Sums of generalized harmonic series and volumes

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Introduction. It is well-known that the Riemann zeta-function (as defined by Euler):

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} \qquad (s \in \mathbf{R},\ s > 1) \tag{1}$$

takes "elementary" values (i.e., values definable in a first-year calculus class) when s is a positive even integer; more precisely:

$$\zeta(2n) = (-1)^{n-1} B_{2n} \frac{1}{2(2n)!} (2\pi)^{2n} \qquad (n \in \mathbf{N}).$$

Here, B_{2n} denotes the $2n$-th Bernoulli number; we use the convention whereby the rational numbers B_n are defined from the Maclaurin series coefficients of the following function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{2}$$

Equivalently, the series (1) can be replaced by one with the same terms restricted to odd values of k, yielding:

$$(1 - 2^{-s})\zeta(s) = \sum_{k=0}^{\infty} (2k+1)^{-s},$$

whence:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n}} = (-1)^{n-1} B_{2n} \frac{2^{2n}-1}{2(2n)!} \pi^{2n} \qquad (n \in \mathbf{N}). \tag{3}$$

If we replace the series of either (1) (for $s = 2n$) or (3) with another of the same terms in absolute value but *alternating in sign*, then there is no similar elementary formula for the sum. However, for *odd* positive integer values of s there is the similar formula due to Euler:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = E_n \frac{1}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}, \tag{4}$$

where the Euler numbers E_n, for $n \in \mathbf{N}_0$, are the natural numbers defined according to:

$$\sec t = \frac{1}{\cos t} = \sum_{n=0}^{\infty} E_n \frac{t^{2n}}{(2n)!} \qquad (|t| < \frac{\pi}{2}). \tag{5}$$

Among the conventional proofs of (3) and (4), most of them involve either residue theory of complex functions, or Mittag-Leffler expansions, or techniques from Fourier analysis. This means that the average undergraduate majoring in mathematics could rarely learn it before the senior year (unless she/he is unusually talented).

The aim of this paper is to present a simultaneous proof of (3) and (4) in the joint formulation:

$$\sum_{k=0}^{\infty} \frac{(-1)^{nk}}{(2k+1)^n} = \delta_n \left(\frac{\pi}{2}\right)^n \qquad (n \in \mathbf{N}). \tag{6}$$

Here each δ_n is a positive rational number, arising as the volume of a certain n-dimensional convex polytope Δ^n with rational vertices. By direct computation of $\mathrm{vol}_n\, \Delta^n$ we find:

$$\sum_{n=1}^{\infty} \delta_n\, t^{n-1} = \frac{1}{2}(\sec t + \tan t) \qquad \left(|t| < \frac{\pi}{2}\right). \tag{7}$$

On expanding the right side of (7) we obtain the expressions for the δ_n in terms of the Bernoulli and Euler numbers as indicated above.

The proof requires no special knowledge beyond what is included in a good second-year calculus class, and it is in four steps. Step I: using standard techniques on convergence we convert the sum of the series into an integral of a rational function over the unit cube in $\mathbf{R}^n$. Step II: by means of a nontrivial substitution of variables, and this is the heart of the matter, the integral is shown to be equal to $\delta_n(\frac{\pi}{2})^n$, with δ_n as above. Step III: by an analysis of some elementary inequalities, we dissect Δ^n into n congruent pyramids; and we express δ_n in terms of the $(n-1)$-dimensional volume of the basis of such a pyramid. Step IV: the volume of this basis is determined by means of a two-step recursion. This requires mathematical induction with respect to n.

From series to integral over a cube. We begin with the following

Lemma 1. *The infinite series (6) converges for each $n \in \mathbf{N}$, and its sum is represented by the integral:*

$$\sum_{k=0}^{\infty} \frac{(-1)^{nk}}{(2k+1)^n} = \int_{\Box^n} \frac{1}{1-(-1)^n (x_1 \cdots x_n)^2}\, dx. \tag{8}$$

Here $\Box^n$ is the open unit cube of $x \in \mathbf{R}^n$ satisfying $0 < x_i < 1$, for $1 \le i \le n$, while $\int_{\Box^n} \cdots dx$ denotes n-dimensional integration over $\Box^n$ with respect to variables $x = (x_1, \ldots, x_n)$.

Proof: First note that the definition of the integral needs some care in the case of n even, as the integrand becomes infinite at the corner $(1, 1, \ldots, 1)$ of the closure of $\Box^n$. Since the integrand is positive in the open set $\Box^n$, in order to prove convergence of the integral it suffices to show the existence of an increasing sequence of sets $K_\lambda \subset \Box^n$ such that $\cup_{0<\lambda<1} K_\lambda = \Box^n$ and that the $\int_{K_\lambda} \frac{1}{1-(-1)^n(x_1\cdots x_n)^2}\, dx$ exist and converge as $\lambda \uparrow 1$. We shall do this, thereby proving our lemma, by choosing the subsets K_λ to be the contracted hypercubes $\lambda\Box^n$ with $0 < \lambda < 1$. Note that:

$$\int_{\lambda\Box^n} \frac{1}{1-(-1)^n (x_1\cdots x_n)^2}\, dx = \int_{\Box^n} \frac{\lambda^n}{1-(-\lambda^2)^n (x_1\cdots x_n)^2}\, dx.$$

Expansion of the right side into a geometric series yields:

$$\int_{\Box^n} \sum_{k=0}^{\infty} \lambda^{n(2k+1)} (-1)^{nk} (x_1 \cdots x_n)^{2k}\, dx.$$

For fixed $0 < \lambda < 1$, the series converges uniformly for $x \in \square^n$, allowing us to interchange integration and summation to obtain:

$$\sum_{k=0}^{\infty} \lambda^{n(2k+1)}(-1)^{nk} \int_{\square^n} (x_1 \cdots x_n)^{2k}\, dx.$$

The integration is easy now in view of:

$$\int_{\square^n} (x_1 \cdots x_n)^{2k}\, dx = \Big(\int_0^1 x_1^{2k}\, dx_1\Big) \cdots \Big(\int_0^1 x_n^{2k}\, dx_n\Big) = \frac{1}{(2k+1)^n}.$$

The final result is:

$$\int_{\lambda\square^n} \frac{1}{1-(-1)^n (x_1 \cdots x_n)^2}\, dx = \sum_{k=0}^{\infty} \lambda^{n(2k+1)} \frac{(-1)^{nk}}{(2k+1)^n}.$$

The series converges uniformly for $\lambda \in [\,0,\ 1\,]$; and hence we can take the limit for $\lambda \uparrow 1$, finding the series of the lemma. But this implies that the limit of the integrals exists for $\lambda \uparrow 1$, and that it is equal to the integral of the lemma. This yields the desired equality. □

From cube to polytope. If $n = 1$ the integral in (8) is immediately seen to be equal to $\frac{1}{2}\frac{\pi}{2}$, hence $\delta_1 = 1/2$. So we assume $n \geq 2$ from now on. In order to evaluate the integral (8) (this is by no means elementary), we make a surprising change of variables. In what follows we shall regard the indices i of the n coordinates of a point in $\mathbf{R}^n$ as integers modulo n, so we compute with them cyclically. The substitution of variables is given by the following equations with the conventions on indices just described:

$$x_i = \frac{\sin y_i}{\cos y_{i+1}} \qquad (i \in \mathbf{N} \mod n). \tag{9}$$

Lemma 2. *The change of variables represented by (9) gives a differentiably invertible one-to-one correspondence between the set $\square^n$ for x and the set $\frac{\pi}{2}\Delta^n$ for y. Here Δ^n is the open n-dimensional convex polytope consisting of the $x \in \mathbf{R}^n$ satisfying:*

$$0 < y_i, \qquad y_i + y_{i+1} < 1 \qquad (i \in \mathbf{N} \mod n). \tag{10}$$

The Jacobian determinant $\frac{\partial x}{\partial y}$ of the correspondence equals $1 - (-1)^n (x_1 \cdots x_n)^2$.

Notice that the Jacobian is exactly the denominator of the integrand in (8) (this is of course the raison d'être of the substitution).

Proof: We denote the correspondence by $\Psi : \frac{\pi}{2}\Delta^n \to \square^n$. For $y \in \frac{\pi}{2}\Delta^n$, we have $0 < y_i < \frac{\pi}{2} - y_{i+1} < \frac{\pi}{2}$ (cyclically); and thus:

$$0 < \sin y_i < \sin\big(\frac{\pi}{2} - y_{i+1}\big) = \cos y_{i+1}.$$

Hence $0 < x_i < 1$, and using this we get $\Psi(y) \in \square^n$. Conversely, given $x \in \square^n$, consider the linear function $\phi : \mathbf{R} \to \mathbf{R}$ given by:

$$\phi(t) = x_n^2(1 - x_1^2(1 - \cdots (1 - x_{n-1}^2(1-t)) \cdots)).$$

Since $|\phi(t)-\phi(t')| = (x_1x_2\cdots x_n)^2|t-t'|$, the mapping ϕ sends $]\,0,1\,[$ into itself contractively, and hence ϕ has a unique fixed point $t_0 \in \square$. Now select the unique element $y \in \frac{\pi}{2}\Delta^n$ satisfying:

$$\sin^2 y_n = t_0, \qquad \sin^2 y_i = x_i^2(1-\sin^2 y_{i+1}) \quad (n-1 \geq i \geq 1).$$

Then $\Psi(y) = x$ comes down to $\sin^2 y_n = x_n^2(1-\sin^2 y_1)$, and this follows from $\phi(\sin^2 y_n) = \sin^2 y_n$. That is, the equation $\Psi(y) = x$ has a unique solution $y \in \frac{\pi}{2}\Delta^n$. This gives that Ψ is differentiable, one-one and onto.

The Jacobian matrix $D\Psi(y)$ has the following simple form, where we use ψ_i and ψ_i' as an abbreviation for $\sin y_i$ and $\cos y_i$ respectively:

$$\begin{pmatrix} \frac{\psi_1'}{\psi_2'} & \frac{\psi_1\psi_2}{\psi_2'^2} & 0 & \cdots & 0 \\ 0 & \frac{\psi_2'}{\psi_3'} & \frac{\psi_2\psi_3}{\psi_3'^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\psi_n\psi_1}{\psi_1'^2} & 0 & 0 & \cdots & \frac{\psi_n'}{\psi_1'} \end{pmatrix}.$$

Consequently the Jacobian $\frac{\partial x}{\partial y} = \det D\Psi(y)$ equals:

$$\prod_{i=1}^n \frac{\psi_i'}{\psi_{i+1}'} - (-1)^n \prod_{i=1}^n \frac{\psi_i\psi_{i+1}}{\psi_{i+1}'^2} = 1-(-1)^n \prod_{i=1}^n \Big(\frac{\psi_i}{\psi_{i+1}'}\Big)^2 = 1-(-1)^n(x_1\cdots x_n)^2.$$

The differentiability of Ψ^{-1} follows from the inverse function theorem (see [MT], p.288) and the fact that the Jacobian is positive on $\frac{\pi}{2}\Delta^n$. $\square$

Note that the equations (9) can be solved explicitly; one obtains:

$$y_i = \arcsin\left(x_i\sqrt{\frac{1+\sum_{j=1}^{n-1}\prod_{k=i+1}^{i+j}(-x_k^2)}{1-(-1)^n(x_1\cdots x_n)^2}}\right) \qquad (i \in \mathbf{N} \mod n).$$

Now transform the integral (8) by the substitution (9), and deduce from Lemma 2:

$$\int_{\square^n} \frac{1}{1-(-1)^n(x_1\cdots x_n)^2}\,dx = \Big(\frac{\pi}{2}\Big)^n \int_{\Delta^n} dy = \Big(\frac{\pi}{2}\Big)^n \mathrm{vol}_n(\Delta^n) = \Big(\frac{\pi}{2}\Big)^n \delta_n.$$

Combining this result with Lemma 1 we find the following

Theorem 3. *The formula (6) above holds with δ_n equal to the volume of the Δ^n of Lemma 2.*

Dissecting the polytope into congruent pyramids. The cyclic permutation of the n coordinates in $\mathbf{R}^n$ defines an orthogonal linear transformation C of $\mathbf{R}^n$. The transformation C preserves the inequalities (10) defining Δ^n, and therefore Δ^n is mapped isometrically onto itself by C.

Now denote by $\Gamma^n = \Gamma^n_n$ the collection of $x \in \Delta^n$ satisfying $x_n < x_i$ for $1 \le i \le n-1$. In addition, define the sets Γ^n_j, for $1 \le j < n$ similarly, but with the index n interchanged with j. Since $C^j \Gamma^n = \Gamma^n_j$, all these sets have equal volume and are pairwise disjunct, and the closure of Δ^n is the union of the closures of the Γ^n_j's. Hence (cf. Theorem 3):

$$\delta_n = \mathrm{vol}_n(\Delta^n) = n\,\mathrm{vol}_n(\Gamma^n). \tag{11}$$

Note that $x \in \Gamma^n$ is characterized by the inequalities:

$$0 < x_n < x_i \quad (1 \le i \le n-1), \qquad x_i + x_{i+1} < 1 \quad (1 \le i \le n-2). \tag{12}$$

Indeed, the two "missing" equations now are a consequence of the given ones, since $x_{n-1} + x_n < x_{n-1} + x_{n-2} < 1$ and $x_n + x_1 < x_2 + x_1 < 1$.

Let Θ^{n-1} be the intersection of the closure $\overline{\Gamma}^n$ with the coordinate hyperplane $x_n = 0$. Obviously $y \in \Theta^{n-1}$ if and only if:

$$0 < y_i \quad (1 \le i \le n-1), \qquad y_n = 0, \qquad y_i + y_{i+1} < 1 \quad (1 \le i \le n-2). \tag{13}$$

The next lemma shows that $\overline{\Gamma}^n$ is the pyramid in $\mathbf{R}^n$ with $a = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ as its apex and Θ^{n-1} as its basis, i.e., $\overline{\Gamma}^n$ is the convex hull of a and Θ^{n-1}.

Lemma 4. $\Psi : \Theta^{n-1} \times]0,1[\to \Gamma^n$ *given by* $x = \Psi(y,t) = (1-t)\,y + t\,a$ *is a differentiably invertible one-to-one correspondence, and its Jacobian is given by* $\dfrac{\partial x}{\partial(y,t)} = \dfrac{1}{2}(1-t)^{n-1}$.

Proof: Suppose x belongs to the pyramid $\mathrm{im}(\Psi)$. Then $0 < x_n = t/2 < (1-t)y_i + t/2 = x_i$, since $0 < y_i$. Furthermore $y_i + y_{i+1} < 1$ implies $x_i + x_{i+1} = (1-t)(y_i + y_{i+1}) + t < 1 - t + t = 1$, for $1 \le i \le n-2$. According to (12) this means $x \in \Gamma^n$. Conversely, assume $x \in \Gamma^n$. Then $x = \Psi(y,t)$ is solved by $t = 2x_n$ and $y_i = (x_i - x_n)/(1 - 2x_n)$. First note that $0 < x_n < 1/2$, for every $x \in \Gamma^n$; hence $0 < t < 1$. The inequality $x_n < x_i$ implies $0 < y_i$, while $x_i + x_{i+1} < 1$ gives $y_i + y_{i+1} < 1$, for $1 \le i \le n-2$. In view of (13) we have $y \in \Theta^{n-1}$, and so $x \in \mathrm{im}(\Psi)$. The computation of the Jacobian is immediate. □

As a consequence of the lemma we find:

$$\mathrm{vol}_n(\Gamma^n) = \frac{1}{2}\int_{\Theta^{n-1}} \Big(\int_0^1 (1-y_n)^{n-1}\, dy_n \Big) dy = \frac{1}{2n}\,\mathrm{vol}_{n-1}(\Theta^{n-1}) =: \frac{1}{2n}\theta_{n-1}. \tag{14}$$

Combining (11) and (14) we see:

$$\delta_n = n\,\mathrm{vol}_n(\Gamma^n) = \frac{1}{2}\theta_{n-1}. \tag{15}$$

Two-step recursion for the volume of the basis of the pyramid. From (13) we see $x \in \Theta_n$ if and only if $0 < x_1 < 1, 0 < x_2 < 1 - x_1, \ldots, 0 < x_n < 1 - x_{n-1}$. Hence:

$$\theta_n = \int_0^1 \left(\int_0^{1-x_1} \cdots \left(\int_0^{1-x_{n-1}} dx_n \right) \cdots dx_2 \right) dx_1.$$

In order to compute this integral we introduce polynomial functions p_n on $\mathbf{R}$ by:

$$p_0(x) = 1, \qquad p_n(x) = \int_0^{1-x} p_{n-1}(t)\, dt; \qquad \text{then} \quad \theta_n = p_n(0).$$

Now we have, for $x \in \mathbf{R}$:

$$\begin{aligned} p_{n+2}(x) &= \int_0^1 p_{n+1}(t)\, dt - \int_{1-x}^1 p_{n+1}(t)\, dt = p_{n+2}(0) - \int_0^x p_{n+1}(1-t)\, dt = \\ &= p_{n+2}(0) - \int_0^x \left(\int_0^t p_n(s)\, ds \right) dt. \end{aligned}$$

Hence by mathematical induction on n we find, for $x \in \mathbf{R}$:

$$p_n(x) = \sum_{0 \le i \le [\frac{n}{2}]} p_{n-2i}(0) \frac{(-1)^i}{(2i)!} x^{2i} - s(n) \frac{(-1)^{[\frac{n}{2}]}}{n!} x^n,$$

where $s(n) = 0$ or 1 if n is even or odd, respectively. Since $p_n(1) = 0$, for $n \in \mathbf{N}$, we obtain:

$$\sum_{i=0}^n \theta_{2n-2i} \frac{(-1)^i}{(2i)!} = 0, \qquad \sum_{i=0}^n \theta_{2n+1-2i} \frac{(-1)^i}{(2i)!} = \frac{(-1)^n}{(2n+1)!} \quad (n \in \mathbf{N}). \tag{16}$$

Notice that (16) gives an algorithm for recursive computation of the θ_n, and hence also of the δ_n. The left hand sides in (16) occur as the coefficient of t^{2n}, and t^{2n+1} respectively, if one performs the multiplication in the left hand side of the following identity of formal power series; and this implies:

$$\left(\sum_{n=0}^\infty \theta_n t^n \right) \left(\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} t^{2n} \right) = 1 + \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} t^{2n+1}.$$

Recognizing the Maclaurin expansion for cos and sin we find in view of (15) the result of (7):

$$2 \sum_{n=1}^\infty \delta_n t^{n-1} = \sum_{n=0}^\infty \theta_n t^n = \frac{1}{\cos t} + \tan t = \sec t + \tan t \qquad (|t| < \frac{\pi}{2}).$$

Recall the expansion (5) for sec. The function tan is an odd function whose odd-order derivatives at 0 belong to $\mathbf{N}$, and it has the power series $\tan t = \sum_{n=1}^\infty T_n t^{2n-1}$, where the T_n are given by (see [C], p.423 for a short proof):

$$T_n = (-1)^{n-1} B_{2n} \frac{2^{2n} - 1}{(2n)!} 2^{2n}.$$

In [KB] one finds tables for the Euler and Bernoulli numbers. Inserting both these expansions in (7) we obtain the following equality of power series, which relates the volumina δ_n to the analytically defined numbers E_n of (5) and B_{2n} of (2):

$$\sum_{n=1}^\infty \delta_n t^{n-1} = \sum_{n=0}^\infty E_n \frac{1}{2(2n)!} t^{2n} + \sum_{n=1}^\infty (-1)^{n-1} B_{2n} \frac{2^{2n} - 1}{2(2n)!} 2^{2n} t^{2n-1} \qquad \left(|t| < \frac{\pi}{2} \right).$$

If we combine this result with (6) we obtain (3) and (4). With this we feel we have completed our short circular excursion through some analysis, some geometry, some combinatorics and back to the tried and familiar grounds of elementary subjects.

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