, 0)$. We have

$$D_{(y,t)} F(0; 0, 0) = (D_y f(0; 0), -1) \qquad \text{and so} \qquad D\psi(0, 0) = -\frac{1}{g(0)} (D_1 f(0; 0), -1),$$

and this leads to the desired formulae.