Exercise 0.1 (**Diffeomorphism from plane onto hyperbolic domain).** We want to parametrize the points belonging to the unbounded open set

$$U = \{ x \in \mathbf{R}^2 \mid |x_1 x_2| < 1 \}$$

by points in all of $\mathbf{R}^2$. Given $x \in U$, note there exists $y \in \mathbf{R}^2$ such that

$$x_1^2 x_2^2 = \frac{y_1^2 y_2^2}{1 + y_1^2 y_2^2}.$$

This suggests to consider

$$\Psi : \mathbf{R}^2 \to \mathbf{R}^2 \qquad \text{given by} \qquad \Psi(y) = f_+(y)\, y \qquad \text{where} \qquad f_\pm(y) = \frac{1}{\sqrt[4]{1 \pm y_1^2 y_2^2}}.$$

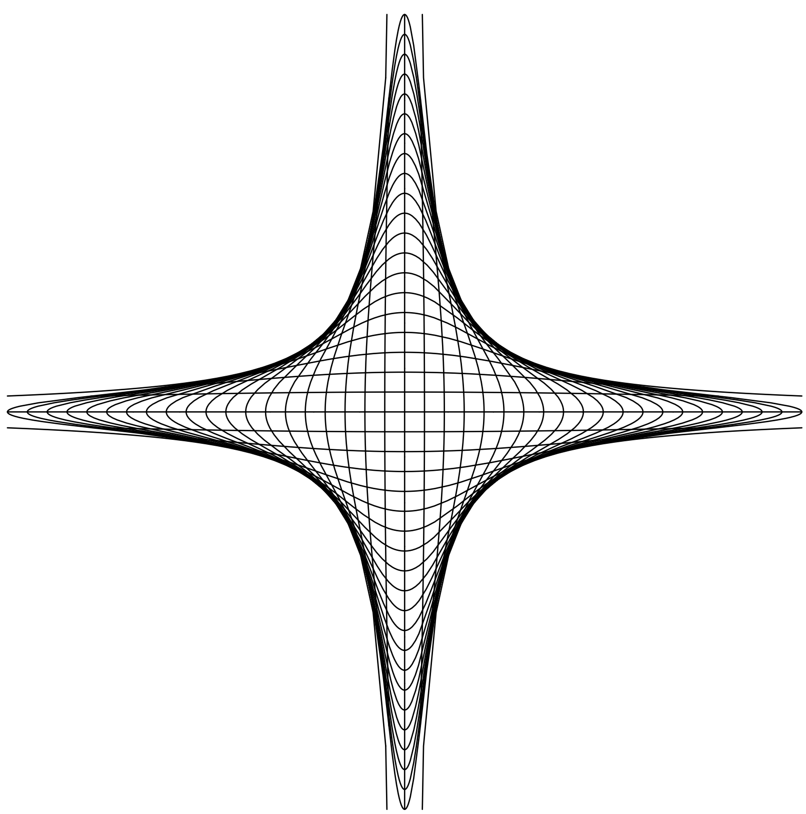
(i) Show that $\Psi : \mathbf{R}^2 \to U$ is a C^∞ diffeomorphism by computing that its inverse $\Phi : U \to \mathbf{R}^2$ satisfies $\Phi(x) = f_-(x)\, x$.

(ii) Prove that $2y_j\, D_j f_+(y) = -y_1^2 y_2^2\, f_+(y)^5$, for $1 \le j \le 2$. Use this to deduce

$$D\Psi(y) = \frac{f_+(y)^5}{2} \begin{pmatrix} 2 + y_1^2 y_2^2 & -y_1^3 y_2 \\ -y_1 y_2^3 & 2 + y_1^2 y_2^2 \end{pmatrix} \qquad \text{and} \qquad \det D\Psi(y) = f_+(y)^6.$$

(iii) Given $y \in \mathbf{R}^2$, consider the curves $s \mapsto \Psi(s, y_2)$ and $t \mapsto \Psi(y_1, t)$ in U. Demonstrate that the curves are C^∞ submanifolds in U of dimension 1. These two submanifolds obviously intersect at the point $\Psi(y)$; show that it is the only point of intersection.

(iv) Verify that the submanifolds from part (iii) are perpendicular at $\Psi(y)$ if and only if $\Psi(y)$ belongs to the intersection of one of the coordinate axes with U.



Solution of Exercise 0.1

(i) Given $x \in U$, consider the equation $x = \Psi(y)$ for $y \in \mathbf{R}^2$. If a solution y exists, then $\mathrm{sgn}(x_j) = \mathrm{sgn}(y_j)$, for $1 \le j \le 2$. Obviously, $y = 0$ is the only solution of $0 = \Psi(y)$. So we may assume that either x_1 or $x_2 \neq 0$, say $x_2 \neq 0$. Then $y_2 \neq 0$ and $\frac{x_1}{x_2} = \frac{y_1}{y_2}$, in other words, $x_1 y_2 = x_2 y_1$. Raising the identity $x_j = \Psi_j(y)$ to the fourth power and taking the indices modulo 2, we obtain

$$x_j^4 = \frac{y_j^4}{1 + y_1^2 y_2^2}, \qquad \text{so} \qquad y_j^4 = x_j^4 + (x_j y_j)^2 (x_j y_{j+1})^2 = x_j^4 + (x_j y_j)^2 (x_{j+1} y_j)^2 = x_j^4 + (x_1^2 x_2^2) y_j^4.$$

In other words,

$$(1 - x_1^2 x_2^2) y_j^4 = x_j^4 \qquad \text{and so} \qquad y_j = f_-(x)\, x_j,$$

where $f_-(x)$ is well-defined because $x \in U$. This proves that there exists a unique solution $y \in \mathbf{R}^2$. In other words, the inverse Φ of $\Psi : \mathbf{R}^2 \to U$ is as given and Ψ is a bijection with an inverse of class C^∞.

(ii) We have

$$D_j f_+(y) = D_j (1 + y_1^2 y_2^2)^{-\frac{1}{4}} = -\frac{1}{4}\, (1 + y_1^2 y_2^2)^{-\frac{5}{4}}\, \frac{2 y_1^2 y_2^2}{y_j} = -\frac{y_1^2 y_2^2}{2 y_j}\, f_+(y)^5.$$

Hence the matrix for $D\Psi(y)$ follows from, for $1 \le i, j \le 2$,

$$D_j \Psi_i(y) = \delta_{ij}\, f_+(y) + D_j f_+(y)\, y_i = \frac{f_+(y)^5}{2} \Big(2\delta_{ij}\, (1 + y_1^2 y_2^2) - \frac{y_i y_1^2 y_2^2}{y_j} \Big).$$

This implies

$$\det D\Psi(y) = \frac{f_+(y)^{10}}{4} (4 + 4 y_1^2 y_2^2) = f_+(y)^6.$$

(iii) All assertions are a direct consequence of the fact that Ψ is a C^∞ diffeomorphism.

(iv) The curves intersect orthogonally at $\Psi(y)$ if and only if the cosine of the angle of intersection is equal to zero. Modulo a strictly positive factor, this cosine is given by

$$\langle\, D_1 \Psi(y),\ D_2 \Psi(y) \,\rangle = -\frac{f_+(y)^{10}}{4} (2 + y_1^2 y_2^2) \|y\|^2\, y_1 y_2.$$

Hence it equals zero if and only if $y_1 y_2 = 0$, and this is the case if and only if $\Psi(y)$ belongs to one of the coordinate axes.