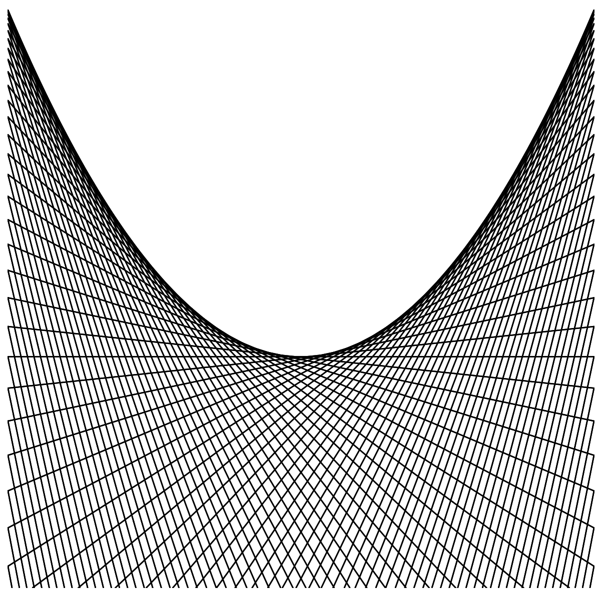
Exercise 0.1 (Viète transformation). Suppose that $y \in \mathbf{R}^2$ satisfies $y_1^2 - y_2 \geq 0$ and let x_1 and $x_2 \in \mathbf{R}$ denote the roots of the monic quadratic polynomial $p(X, y) := X^2 + 2y_1 X + y_2$ in the variable X with coefficients $2y_1$ en y_2.

(i) Prove the following *Viète formulae*: $y_1 = -\frac{1}{2}(x_1 + x_2)$ and $y_2 = x_1 x_2$.

Next consider the *Viète transformation* (from the plane of roots to the plane of coefficients)

$$\Phi : \mathbf{R}^2 \to \mathbf{R}^2 \qquad \text{given by} \qquad \Phi(x) = y = \Big(-\frac{1}{2}(x_1 + x_2), x_1 x_2 \Big).$$

In the illustration below we see the image under Φ of a grid of equidistant straight lines parallel to the coordinate axes (in other words: squared paper). Apparently these lines are mapped under Φ to lines all of which are tangent to a parabola. We shall prove this remarkable result in the following.



(ii) Show that $\Phi(x_1, x_2) = \Phi(x_2, x_1)$ and deduce from this that it is sufficient to prove the result for horizontal lines only.

(iii) Consider $\{ (x_1, x_2) \in \mathbf{R}^2 \mid x_1 \in \mathbf{R} \}$, the horizontal line at the level $x_2 \in \mathbf{R}$. Verify that the image of this line under Φ is equal to

$$L(x_2) = \{ y \in \mathbf{R}^2 \mid p(x_2, y) = 0 \}.$$

Show that $L(x_2)$ is a straight line in $\mathbf{R}^2$ of slope $-2x_2$.

(iv) Determine the set $S \subset \mathbf{R}^2$ of singular points of Φ (i.e., $x \in S$ if and only if $\det D\Phi(x) = 0$) and verify that $P = \Phi(S)$ is a parabola in $\mathbf{R}^2$.

Define $V = \{ y \in \mathbf{R}^2 \mid y_1^2 > y_2 \}$.

(v) Prove that V is the set of points in $\mathbf{R}^2$ that lie below P. Show that $\Phi : \mathbf{R}^2 \setminus S \to V$ is surjective; in particular, demonstrate that we have the C^∞ diffeomorphism

$$\Phi : \{ x \in \mathbf{R}^2 \mid x_1 > x_2 \} \to V.$$

Conclude that $y \in V$ implies $y \in L(x_1) \cap L(x_2)$ with $x_1 \neq x_2$.

(vi) Let $x_2 \in \mathbf{R}$ be a fixed but arbitrarily chosen element. Prove that $L(x_2) \cap P = \Phi(x_2, x_2)$, compute the geometric tangent line of P at this point, and show that this line is equal to $L(x_2)$.

(vii) For every $y \in L(x_2)$ verify that $y_2 = y_1^2 - (y_1 + x_2)^2$; and using this identity give another proof of the statements from part (vi).

(viii) Deduce from the preceding results that passing through every point $y \in V$ there are exactly two distinct lines tangent to P and that these tangents have slopes equal to minus twice the roots of the polynomial $p(X, y)$ in X.

Solution of Exercise 0.1

(i) $x^2 + 2y_1 x + y_2 = (x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 x_2$ implies $y_1 = -\frac{1}{2}(x_1 + x_2)$ and $y_2 = x_1 x_2$.

(ii) The coefficients of $\Phi(x)$ are symmetric in x_1 and x_2. Horizontal lines are of the form $\{x \in \mathbf{R}^2 \mid x_2 = \text{constant}\}$.

(iii) Suppose $y = \Phi(x)$, that is, $2y_1 = -x_1 - x_2$ and $y_2 = x_1 x_2$. Then

$$2x_2 y_1 = -x_1 x_2 - x_2^2 = -y_2 - x_2^2, \qquad \text{so} \qquad p(x_2, y) = 0, \qquad \text{that is} \qquad y_2 = -2x_2 y_1 - x_2^2;$$

and this shows that y belongs to the straight line $L(x_2)$ in $\mathbf{R}^2$ of slope $-2x_2$.

(iv) We have

$$D\Phi(x) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ x_2 & x_1 \end{pmatrix}, \qquad \det D\Phi(x) = -\frac{1}{2}(x_1 - x_2) = 0 \qquad \Longrightarrow \qquad x_1 = x_2.$$

Hence $S = \{ (x_2, x_2) \in \mathbf{R}^2 \mid x_2 \in \mathbf{R} \}$, the diagonal in $\mathbf{R}^2$. Now $(y_1, y_2) = \Phi(x_2, x_2) = (-x_2, x_2^2)$ satisfies $y_1^2 = x_2^2 = y_2$, which implies

$$P = \Phi(S) \subset \{ y \in \mathbf{R}^2 \mid y_1^2 - y_2 = 0 \} =: \widetilde{P}.$$

Conversely, if $y_1^2 = y_2$, then we have $y_2 \geq 0$; hence there exists $x_2 \in \mathbf{R}$ satisfying $y_2 = x_2^2$. Then $y_1^2 = x_2^2$, having a solution $y_1 = -x_2$, that is, $y = \Phi(x_2, x_2)$. It follows that $\widetilde{P} \subset P$ and therefore $P = \widetilde{P}$.

(v) Indeed, given $y \in V$, the system of equations $x_1 + x_2 = -2y_1$ and $x_1 x_2 = y_2$ for $x \in \mathbf{R}^2$ is equivalent to the system $x_1^2 + 2y_1 x_1 + y_2 = 0$ and $x_2 = -x_1 - 2y_1$. The latter system has a solution $x \in \mathbf{R}^2 \setminus S$, because $y \in V$ represents the well-known discriminant criterion for $p(X, y)$ having two distinct real roots. Hence $y = \Phi(x)$, and therefore $y \in L(x_1) \cap L(x_2)$ with $x_1 \neq x_2$.

(vi) Consider $y \in L(x_2) \cap P$. According to part (iv) the condition $y \in P$ implies the existence of $\widetilde{x}_2 \in \mathbf{R}$ such that $y = \Phi(\widetilde{x}_2, \widetilde{x}_2)$. Furthermore, the condition $y \in L(x_2)$ now gives

$$0 = x_2^2 - 2x_2\widetilde{x}_2 + \widetilde{x}_2^2 = (x_2 - \widetilde{x}_2)^2, \qquad \text{so} \qquad x_2 = \widetilde{x}_2, \qquad \text{hence} \qquad y = \Phi(x_2, x_2).$$

The tangent line of P at $\Phi(x_2, x_2)$ is the set of $y \in \mathbf{R}^2$ satisfying

$$(2y_1, -1)\Big|_{y\,=\,(-x_2,\, x_2^2)} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -(2x_2 y_1 + y_2) = 0.$$

As a consequence, the geometric tangent line of P at $\Phi(x_2, x_2)$ equals $\{ y \in \mathbf{R}^2 \mid 2x_2 y_1 + y_2 = c \}$ where $c \in \mathbf{R}$ is determined by $c = -2x_2^2 + x_2^2 = -x_2^2$; in other words, the geometric tangent line equals $L(x_2)$.

(vii) A point $y \in L(x_2)$ satisfies

$$y_2 = -2x_2y_1 - x_2^2, \qquad \text{so} \qquad y_2 - y_1^2 = -(y_1^2 + 2y_1x_2 + x_2^2) = -(y_1 + x_2)^2 \leq 0.$$

This yields that $L(x_2) \subset V \cup P$. Furthermore, $y \in P$ if and only if $y_1 = -x_2$; but then $y_2 = x_2^2$, that is, $y = \Phi(x_2, x_2)$. We have proved that the line $L(x_2)$ lies at one side of the parabola P and intersects P in the point $\Phi(x_2, x_2)$, and this proves the claim.

(viii) Obvious.