Final exam UCU SCI211, December 19, 2001

1 Let $(x(t),\, y(t))$ be the solution of the differential equations

$$\frac{dx(t)}{dt} = y(t), \quad \frac{dy(t)}{dt} = -x(t),$$

with the initial condition

$$x(0) = 1, \quad y(0) = 0.$$

a) Prove that the function $t \mapsto x(t)^2 + y(t)^2$ is constant. What is its value?

b) Write down the Euler method with step length $h = t/N$. The value $(x_N,\, y_N)$ after N steps is the corresponding numerical approximation of $(x(t),\, y(t))$. Prove that, for every positive integer n, ${x_n}^2 + {y_n}^2 = \left(1 + h^2\right)^n$.

c) Prove that

$$1 + \frac{t^2}{2N} \leq \left({x_N}^2 + {y_N}^2\right)^{1/2} \leq \mathrm{e}^{t^2/2N}.$$

(Hint: you may use the well-known theorem that if f is a differentiable function, then $f(a) = f(0) + f'(b)\,a$, for some b between 0 and a. For the first inequality use $f(a) = (1+a)^{N/2}$ and for the second inequality use $f(a) = \ln(1+a)$.)

Prove that the error in the distance to the origin, the number $\left({x_N}^2 + {y_N}^2\right)^{1/2} - \left(x(t)^2 + y(t)^2\right)^{1/2}$, is at least equal to $t^2/2N$ and at most equal to $\mathrm{e}^{t^2/2N} - 1$.

2 Consider the wave equation

$$\frac{\partial^2 u(x,\, t)}{\partial t^2} = \frac{\partial^2 u(x,\, t)}{\partial x^2}, \quad -\infty < x < \infty,$$

on the real line.

a) Give d'Alembert's formula for the solution $u(x,\, t)$ in terms of the initial profile $u(x,\, 0) = f(x)$ and the initial velocity $\frac{\partial u(x,\, t)}{\partial t}\big|_{t=0} = g(x)$.

b) Let $f(x)$ be periodic with period 2 and given by $f(x) = x^2$ when $-1 \leq x \leq 1$. Let $g(x) \equiv 0$. Make a sketch of the function $x \mapsto u(x,\, t)$ given by d'Alembert's formula, for $-4 \leq x \leq 4$, and for $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. (Five graphs, make these distinguishable if you put these into one picture.)

c) Returning to general initial conditions, prove that $u(x,\, t) = f(x+t)$ if $g(x) = df(x)/dx$. Sketch the graph of $x \mapsto u(x,\, 1/2)$, for $-4 \leq x \leq 4$, if $f(x)$ is as in b) and $g(x)$ is periodic with period 2 and $g(x) = 2x$ when $-1 < x < 1$. (The discontinuities of $g(x)$ at the odd integers are no obstacle to the application of d'Alembert's formula.)

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3 Denote by

$$D := \{(x,\, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \quad \text{and} \quad C := \{(x,\, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

the unit disc in the plane and its boundary circle, respectively. Our aim is to solve the diffusion equation

$$\frac{\partial u(x,\, y,\, t)}{\partial t} = \frac{\partial^2 u(x,\, y,\, t)}{\partial x^2} + \frac{\partial^2 u(x,\, y,\, t)}{\partial y^2}, \quad (x,\, y) \in D,\ t > 0,$$

with boundary condition

$$u(x,\, y,\, t) = 0 \quad \text{when} \quad (x,\, y) \in C,\ t \geq 0,$$

and prescribed initial profile $u(x,\, y,\, 0) = f(x,\, y)$.
For this purpose we use the substitution of polar coordinates

$$U(r,\, \theta,\, t) := u(r\, \cos\theta,\, r\, \sin\theta,\, t), \quad F(r,\, \theta) := f(r\, \cos\theta,\, r\, \sin\theta).$$

It is known (and you don't have to verify this here) that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

a) Expand the 2π-periodic functions $\theta \mapsto U(r,\, \theta,\, t)$ and $\theta \mapsto F(r,\, \theta)$ into the Fourier series

$$U(r,\, \theta,\, t) = \sum_{k=-\infty}^{\infty} U_k(r,\, t)\, \mathrm{e}^{i\,k\,\theta} \quad \text{and} \quad F(r,\, \theta) = \sum_{k=-\infty}^{\infty} F_k(r)\, \mathrm{e}^{i\,k\,\theta},$$

respectively. Prove that the diffusion equation in $D \setminus \{(0,\, 0)\}$ is equivalent to

$$\frac{\partial U_k(r,\, t)}{\partial t} = \frac{\partial^2 U_k(r,\, t)}{\partial r^2} + \frac{1}{r}\frac{\partial U_k(r,\, t)}{\partial r} - \frac{k^2}{r^2}\, U_k(r,\, t), \quad k \in \mathbb{Z},\ r,\, t > 0, \tag{1}$$

that the boundary condition is equivalent to

$$U_k(1,\, t) = 0, \quad k \in \mathbb{Z},\ t \geq 0, \tag{2}$$

and that the initial condition is equivalent to

$$U_k(r,\, 0) = F_k(r), \quad k \in \mathbb{Z},\ 0 \leq r \leq 1. \tag{3}$$

b) For any integer k, write $n = |k|$. Note that $n^2 = k^2$ and $n \geq 0$. Let $J_n(\rho)$ denote the Bessel function of order n. Write down the second order differential equation, the Bessel equation, which is satisfied by $J_n(\rho)$.

Let $R_{n,\,m}$ denote the m-th strictly positive zero of $J_n(\rho)$. Prove that

$$U_{k,\,m}(r,\, t) = \mathrm{e}^{-R^2\, t}\, J_n(R\, r), \quad R = R_{n,\,m},$$

satisfies the differential equation (1), the boundary condition (2), and the initial condition (3) with

$$F_k(r) = J_n\left(R_{n,\,m}\, r\right).$$

(Hint: use the substitution $\rho = R\, r$.)