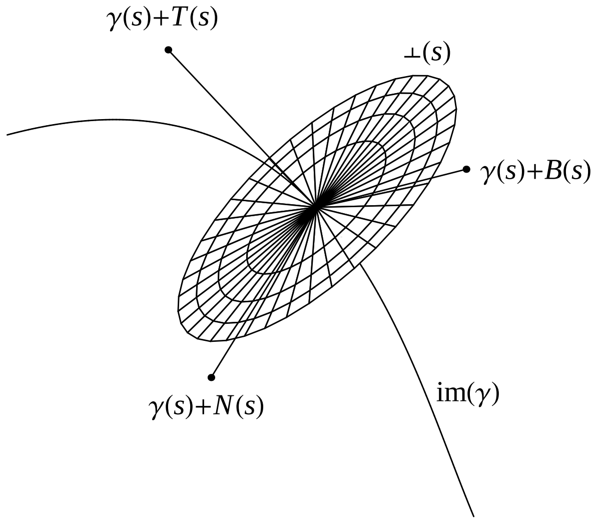
HERKANSINGSTENTAMEN WISB 212
Analyse in Meer Variabelen

29–08–2005 9–12 uur

- *Zet uw naam en collegekaartnummer op elk blad, en op het eerste blad het totaal aantal ingeleverde bladzijden.*
- *De verschillende onderdelen van de vraagstukken zijn zoveel als mogelijk is, onafhankelijk van elkaar. Indien u een bepaald onderdeel niet of slechts ten dele kunt maken, mag u de resultaten daaruit gebruiken bij het maken van de volgende onderdelen. Raak dus niet ontmoedigd indien het u niet lukt een bepaald onderdeel te maken en ga gewoon door.*
- *De vraagstukken tellen* **NIET** *evenzwaar: Vraagstuk 1 telt voor 40 punten en Vraagstuk 2 voor 60 punten.*
- *De antwoorden mag u uiteraard in het Nederlands geven, ook al zijn de vraagstukken in het Engels geformuleerd.*
- *Bij dit tentamen mogen syllabi, aantekeningen en/of rekenmachine* **NIET** *worden gebruikt.*

Exercise 1.1 (Formulae of Serret–Frenet). Let $J \subset \mathbf{R}$ be an open interval in $\mathbf{R}$ and let $\gamma : J \to \mathbf{R}^3$ be a C^∞ curve in $\mathbf{R}^3$. For any $s \in J$, denote by $\perp(s)$ the plane in $\mathbf{R}^3$ that contains the point $\gamma(s)$ and is perpendicular to the tangent vector $T(s) := \gamma'(s) \in \mathbf{R}^3$ of $\mathrm{im}(\gamma)$ at $\gamma(s)$. In this exercise, $'$ denotes the derivative of a mapping defined on J with respect to the variable in J.



(i) Prove $\perp(s) = \{\, x \in \mathbf{R}^3 \mid \langle x - \gamma(s),\, T(s)\rangle = 0 \,\}$.

(ii) Consider $x \in \mathbf{R}^3$ and suppose the function $x \mapsto \|x - \gamma(s)\|$ attains a minimum at $s_0 \in J$. Show $x \in \perp(s_0)$.

Now suppose that γ be parametrized by arc length, in other words, that $\|T(s)\| = 1$, and furthermore, that $\gamma''(s) \neq 0$, for all $s \in J$. Write $N(s) \in \mathbf{R}^3$ for the unit vector in the direction $\gamma''(s)$.

(iii) Prove that $N(s)$ is perpendicular to $T(s)$. (This fact justifies calling $N(s)$ the *principal normal* to $\mathrm{im}(\gamma)$ at $\gamma(s)$.)

Define the *binormal* $B(s) \in \mathbf{R}^3$ by $B(s) := T(s) \times N(s)$. Note that $\|B(s)\| = 1$ and that the triple of mutually orthogonal unit vectors $(T(s)\,N(s)\,B(s))$ in $\mathbf{R}^3$ is positively oriented, in other words, the matrix

$$O(s) := (T(s)\;\; N(s)\;\; B(s)) \in \mathbf{SO}(3, \mathbf{R}) \qquad (s \in J).$$

(iv) Deduce that $N(s) \times B(s) = T(s)$ and $B(s) \times T(s) = N(s)$, for all $s \in J$.

(v) Show that $\perp(s) = \{\, \gamma(s) + \lambda_1 N(s) + \lambda_2 B(s) \in \mathbf{R}^3 \mid \lambda \in \mathbf{R}^2 \,\}$.

Next we study the rate of change of the matrix-valued mapping $J \ni s \mapsto O(s) \in \mathrm{Mat}(3, \mathbf{R})$.

(vi) By differentiating the identity $O(s)^t\, O(s) = I$ (where ^{t} denotes the transpose) verify

$$(O(s)^t\, O'(s))^t + O(s)^t\, O'(s) = 0 \qquad (s \in J),$$

and deduce that there exists a mapping $J \to \mathrm{A}(3, \mathbf{R})$, the linear subspace in $\mathrm{Mat}(3, \mathbf{R})$ consisting of antisymmetric matrices, with $s \mapsto A(s)$ such that

$$O(s)^t\, O'(s) = A(s), \qquad \text{hence} \qquad O'(s) = O(s)A(s) \qquad (s \in J). \tag{1}$$

(vii) Show that we can find a mapping $a : J \to \mathbf{R}^3$ so that we have the following equality of matrix-valued mappings on J:

$$(T'\,N'\,B') = (T\,N\,B)\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

In particular, $\gamma''(s) = T'(s) = a_3(s)N(s) - a_2(s)B(s)$. On the other hand, by definition, $\gamma''(s)$ is a scalar multiple, say $\kappa(s) \geq 0$, of $N(s)$, and this implies $\kappa(s) = a_3(s)$ and $a_2(s) = 0$. We call $\kappa(s)$ the *curvature* and $\tau(s) := a_1(s)$ the *torsion* of $\mathrm{im}(\gamma)$ at $\gamma(s)$. We now have obtained the following *formulae of Frenet–Serret*:

$$(\star) \qquad (T'\,N'\,B') = (T\,N\,B)\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}, \qquad \text{that is} \qquad \begin{aligned} T' &= \kappa\,N \\ N' &= -\kappa\,T + \tau\,B \\ B' &= -\tau\,N \end{aligned}$$

Finally, consider the special case of the *helix* $\gamma : \mathbf{R} \to \mathbf{R}^3$ given by $\gamma(s) = \frac{1}{\sqrt{2}}(\cos s, \sin s, s)$.

(viii) Under this assumption, compute $T(s)$, $N(s)$, $B(s)$, $\kappa(s)$ and $\tau(s)$, for all $s \in \mathbf{R}$.
Hint: $\kappa(s) = \tau(s)$.

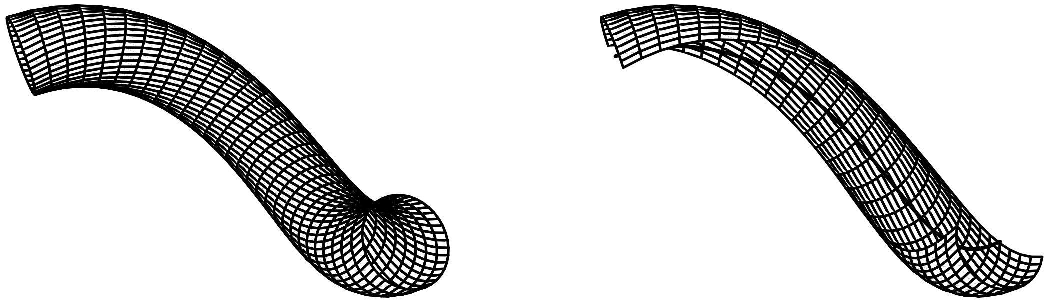


Illustration for Exercise 2.

Exercise 1.2 (Tubular neighborhood of curve). All the notation is as in the preceding exercise. Now define $\mathrm{tub}(r)$, the *tubular surface* at a distance $r > 0$ from the curve γ, by means of

$$\mathrm{tub}(r) := \bigcup_{s\in J} \mathrm{tub}(s,r) := \bigcup_{s\in J} \{\, x \in \perp(s) \mid \|x-\gamma(s)\| = r \,\}.$$

See the illustration on the previous page.

(i) Prove that $\mathrm{tub}(r) = \mathrm{im}(\phi)$ where

$$\phi : J \times \,]-\pi,\, \pi\,] \to \mathbf{R}^3 \qquad \text{is given by} \qquad \phi(s,\alpha) = \gamma(s) + r\cos\alpha\, N(s) + r\sin\alpha\, B(s).$$

(ii) Using the formulae $(\star)$ of Frenet–Serret from the preceding exercise show

$$\frac{\partial\phi}{\partial s}(s,\alpha) \;=\; (1 - r\,\kappa(s)\cos\alpha)T(s) - r\,\tau(s)\sin\alpha\, N(s) + r\,\tau(s)\cos\alpha\, B(s)$$

$$\frac{\partial\phi}{\partial\alpha}(s,\alpha) \;=\; -r\sin\alpha\, N(s) + r\cos\alpha\, B(s), \qquad \Big\| \frac{\partial\phi}{\partial s} \times \frac{\partial\phi}{\partial\alpha}(s,\alpha) \Big\| = r(1 - r\,\kappa(s)\,\cos\alpha).$$

(iii) Verify that ϕ is an immersion under the assumption that $\kappa(s) < \frac{1}{r}$, for all $s \in J$. Deduce that for every point in $J \times \,]-\pi,\, \pi\,]$ there exists a neighborhood D such that $\phi(D) \subset \mathrm{tub}(r)$ is a C^∞ submanifold in $\mathbf{R}^3$ of dimension 2.

(iv) Suppose that γ is an embedding and that, for every $x \in \mathrm{tub}(r)$, there exists only one $s \in J$ such that $\|x - \gamma(s)\| \le r$. Use part (ii) of Exercise 1.1 to prove that ϕ is an embedding.

From now on assume that γ and ϕ are embeddings and that γ is of finite length.

(v) Conclude $\mathrm{area}_2(\mathrm{tub}(r)) = 2\pi r\,\mathrm{length}(\gamma)$.

Furthermore, define $\mathrm{Tub}(r)$, the open *tubular neighborhood* of radius r of the curve γ, by means of

$$\mathrm{Tub}(r) := \bigcup_{0\le\rho<r} \mathrm{tub}(\rho).$$

(vi) Prove $\mathrm{vol}_3(\mathrm{Tub}(r)) = \pi r^2\,\mathrm{length}(\gamma)$.

Finally, consider the C^∞ mapping

$$\Psi : J \times \mathbf{R}^2 \to \mathbf{R}^3 \qquad \text{given by} \qquad \Psi(s,t,u) = \gamma(s) + t\,N(s) + u\,B(s).$$

(vii) Compute

$$\det D\Psi(s,t,u) = \Big\langle \frac{\partial\Psi}{\partial s},\, \frac{\partial\Psi}{\partial t} \times \frac{\partial\Psi}{\partial u} \Big\rangle(s,t,u) = 1 - \kappa(s)\,t.$$

Next, suppose $D(s) \subset \mathbf{R}^2$ is an open and Jordan measurable set and introduce the planar sets $U(s) \subset \mathbf{R}^3$, for $s \in J$, and the solid $U \subset \mathbf{R}^3$ by

$$U(s) = \{\, \Psi(s,t,u) \in \mathbf{R}^3 \mid (t,u) \in D(s) \,\} \qquad \text{and} \qquad U = \bigcup_{s\in J} U(s).$$

(viii) Assume that $\Psi : \bigcup_{s\in J}\{s\} \times D(s) \to U$ is a C^∞ diffeomorphism with positive Jacobi determinant. Prove

$$\mathrm{vol}_3(U) = \int_J \Big(\mathrm{area}\,(D(s)) - \kappa(s) \int_{D(s)} t\,d(t,u) \Big)\,ds = \int_{\mathrm{im}(\gamma)} \Big(\mathrm{area}\,(U(s)) - \kappa(s) \int_{U(s)} y_1\,d_2y \Big)\,d_1s.$$

(ix) **(Extra, no part of the examination.)** Apply the formula from the previous part in the case of the helix $\gamma : J =: \,]-\pi,\, \pi\,[\, \to \mathbf{R}^3$ as in part (viii) of Exercise 1.1 and $D(s) = \{\, (t, u) \in \mathbf{R}^2 \mid 0 < t,\, u < 1 \,\}$, for all $s \in J$, to show that $\mathrm{vol}_3(U) = 2\pi(1 - \frac{\sqrt{2}}{4}) = 4.061\,743 \cdots$ in this case.

Background. The result in part (vi) above is a very special case of a result of H. Weyl: On the volume of tubes, Amer. J. Math. **61** (1939) 461-472. This paper has been very influential in modern differential geometry. Remarkable is that the formulae in parts (v) and (vi) are independent of the amount of "twisting" of the curve $\mathrm{im}(\gamma)$.

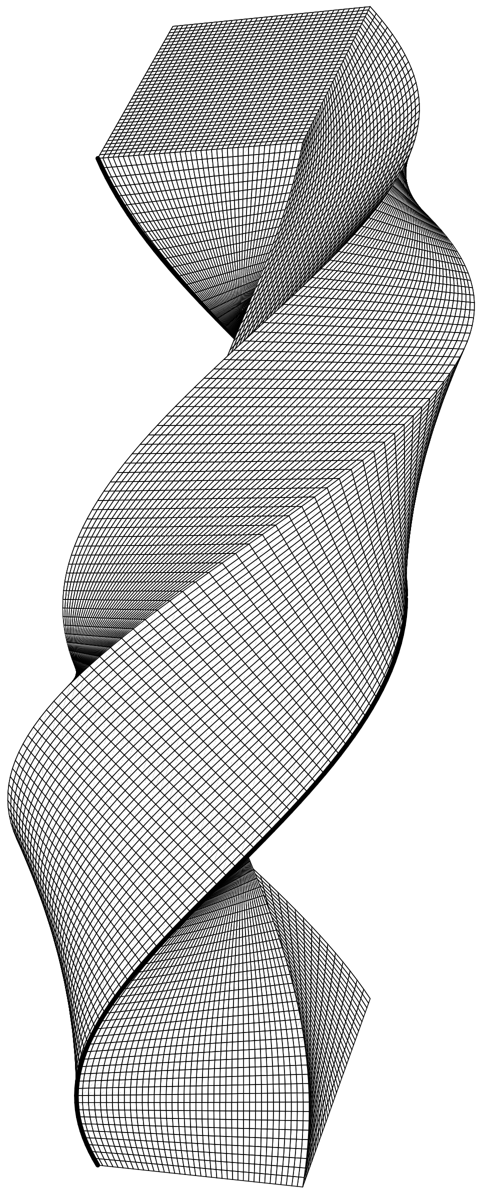


Illustration for part (ix).

Solution of Exercise 1.1

(i) Straightforward application of linear algebra.

(ii) Consider the function $s \mapsto \|x-\gamma(s)\|^2 = \langle x-\gamma(s),\, x-\gamma(s)\rangle$. If it attains a minimum at s_0, its derivative has to vanish at s_0, in other words,

$$\langle\, x-\gamma(s_0),\, \gamma'(s_0)\,\rangle = \langle\, x-\gamma(s_0),\, T(s_0)\,\rangle = 0, \qquad \text{that is} \qquad x \in \perp(s_0).$$

(iii) One obtains

$$\langle\, \gamma'(s),\, \gamma'(s)\,\rangle = 1 \quad\Longrightarrow\quad 2\langle\, \gamma''(s),\, \gamma'(s)\,\rangle = 0 \quad\Longrightarrow\quad \langle\, N(s),\, T(s)\,\rangle = 0.$$

(iv) The matrix $O(s)$ maps the standard basis vectors e_1, e_2 and e_3 in $\mathbf{R}^3$ to $T(s)$, $N(s)$ and $B(s)$, respectively, and preserves exterior products as an element of $\mathbf{SO}(3,\mathbf{R})$. As $e_j \times e_{j+1} = e_{j+2}$ where the indices are taken modulo 3, the desired identities follow.

Solution of Exercise 1.2

(i) If $x = \phi(s,\alpha)$, then $x = \gamma(s) + \lambda_1 N(s) + \lambda_2 B(s) \in \perp(s)$ according to Exercise ***. Furthermore

$$\|x-\gamma(s)\| = r\|\cos\alpha\, N(s) + \sin\alpha\, B(s)\| = r,$$

since $N(s)$ and $B(s)$ are mutually perpendicular unit vectors. Thus, $\mathrm{im}\,\phi \subset \mathrm{tub}(r)$. Conversely, suppose $x \in \mathrm{tub}(r)$, then $x \in \mathrm{tub}(s,r)$, for some $s \in J$. Hence $x \in \perp(s)$ and $\|x-\gamma(s)\| = r$, that is

$$x = \gamma(s) + r\cos\alpha\, N(s) + r\sin\alpha\, B(s) = \phi(s,\alpha),$$

for some $\alpha \in\,]-\pi,\, \pi\,]$. Therefore, $\mathrm{tub}(r) \subset \mathrm{im}\,\phi$.

(ii) We have

$$\begin{aligned}
\frac{\partial\phi}{\partial s}(s,\alpha) &= \gamma'(s) + r\cos\alpha\, N'(s) + r\sin\alpha\, B'(s)\\
&= T(s) + r\cos\alpha\big(-\kappa(s)\,T(s) + \tau(s)\,B(s)\big) - r\sin\alpha\,\tau(s)\,N(s)\\
&= (1 - r\,\kappa(s)\cos\alpha)T(s) - r\,\tau(s)\sin\alpha\, N(s) + r\,\tau(s)\cos\alpha\, B(s),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\phi}{\partial\alpha}(s,\alpha) &= -r\sin\alpha\, N(s) + r\cos\alpha\, B(s),\\
\frac{\partial\phi}{\partial s}\times\frac{\partial\phi}{\partial\alpha}(s,\alpha) &= -r\sin\alpha\big(1 - r\cos\alpha\,\kappa(s)\big)B(s) - r\cos\alpha\big(1 - r\cos\alpha\,\kappa(s)\big)N(s)\\
&\qquad -r^2\tau(s)\sin\alpha\,\cos\alpha\, T(s) + r^2\tau(s)\sin\alpha\,\cos\alpha\, T(s)\\
&= -r\big(1 - r\cos\alpha\,\kappa(s)\big)\big(\sin\alpha\, B(s) + \cos\alpha\, N(s)\big)\\
\Big\|\frac{\partial\phi}{\partial s}\times\frac{\partial\phi}{\partial\alpha}(s,\alpha)\Big\| &= r(1 - r\,\kappa(s)\,\cos\alpha).
\end{aligned}$$

(iii)

$$r\,\kappa(s) < 1 \quad\Longrightarrow\quad 1 - r\,\kappa(s)\cos\alpha > 0 \quad\Longrightarrow\quad \frac{\partial\phi}{\partial s}\times\frac{\partial\phi}{\partial\alpha}(s,\alpha) \neq 0,$$

which implies that $\frac{\partial\phi}{\partial s}(s,\alpha)$ and $\frac{\partial\phi}{\partial\alpha}(s,\alpha)$ are linearly independent, that is $\mathrm{rank}\,D\phi = 2$, in other words, ϕ is an immersion. The second assertion is the Immersion Theorem **??**.

(iv)

$$\begin{aligned}\operatorname{area}\big(\operatorname{tub}(r)\big) &= \int_{J\times\,]-\pi,\pi]} \Big\| \frac{\partial\phi}{\partial s}\times\frac{\partial\phi}{\partial\alpha}(s,\alpha)\Big\|\, d(s,\alpha)\\ &= \int_J \Big(\int_{-\pi}^{\pi} r(1-r\,\kappa(s)\,\cos\alpha)\,d\alpha\Big)\,dr = \pi r^2\,\operatorname{length}(\gamma).\end{aligned}$$

(v) The Change of Variables Theorem **??** implies

$$\begin{aligned}\operatorname{vol}_3(U) &= \int_U dx = \int_{\bigcup_{s\in J}(\{s\}\times D(s))} (1-\kappa(s)\,t)\,d(s,t,u)\\ &= \int_J\Big(\int_{D(s)}(1-\kappa(s)\,t)\,d(t,u)\,ds = \int_J\Big(\operatorname{area}(D(s)) - \kappa(s)\int_{D(s)} t\,d(t,u)\Big)\,ds\\ &= \int_{\operatorname{im}(\gamma)}\Big(\operatorname{area}(U(s)) - \kappa(s)\int_{U(s)} y_1\,d_2y\Big)\,d_1s.\end{aligned}$$