Exercise 0.1 (Left-invariant integration on $\mathrm{Mat}(n, \mathbf{R})$**).** As usual, we write $C_0(\mathbf{R}^n)$ for the linear space of continuous functions $f : \mathbf{R}^n \to \mathbf{R}$ having bounded support. Furthermore, we identify the linear space $\mathrm{Mat}(n, \mathbf{R})$ of $n \times n$ matrices over $\mathbf{R}$ with $\mathbf{R}^{n^2}$; in this way, by using n^2-dimensional integration, we assign a meaning to

$$\int_{\mathrm{Mat}(n,\mathbf{R})} f(X)\,dX \qquad \big(f \in C_0(\mathrm{Mat}(n, \mathbf{R}))\big).$$

(i) In particular, suppose $n = 2$ and consider the subgroup

$$\mathbf{SO}(2, \mathbf{R}) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{Mat}(2, \mathbf{R}) \,\middle|\, -\pi < \alpha \le \pi \right\}$$

of all orthogonal matrices in $\mathrm{Mat}(2, \mathbf{R})$ of determinant 1. Without proof one may use that ϕ is a C^∞ embedding if we define

$$\phi : \,]-\pi, \pi[\, \to \mathbf{R}^4 \qquad \text{by} \qquad \phi(\alpha) = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha).$$

Now prove $\mathrm{vol}_1(\mathbf{SO}(2, \mathbf{R})) = 2\pi\sqrt{2}$.

(ii) Prove, for any $f \in C_0(\mathbf{R})$ with $0 \notin \mathrm{supp}\, f$ and any $0 \neq y \in \mathbf{R}$,

$$\int_{\mathbf{R}} \frac{f(y\,x)}{x}\,dx = \int_{\mathbf{R}} \frac{f(x)}{x}\,dx.$$

We now generalize the identity in part (ii) to $\mathrm{Mat}(n, \mathbf{R})$. We shall prove, for every $f \in C_0(\mathrm{Mat}(n, \mathbf{R}))$ with $\mathrm{supp}\, f \subset \mathbf{GL}(n, \mathbf{R})$ (= the group of invertible matrices in $\mathrm{Mat}(n, \mathbf{R})$) and $Y \in \mathbf{GL}(n, \mathbf{R})$,

$$(\star) \qquad \int_{\mathrm{Mat}(n,\mathbf{R})} \frac{f(YX)}{|\det X|^n}\,dX = \int_{\mathrm{Mat}(n,\mathbf{R})} \frac{f(X)}{|\det X|^n}\,dX.$$

Given $Y \in \mathbf{GL}(n, \mathbf{R})$, define

$$\Phi_Y : \mathrm{Mat}(n, \mathbf{R}) \to \mathrm{Mat}(n, \mathbf{R}) \qquad \text{by} \qquad \Phi_Y(X) = Y\,X.$$

(iii) Show that Φ_Y is a C^∞ diffeomorphism satisfying $D\Phi_Y(X) = \Phi_Y$, for all $X \in \mathrm{Mat}(n, \mathbf{R})$.

Denote by $e_1, \dots, e_n$ the standard basis (column) vectors in $\mathbf{R}^n$, then a basis for $\mathrm{Mat}(n, \mathbf{R})$ is formed by the matrices

$$E_{i,j} = (0 \cdots 0\ e_i\ 0 \cdots 0) \qquad (1 \le i,\, j \le n),$$

where e_i occurs in the j-th column. The ordering is lexicographic, but first with respect to j and then to i. In the case of $n = 2$ we thus obtain, in the following order:

$$E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iv) Verify $\Phi_Y(E_{i,j}) = (0 \dots 0\ Ye_i\ 0 \dots 0)$. Deduce that the matrix of Φ_Y with respect to the $(E_{i,j})$ is given in block diagonal form with a copy of Y in each block and that $\det \Phi_Y = (\det Y)^n$.
Hint: First consider explicitly the case of $n = 2$, where the matrix of Φ_Y belongs to $\mathrm{Mat}(4, \mathbf{R})$. Then treat the general case.

(v) Prove $\Phi_Y(\mathbf{GL}(n, \mathbf{R})) \subset \mathbf{GL}(n, \mathbf{R})$. Now show the validity of $(\star)$ above by applying parts (iii) and (iv).

(vi) Select $Y \in \mathbf{GL}(n, \mathbf{R})$ satisfying $\det Y = -1$ and set $f(X) = \det X$. With these data $(\star)$ implies $-1 = 1$. Explain!

Solution of Exercise 0.1

(i) We have

$$\|D\phi(\alpha)\| = \|(-\sin\alpha, \cos\alpha, -\cos\alpha, -\sin\alpha)\| = \sqrt{2}.$$

Therefore integration of the constant function 1 over the submanifold $\mathbf{SO}(2, \mathbf{R})$ with respect to the Euclidean density gives $\int_{-\pi}^{\pi} \sqrt{2}\, d\alpha = 2\pi\sqrt{2}$.

(ii) The formula is a direct consequence of the substitution $x \mapsto y\,x$ in the right-hand side of the given formula.

(iii) The coefficients of the product matrix $Y\,X$ are given by polynomial functions in the coefficients of Y and X, therefore Φ_Y is a C^∞ mapping. As $Y \in \mathbf{GL}(n, \mathbf{R})$, the mapping Φ_Y is invertible, with $\Phi_{Y^{-1}}$ as its inverse; and this shows that Φ_Y is a C^∞ diffeomorphism. The formula for $D\Phi_Y$ follows from Example 2.2.5, because Φ_Y is a linear mapping.

(iv) On account of the properties of matrix multiplication we have

$$\begin{aligned}\Phi_Y(E_{i,j}) &= Y\,E_{i,j} = Y\,(0 \cdots 0\; e_i\; 0 \cdots 0) = (Y0 \cdots Y0\; Ye_i\; Y0 \cdots Y0)\\ &= (0 \cdots 0\; Ye_i\; 0 \cdots 0).\end{aligned}$$

The matrix of Φ_Y is obtained by successively applying Φ_Y to all the basis vectors in $\mathrm{Mat}(n, \mathbf{R})$. Since the resulting $n^2 \times n^2$ matrix contains n identical blocks along the diagonal, the formula for $\det \Phi_Y$ follows.

(v) The inclusion is a consequence of the multiplicative property of the determinant. Application of the Change of Variables Theorem 6.6.1 with $\Psi = \Phi_Y$ leads to $(*)$, because $|\det D\Phi_Y(X)| = |\det \Phi_Y| = |\det Y|^n$, for all $X \in \mathrm{Mat}(n, \mathbf{R})$.

(vi) In this case, the function f has no bounded support. Actually, the integral on the right-hand side of $(*)$ is divergent.