Final exam UCU SCI 211, December 20, 2002

In every part of a problem you may use the conclusion of a previous part, even if you did not prove that one (yet). Lots of success!

1) Consider the wave equation

$$\frac{\partial^2 v(x,\, t)}{\partial t^2} = \frac{\partial^2 v(x,\, t)}{\partial x^2}, \quad 0 < x < \infty,$$

on the positive x-axis, with the boundary condition

$$v(0, t) = 0 \quad \text{for all} \quad t \in \mathbf{R},$$

and the initial conditions

$$v(x,\, 0) = \varphi(x) \quad \text{and} \quad \left.\frac{\partial v(x,\, t)}{\partial t}\right|_{t=0} = 0 \quad \text{for all} \quad x \geq 0,$$

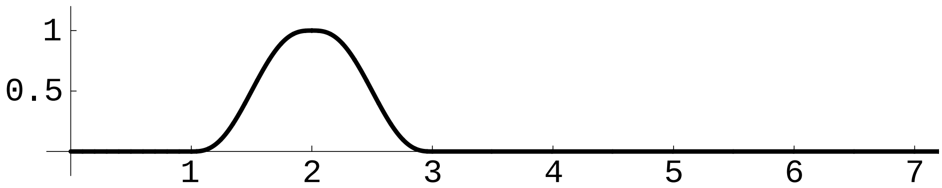
where φ is a given function on the positive x-axis. Assume that φ is twice continuously differentiable, that $\varphi(0) = 0$ and that $\varphi''(0) = 0$. Define the function f on the whole real axis by $f(x) = \varphi(x)$ when $x \geq 0$ and $f(x) = -\varphi(-x)$ when $x < 0$. It follows that f is twice continuously differentiable on the whole real axis and that f is odd. (You don't have to prove this here. Also, don't be irritated that we used a different notation for φ and its odd extension f to the whole real axis.)

a) Let $u(x,\, t)$ be the solution of the wave equation on the whole real axis such that

$$u(x,\, 0) = f(x) \quad \text{and} \quad \left.\frac{\partial u(x,\, t)}{\partial t}\right|_{t=0} = 0 \quad \text{for all} \quad x \in \mathbf{R}.$$

Use d'Alembert's formula in order to prove that $u(0, t) = 0$ for every $t \in \mathbf{R}$. Therefore, if $v(x,\, t) := u(x,\, t)$ for every $x \geq 0$ and every $t \in \mathbf{R}$, then $v(x,\, t)$ is a solution of the wave equation on the positive real axis, which satisfies the required boundary condition and the required initial conditions. (Again, don't be irritated that we used a different notation for u and its restriction v to the positive axis.)

b) Suppose that the graph of the function $\varphi(x)$ is given by



Draw the graph of the function $x \mapsto v(x,\, 1)$ and of the function $x \mapsto v(x,\, 4)$ for $x \geq 0$. Show the scale on the axes!

Turn page!

2) Consider the partial differential equation

$$\frac{\partial^2 u(x,\, y)}{\partial x^2} + \frac{\partial^2 u(x,\, y)}{\partial y^2} = 1, \quad 0 < x < 1,\ 0 < y < 1 \tag{1}$$

in the square in the plane, with the boundary conditions

$$u(0,\, y) = u(1,\, y) = 0 \quad \text{when} \quad 0 \le y \le 1 \quad \text{and} \quad u(x,\, 0) = u(x,\, 1) = 0 \quad \text{when} \quad 0 \le x \le 1. \tag{2}$$

a) Let $f(t)$ be periodic with period 2, odd, and such that $f(t) = 1$ when $0 < t < 1$. Let b_k be the coefficients in Theorem 1.2 in the Guide Book. Show that $b_k = 4/(\pi\, k)$ when k is odd and $b_k = 0$ when k is even. Show that

$$\sum_{l\ge 1,\, l\,\text{odd}} \ \sum_{k\ge 1,\, k\,\text{odd}} \frac{4}{\pi\, k} \sin(k\,\pi\, x)\, \frac{4}{\pi\, l} \sin(l\,\pi\, y) = 1, \quad 0 < x < 1,\ 0 < y < 1.$$

b) Define

$$u(x,\, y) := -\sum_{l\ge 1,\, l\,\text{odd}} \ \sum_{k\ge 1,\, k\,\text{odd}} \frac{16}{\pi^4\, k\, l\, (k^2 + l^2)} \sin(k\,\pi\, x)\, \sin(l\,\pi\, y).$$

Show that $u(x,\, y)$ satisfies the differential equation (1) and the boundary conditions (2).

3) Let $x(t)$ be the solution of the differential equation $\mathrm{d}x(t)/\,\mathrm{d}t = c\, x(t)$, with the initial condition $x(0) = a$. Here c and a are positive constants.

a) For any $t > 0$, write down the Euler method with step length $h = t/N$. (Here we treat t as a positive constant.) The value x_N after N steps is the corresponding numerical approximation of $x(t)$. Prove by induction on n that, for every positive integer n, $x_n = a\,(1 + h\,c)^n$.

b) Now treat t again as a variable. Write $x_N = x_N(t)$, in order to emphasize the dependence of x_N on t. Define $v(t) := \ln x(t) - \ln x_N(t)$. Prove that $v(0) = 0$ and that

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{c^2\, t}{N + c\, t}.$$

Let T be a given positive number. Deduce that, for every $0 \le t \le T$,

$$\frac{c^2\, t}{N + c\, T} \le \frac{\mathrm{d}v(t)}{\mathrm{d}t} \le \frac{c^2\, t}{N},$$

and, as a consequence,

$$\frac{c^2\, t^2}{2(N + c\, T)} \le v(t) \le \frac{c^2\, t^2}{2N} \quad \text{when} \quad 0 \le t \le T.$$

(This implies that the difference between $x_N(t)$ and $x(t)$ is of order $1/N$ as $N \to \infty$. However, you don't have to prove this here.)