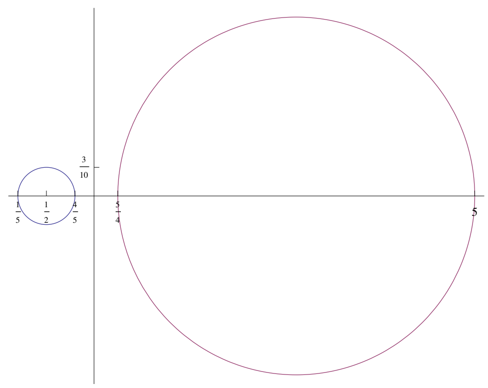
**Exercise 0.1 (Invariance of harmonicity under Kelvin transform – sequel to Exercise 2.40 ).** Set $U = \mathbf{R}^n \setminus \{0\}$ and define the *inversion* $\iota : U \to U$ by $\iota(x) = \frac{1}{\|x\|^2}\, x$.

(i) Compute $\iota^2$ and deduce that $\iota$ is a $C^\infty$ diffeomorphism.

(ii) Verify that $\iota$ preserves the collection of spheres and hyperplanes in $U$.
**Hint:** $S \subset \mathbf{R}^n$ is a nondegenerate sphere or hyperplane if and only if

$$S = \{\, x \in \mathbf{R}^n \mid a\|x\|^2 + \langle\, b, x \,\rangle + c = 0 \,\},$$

where $b \in \mathbf{R}^n$ and $a$ and $c \in \mathbf{R}$ satisfy $\|b\|^2 - 4ac > 0$.




The smaller circle is $\{\, x \in \mathbf{R}^2 \mid 25\|x\|^2 + \langle\, (-25, 0), x \,\rangle + 4 = 0 \,\}$

(iii) Prove that $\iota$ is *conformal*, see Exercise 5.29. This is the case if and only if the derivative $D\iota(x)$ is a scalar multiple of an orthogonal linear transformation, for all $x \in U$.
**Hint:** Fix $x \in U$ and select an orthogonal transformation $A$ such that $Ax = (\|x\|, 0, \dots, 0)$. Clearly $\iota = A^{-1} \circ \iota \circ A$, which entails $D\iota(x) = A^{-1} \circ D\iota(Ax) \circ A$. Now complete the proof by showing that $D\iota(Ax)$ is a scalar multiple of an orthogonal transformation.

(iv) Show that $\det D\iota(x) = -\frac{1}{\|x\|^{2n}}$, for all $x \in U$.

We define the *Kelvin transform* $\mathcal{K} : C(U) \to C(U)$ by

$$\mathcal{K}f(x) = \|x\|^{2-n} f(\iota(x)) \qquad (f \in C(U), x \in U).$$

In the following we will verify the assertion that the Kelvin transform of every harmonic function is harmonic again. In particular, applying the assertion with the constant function $1$ on $U$, we find $\Delta(\frac{1}{\|\cdot\|^{n-2}}) = 0$ on $U$ as in Exercise 2.40.(iv) and Example 7.8.4.
First we demonstrate the result for polynomial functions $p$ that are homogeneous of degree $d \in \mathbf{N}_0$. From Exercise 2.40.(iv) we recall

$$\Delta(\|\cdot\|^{2-n-2d}\, p) = \|\cdot\|^{2-n-2d}\, \Delta p.$$

(v) Show

$$\Delta(\mathcal{K}p) = \mathcal{K}(\|\cdot\|^4\, \Delta p).$$

(vi) Establish the general case of the assertion for $f \in C^2(U)$ by means of the Weierstrass Approximation Theorem on $\mathbf{R}^n$, see Exercises 1.55 and 6.103.

**Solution of Exercise 0.1**

(i) We have

$$\|\iota(x)\| = \frac{1}{\|x\|}, \qquad \text{hence} \qquad \iota^2(x) = \iota(\iota(x)) = \frac{1}{\frac{1}{\|x\|^2}} \frac{1}{\|x\|^2}\, x = x.$$

This proves that $\iota$ is bijective and its own inverse. As the component functions of $\iota$ are of class $C^\infty$, so is $\iota$ and its inverse. This gives the desired result.

(ii) If $x \in U \cap S$, then $c\|\iota(x)\|^2 + \langle\, b, \iota(x)\,\rangle + a = 0$.

(iii) To simplify notation we assume that $x = Ax = (\|x\|, 0, \dots, 0)$ as in the Hint and let $h \in \mathbf{R}^n$ be sufficiently small. Then a straightforward calculation gives the equality

$$\iota(x+h) - \iota(x) = \frac{1}{\|x+h\|^2}\Big( -h_1 - \frac{\|h\|^2}{\|x\|}, h_2, \dots, h_n \Big).$$

Here we used that $\|x+h\|^2 = (\|x\| + h_1)^2 + \|(h_2, \dots, h_n)\|^2$. The equality implies that the matrix of $D\iota(Ax)$ is diagonal with diagonal $\frac{1}{\|x\|^2}(-1, 1, \dots, 1)$ .

(iv) The desired formula is an immediate consequence of the preceding argument.

(v) Since $\Delta p$ is homogeneous of degree $d-2$, we have

$$\begin{aligned}\Delta(\mathcal{K}p) &= \Delta\big(x \mapsto \|x\|^{2-n}\, p(\tfrac{1}{\|x\|^2}x)\big) = \Delta(\|\cdot\|^{2-n-2d}\, p) = \|\cdot\|^{2-n-2d}\,\Delta p \\ &= \|\cdot\|^{2-n} \tfrac{1}{\|\cdot\|^4} \tfrac{1}{\|\cdot\|^{2(d-2)}}\, \Delta p = \mathcal{K}(\|\cdot\|^4\, \Delta p).\end{aligned}$$

(vi) The preceding part shows that the assertion holds for all polynomials (by linearity). On account of the Weierstrass Approximation Theorem arbitrary $C^2$ functions can be locally uniformly approximated by polynomials, which establishes the truth of the assertion for such functions.