Exercise 0.1 (Laplacian of composition of norm and linear mapping). For x and $y \in \mathbf{R}^n$, recall that $\langle x,\, y\rangle = x^t\, y$ where x^t denotes the transpose of the column vector $x \in \mathbf{R}^n$; and furthermore, that $\|x\| = \sqrt{\langle x,\, x\rangle}$. Fix $A \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R}^p)$ and recall $\ker A = \{\, x \in \mathbf{R}^n \mid Ax = 0 \,\}$. Now define

$$f : \mathbf{R}^n \backslash \ker A \to \mathbf{R} \qquad \text{by} \qquad f = \|\cdot\| \circ A, \qquad \text{i.e.} \qquad f(x) = \|Ax\|; \qquad \text{and set} \qquad f^2(x) = f(x)^2.$$

(i) Give an argument without computations that f is a positive C^∞ function.

(ii) By application of the chain rule to f^2 show, for $x \in \mathbf{R}^n \setminus \ker A$ and $h \in \mathbf{R}^n$,

$$Df(x)h = \frac{\langle Ax,\, Ah\rangle}{f(x)}.$$

Deduce that

$$Df(x) \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R}) \qquad \text{is given by} \qquad Df(x) = \frac{1}{f(x)} x^t\, A^t\, A.$$

Denote by $(e_1, \dots, e_n)$ the standard basis vectors in $\mathbf{R}^n$.

(iii) For $1 \le j \le n$, derive from part (ii) that

$$D_j f(x) = \frac{\langle Ax,\, Ae_j\rangle}{f(x)} \qquad \text{and deduce} \qquad D_j^2 f(x) = \frac{\|Ae_j\|^2}{f(x)} - \frac{\langle Ax,\, Ae_j\rangle^2}{f^3(x)}.$$

As usual, write $\Delta = \sum_{1\le j\le n} D_j^2$ for the Laplace operator acting in $\mathbf{R}^n$ and $\|A\|_{\text{Eucl}}^2 = \sum_{1\le j\le n} \|Ae_j\|^2$.

(iv) Now demonstrate

$$\Delta(\|\cdot\| \circ A)(x) = \frac{\|A\|_{\text{Eucl}}^2\, \|Ax\|^2 - \|A^t\, Ax\|^2}{\|Ax\|^3}.$$

(v) Which form takes the preceding identity if A equals the identity mapping in $\mathbf{R}^n$?

Solution of Exercise 0.1

(i) The function $\sqrt{\cdot} : \,]\,0, \infty\,[\, \to \mathbf{R}$ is of class C^∞. Hence, f is the composition of C^∞ functions, therefore the assertion follows from the chain rule.

(ii) $f^2(x) = \langle Ax,\, Ax\rangle$ implies $Df^2(x)h = 2\langle Ax,\, Ah\rangle$ according to Corollary 2.4.3.(ii). Hence the desired formula follows from $2f(x)\, Df(x)h = Df^2(x)h = 2\langle Ax,\, Ah\rangle$ on account of the chain rule. Furthermore

$$Df(x)h = \frac{\langle Ax,\, Ah\rangle}{f(x)} = \frac{1}{f(x)}\, (Ax)^t\, Ah = \frac{1}{f(x)} x^t\, A^t\, A\, h.$$

(iii) We have

$$D_j f(x) = Df(x)e_j = \frac{\langle Ax,\, Ae_j\rangle}{f(x)}.$$

Application of Corollary 2.4.3.(iii) and (ii) as well as part (ii) implies

$$D_j^2 f(x) = D(D_j f)(x)e_j = \frac{\|Ae_j\|^2}{f(x)} - \frac{\langle Ax,\, Ae_j\rangle^2}{f^3(x)}.$$

(iv) Summation of the preceding identity for j running from 1 to n and $\langle Ax,\, Ae_j\rangle = \langle A^t\, Ax,\, e_j\rangle$ gives

$$\Delta f(x) = \sum_{1\le j\le n} D_j^2 f(x) = \frac{1}{f(x)} \sum_{1\le j\le n} \|Ae_j\|^2 - \frac{1}{f^3(x)} \sum_{1\le j\le n} \langle A^t\, Ax,\, e_j\rangle^2.$$

Furthermore, note that, for all $y \in \mathbf{R}^n$,

$$\sum_{1\le j\le n} \langle y,\, e_j\rangle^2 = \Big\| \sum_{1\le j\le n} \langle y,\, e_j\rangle e_j \Big\|^2 = \|y\|^2.$$

(v) In this case we obtain $\Delta(\|\cdot\|)(x) = \frac{n-1}{\|x\|}$, for $x \in \mathbf{R}^n \setminus \{0\}$ (compare with Exercise 2.40.(iii)).