Exercise 0.1 (Computation of $\zeta(2)$ by successive integration). Define the open set $J = \left]\,0, \sqrt{2}\,\right[\subset \mathbf{R}$ and the function $m : J \to \mathbf{R}$ by $m(y_1) = \min(y_1, \sqrt{2} - y_1)$.

(i) Sketch the graph of m. Verify that the open subset $\diamondsuit$ of $\mathbf{R}^2$ is a square of area 1 if we set

$$\diamondsuit = \{\, y \in \mathbf{R}^2 \mid y_1 \in J,\ -m(y_1) < y_2 < m(y_1) \,\}.$$

(ii) Define

$$f : \diamondsuit \to \mathbf{R} \qquad \text{by} \qquad f(y) = \frac{1}{2 - y_1^2 + y_2^2}.$$

Compute by successive integration

$$\int_{\diamondsuit} f(y)\, dy = \frac{\pi^2}{12}.$$

At $(\sqrt{2}, 0)$, which belongs to the closure in $\mathbf{R}^2$ of $\diamondsuit$, the integrand f is unbounded. Yet, without proof one may take the convergence of the integral for granted.
Hint: Write the integral the sum of two integrals, one involving $\left]\,0, \frac{1}{2}\sqrt{2}\,\right[$ and one $\left]\,\frac{1}{2}\sqrt{2}, \sqrt{2}\,\right[$, which can be computed to be $\frac{\pi^2}{36}$ and $\frac{\pi^2}{18}$, respectively. In doing so, use that $f(y) = f(y_1, -y_2)$. Furthermore, without proof one may use the following identities, which easily can be verified by differentiation:

$$\begin{aligned}
\int f(y_1, y_2)\, dy_2 &= : g(y_1, y_2) := \frac{1}{\sqrt{2 - y_1^2}} \arctan\Big(\frac{y_2}{\sqrt{2 - y_1^2}}\Big),\\
\int g(y_1, y_1)\, dy_1 &= \frac{1}{2} \arctan^2\Big(\frac{y_1}{\sqrt{2 - y_1^2}}\Big),\\
\int g(y_1, \sqrt{2} - y_1)\, dy_1 &= -\arctan^2\Bigg(\sqrt{\frac{\sqrt{2} - y_1}{\sqrt{2} + y_1}}\Bigg).
\end{aligned}$$

Introduce the open set $I = \left]\,0, 1\,\right[\subset \mathbf{R}$, and furthermore the counterclockwise rotation of $\mathbf{R}^2$ about the origin by the angle $\frac{\pi}{4}$ by

$$\Psi \in \mathrm{End}(\mathbf{R}^2) \qquad \text{with} \qquad \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \text{set} \qquad \square = I^2 \subset \mathbf{R}^2.$$

(iii) Show that $\Psi : \diamondsuit \to \square$ is a C^∞ diffeomorphism and using this fact deduce from part (ii)

$$\int_{\square} \frac{1}{1 - x_1 x_2}\, dx = \frac{\pi^2}{6}.$$

(iv) Conclude from part (iii)

$$\int_I \frac{\log(1 - x)}{x}\, dx = -\frac{\pi^2}{6}.$$

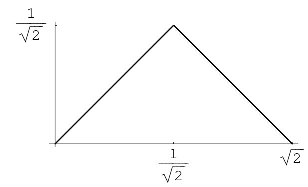
Give arguments that the integrand is a bounded continuous function on I near 0.

(v) Compute $\int_{\square} (x_1 x_2)^{k-1}\, dx$, for $k \in \mathbf{N}$. Assuming without proof that in this particular case summation of an infinite series and integration may be interchanged, use part (iii) (or part (iv)) to show Euler's celebrated identity

$$\zeta(2) := \sum_{k \in \mathbf{N}} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Solution of Exercise 0.1

(i) $\mathrm{graph}(m)$ is given by



This is an isosceles rectangular triangle of hypothenuse $\sqrt{2}$, hence its area equals $\frac{1}{2}$.

(ii) Note $J = \frac{1}{2}J \cup (\frac{1}{2}\sqrt{2} + \frac{1}{2}J)$ while the two subintervals have only one point in common. On $\frac{1}{2}J$ and $\frac{1}{2}\sqrt{2} + \frac{1}{2}J$ one has $m(y_1) = y_1$ and $m(y_1) = \sqrt{2} - y_1$, respectively. Furthermore $f(y) = f(y_1, -y_2)$. Therefore, using a generalization of Corollary 6.4.3 on interchanging the order of integration and the antiderivatives as given in the hint, one obtains

$$\begin{aligned}
\int_{\Diamond} f(y)\,dy &= 2\int_0^{\frac{1}{2}\sqrt{2}}\int_0^{y_1} f(y)\,dy_2\,dy_1 + 2\int_{\frac{1}{2}\sqrt{2}}^{\sqrt{2}}\int_0^{\sqrt{2}-y_1} f(y)\,dy_2\,dy_1 \\
&= 2\int_0^{\frac{1}{2}\sqrt{2}} g(y_1, y_1)\,dy_1 + 2\int_{\frac{1}{2}\sqrt{2}}^{\sqrt{2}} g(y_1, \sqrt{2}-y_1)\,dy_1 \\
&= \arctan^2\left(\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}}}\right) + 2\arctan^2\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi^2}{36} + \frac{\pi^2}{18} = \frac{\pi^2}{12},
\end{aligned}$$

because $\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$.

(iii) The rotations Ψ and Ψ^{-1} are bijective and C^∞; hence, Ψ is a C^∞ diffeomorphism. From the description of Ψ as a specific rotation one gets $\Psi(\Diamond) = \Box$. Thus, $\Psi : \Diamond \to \Box$ is a C^∞ diffeomorphism. Observe that, for $y \in \Diamond$ and $x = \Psi(y) \in \Box$,

$$\frac{1}{1 - x_1x_2} = \frac{1}{1 - \frac{1}{2}(y_1 - y_2)(y_1 + y_2)} = 2f(y) \qquad \text{and} \qquad |\det D\Psi(y)| = 1.$$

Application of the Change of Variables Theorem 6.6.1 now leads to the desired equality.

(iv) Note that

$$\int_I \frac{1}{1 - x_1x_2}\,dx_2 = \Big[-\frac{\log(1 - x_1x_2)}{x_1}\Big]_0^1 = -\frac{\log(1 - x_1)}{x_1}.$$

Since $\Box = I \times I$, one obtains the desired formula by means of Corollary 6.4.3 once more. Taylor series expansion of the integrand about 0 shows that it equals $-1 + \mathcal{O}(x)$, for $x \downarrow 0$.

(v) Obviously

$$\int_\Box x_1^{k-1}x_2^{k-1}\,dx = \Big(\int_I x^{k-1}\,dx\Big)^2 = \frac{1}{k^2}.$$

Summation of the geometric series leads to

$$\sum_{k \in \mathbf{N}} (x_1x_2)^{k-1} = \frac{1}{1 - x_1x_2}.$$

Integrating the equality over $\square$ and interchanging summation of an infinite series and integration one finds, on the basis of part (iii)

$$\sum_{k\in\mathbf{N}} \frac{1}{k^2} = \sum_{k\in\mathbf{N}} \int_{\square} (x_1x_2)^{k-1}\,dx = \int_{\square} \frac{1}{1-x_1x_2}\,dx = \frac{\pi^2}{6}.$$

Background. Compare this exercise with Exercise 6.39. Note that the definition of the integral in part (ii) needs some care, as the integrand f becomes infinite at the corner $(\sqrt{2},0)$ of the closure of $\lozenge$. Since f is continuous and positive on the open set $\lozenge$, in order to prove convergence of the integral it suffices to show the existence of an increasing sequence of compact Jordan measurable sets $K_k \subset \lozenge$ such that $\cup_{k\in\mathbf{N}} K_k = \lozenge$ and that the $\int_{K_k} f(y)\,dy$ exist and converge as $k \to \infty$, see Theorem 6.10.6. One may do this, by choosing the subsets K_k to be the closures of the contracted squares $\frac{k-1}{k}\lozenge$.

Next, the antiderivatives in part (ii) may be computed as follows. For the first one, write

$$f(y) = \frac{1}{\sqrt{2-y_1^2}}\,\frac{1}{1+\big(\frac{y_2}{\sqrt{2-y_1^2}}\big)^2}\,\frac{d}{dy_2}\frac{y_2}{\sqrt{2-y_1^2}} \qquad \text{and set} \qquad u = u(y_2) = \frac{y_2}{\sqrt{2-y_1^2}};$$

further, use $\int \frac{1}{1+u^2}\,du = \arctan u$. For the second antiderivative, apply the change of variables

$$v = v(y_1) = \frac{y_1}{\sqrt{2-y_1^2}}, \quad \text{so} \quad y_1 = \sqrt{2}\,\frac{v}{\sqrt{1+v^2}}, \qquad \sqrt{2-y_1^2} = \frac{\sqrt{2}}{(1+v^2)^{\frac{1}{2}}}, \qquad \frac{dy_1}{dv} = \frac{\sqrt{2}}{(1+v^2)^{\frac{3}{2}}}.$$

Thus,

$$\int g(y_1,y_1)\,dy_1 = \int \frac{\arctan v}{1+v^2}\,dv = \frac{1}{2}\arctan^2 v.$$

For the third antiderivative, apply the change of variables

$$w = w(y_1) = \frac{\sqrt{2}-y_1}{\sqrt{2-y_1^2}}, \quad \text{so} \quad y_1 = \sqrt{2}\,\frac{1-w^2}{1+w^2}, \qquad \sqrt{2-y_1^2} = \frac{2\sqrt{2}w}{1+w^2}, \qquad \frac{dy_1}{dv} = -\frac{4\sqrt{2}w}{(1+w^2)^2}.$$

Thus,

$$\int g(y_1,y_1)\,dy_1 = -2\int \frac{\arctan w}{1+w^2}\,dv = -\arctan^2 w.$$