Exercise 0.1 (Vector field on open set is uniquely determined by its curl, divergence and restriction to the boundary of its normal component). We call an open set $\Omega \subset \mathbf{R}^3$ *admissible* if it satisfies the conditions of the Theorem on Integration of a Total Derivative. Let g be a C^2 function on an open neighborhood of an admissible set Ω and denote by $f : \Omega \to \mathbf{R}^3$ the gradient vector field associated to g. Suppose

$$\operatorname{div} f = 0 \quad \text{on } \Omega \qquad \text{and} \qquad \langle\, f, \nu \,\rangle = 0 \quad \text{on } \partial\Omega.$$

Here $\nu(y)$ denotes, as usual, the outer normal to $\partial\Omega$ at $y \in \partial\Omega$.

(i) Prove $\operatorname{curl} f = 0$ on Ω.

(ii) Using Green's first identity show that $f = 0$ on Ω.

Next, consider the special case of

$$g : \mathbf{R}^3 \setminus \{0\} \to \mathbf{R} \qquad \text{with} \qquad g(x) = -\frac{1}{\|x\|} \qquad \text{and set} \qquad f(x) = \operatorname{grad} g(x) = \frac{1}{\|x\|^3}\, x.$$

(iii) Verify $\operatorname{div} f = 0$ on $\mathbf{R}^3 \setminus \{0\}$.

(iv) Deduce from the preceding two parts that there exists no admissible open set $\Omega \subset \mathbf{R}^3 \setminus \{0\}$ having the property that $\mathbf{R}y$ is contained in the tangent space of $\partial\Omega$ at y, for all $y \in \partial\Omega$.

(v) Can you give an example of an admissible set open $\Omega \subset \mathbf{R}^3 \setminus \{0\}$ having the property in part (iv) for "more or less half" of the points $y \in \partial\Omega$?

Background. The conditions $\operatorname{div} f$ and $\langle\, f, \nu \,\rangle = 0$ on the vector field f assert that it is incompressible and that it has no flux through the boundary of Ω. Loosely speaking, these conditions force f to be the vector field of a circulation within Ω, but that is ruled out by the condition that f be irrotational.

Solution of Exercise 0.1

(i) For every $x \in \Omega$, the matrix of $Df(x) \in \operatorname{End}(\mathbf{R}^3)$ is given by $(D_j D_i g(x))_{1 \le i,j \le 3}$, which is symmetric on account of Theorem 2.7.2. Therefore $Af(x) = 0$, and this leads to $\operatorname{curl} f = 0$ on Ω.

(ii) Green's first identity implies

$$\int_\Omega (g\, \Delta g)(x)\, dx + \int_\Omega \|\operatorname{grad} g(x)\|^2\, dx = \int_{\partial\Omega} \Big(g\, \frac{\partial g}{\partial \nu} \Big)(y)\, d_1 y.$$

By our assumptions on f we have $\Delta g = \operatorname{div} \operatorname{grad} g = \operatorname{div} f = 0$ on Ω and $\frac{\partial g}{\partial \nu} = \langle \operatorname{grad} g, \nu \,\rangle = \langle\, f, \nu \,\rangle = 0$ on $\partial\Omega$. It follows that $\int_\Omega \|\operatorname{grad} g(x)\|^2\, dx = 0$. Since the integrand is a nonnegative continuous function on Ω, it follows that $f = \operatorname{grad} g = 0$ on Ω.

(iii) See Example 7.8.4 in the case of $n = 3$.

(iv) Note that the vector $f(y)$ is proportional to y, for all $y \in \partial\Omega$. Now argue by contradiction. Indeed, suppose Ω is a set having the properties described in this part. Then the outer normal $\nu(y)$ is perpendicular to y, and so to $f(y)$, for $y \in \partial\Omega$; but this means $\langle\, f, \nu \,\rangle = 0$ on $\partial\Omega$. Part (iii) then implies that the conclusion of part (ii) holds; in other words, $f = 0$ on Ω. This is a contradiction because f is nowhere zero on Ω.