Answers final exam UCU SCI 211, December 18, 2003

1) According to the theory, for instance of Section 8.2 in the Guide Book wit $\gamma = 1$ and $L = 1$, the solution is of the form

$$u(x,\, t) = \sum_{k=0}^{\infty} a_k\ \mathrm{e}^{-\pi^2\, k^2\, t}\ \cos(k\pi x),$$

in which substitution of $t = 0$ yields

$$1 + \cos(\pi x) = u(x,\, 0) = \sum_{k=0}^{\infty} a_k\ \cos(k\pi x).$$

Here the left hand side is a cosine series of the same form as appears in the right hand side, with $a_0 = 1$, $a_1 = 1$ and $a_k = 0$ for every $k > 1$. (One can also use the integral formulas for the Fourier cofficients a_k, which is a little more work.) The conclusion is that

$$u(x,\, t) = 1 + \mathrm{e}^{-\pi^2\, t}\ \cos(\pi x).$$

It follows that

$$|u(x,\, t) - 1| = \left|\mathrm{e}^{-\pi^2\, t}\ \cos(\pi x)\right| = \mathrm{e}^{-\pi^2\, t}\ |\cos(\pi x)| \leq \mathrm{e}^{-\pi^2\, t},$$

because $|\cos(\pi x)| \leq 1$. Substitution of $t = 1$ yields the desired inequality.

2) Let $f(x)$ be a twice differentiable function on the real axis which is periodic with period $2L$ and has the property that $f(x) = 0$ when $-L \leq x \leq 0$. Define

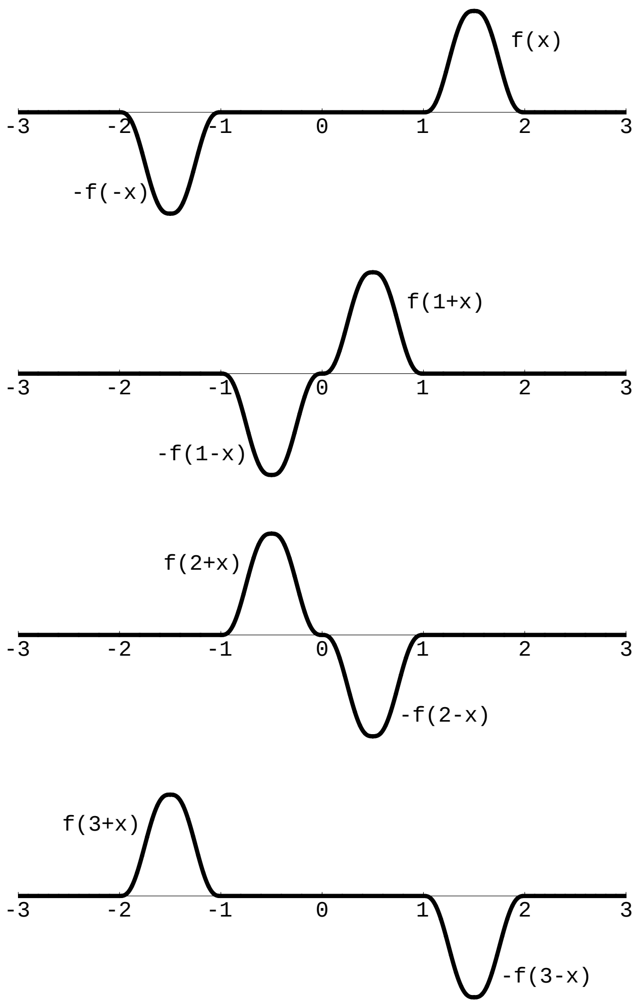
$$u(x,\, t) = f(t + x) - f(t - x).$$

a) Differentiating $u(x,\, t)$ twice with respect to t we obtain $f''(t + x) - f''(t - x)$. Differentiating $u(x,\, t)$ with respect to x we obtain $f'(t + x) + f'(t - x)$, and differentiating this again with respect to x we obtain $f''(t + x) - f''(t - x)$, which is equal to the second order derivative of $u(x,\, t)$ with respect to t. This shows that $u(x,\, t)$ satisfies the wave equation.

Putting $x = 0$ we get $u(0,\, t) = f(t) - f(t) = 0$. Putting $x = L$ we obtain $f(t + L) - f(t - L) = 0$, because $x + L = (x - L) + 2L$ and f was assumed to be periodic with period $2L$. This shows that $u(x,\, t)$ satisfies the boundary conditions.

Finally, if $0 < x < 3$ then $u(x,\, 0) = f(x) - f(-x) = f(x)$, because $x > 0$ implies $-x < 0$ and therefore $f(-x) = 0$. If we put $t = 0$ in $\partial u(x,\, t)/\partial t = f'(t + x) - f'(t - x)$ we obtain that the initial velocity is equal to $f'(x) - f'(-x) = f'(x)$, because $f(y) = 0$ for all $y < 0$ implies that $f'(y) = 0$ for all $y < 0$. This shows that $u(x,\, t)$ satisfies the initial conditions.

b) The pictures should look like the following.



In the interval $0 \leq x \leq 3$, the bump is translated to the left with speed 1 until it hits the left end point of the interval, whereas after time $t = 2$ it has reappeared upside down and is shifting to the right with speed equal to 1.

c) We have $u(x,\, 1.5) = f(1.5 + x) - f(1.5 - x) = 0$, in which the second equation just expresses the symmetry of the function f with repect to the reflection $1.5 + x \mapsto 1.5 - x$. This is not such a paradox as it may seem, because at time $t = 1.5$ the velocity $\partial u(x,\, t)/\partial t$ is nonzero, the string just moves through the horizontal position at time $t = 1.5$.

3) If we substitute $u(x,\, y) = f(x)\, g(y)$ in the partial differential equation, we obtain

$$f''(x)\, g(y) + f(x)\, g''(y) + 2f'(x)\, g(y) + f(x)\, g(y) = 0.$$

Assuming that $u(x,\, y) = f(x)\, g(y)$ is nonzero we can divide by $f(x)\, g(y)$ and obtain the equivalent equation

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} + 2\frac{f'(x)}{f(x)} + 1 = 0,$$

or equivalently

$$\frac{f''(x)}{f(x)} + 2\frac{f'(x)}{f(x)} + 1 = -\frac{g''(y)}{g(y)}.$$

The left hand side does not depend on y, and is equal to the right hand side which does not depend on x, and therefore it is equal to a constant λ neither depending on y nor on x. This implies that our equation is equivalent to the two equations

$$\frac{f''(x)}{f(x)} + 2\frac{f'(x)}{f(x)} + 1 = \lambda = -\frac{g''(y)}{g(y)},$$

or equivalently

$$f''(x) + 2f'(x) + (1 - \lambda)\, f(x) = 0, \quad g''(y) + \lambda\, g(y) = 0,$$

in which λ is an arbitrary constant. The exponential function $\mathrm{e}^{p\,x}$ is a solution for the equation for $f(x)$ if and only if $p^2 + 2p + (1 - \lambda) = 0$, or $p = -1 \pm \sqrt{\lambda}$. If $\lambda \neq 0$ then, using the two-dimensionality of the space of solutions of the differential equation for $f(x)$, we obtain that the general solution is

$$f(x) = c_+\, \mathrm{e}^{(-1+\sqrt{\lambda})\,x} + c_-\, \mathrm{e}^{(-1-\sqrt{\lambda})\,x},$$

in which c_+ and c_- are arbitrary constants. Similarly $\mathrm{e}^{q\,y}$ is a solution of the differential equation for $g(y)$ if and only if $q^2 + \lambda = 0$, and we have

$$g(y) = d_+\, \mathrm{e}^{\sqrt{-\lambda}\,y} + d_-\, \mathrm{e}^{-\sqrt{-\lambda}\,y},$$

in which d_+ and d_- are arbirary constants, and therefore the answer is

$$u(x,\, y) = \left(c_+\, \mathrm{e}^{(-1+\sqrt{\lambda})\,x} + c_-\, \mathrm{e}^{(-1-\sqrt{\lambda})\,x}\right) \left(d_+\, \mathrm{e}^{\sqrt{-\lambda}\,y} + d_-\, \mathrm{e}^{-\sqrt{-\lambda})\,y}\right),$$

in which λ, c_-, c_+, d_-, d_+ are arbitrary complex numbers. The only exceptional case occurs for $\lambda = 0$, in which case this should be replaced by

$$u(x,\, y) = \left(a\, \mathrm{e}^{-x} + b\, x\, \mathrm{e}^{-x}\right)\, (c + d\, y)\,,$$

in which a, b, c, d are arbitrary constants.

4)

a) We have

$$\mathrm{d}\big(x(t)^2 - y(t)^2\big) \,/\, \mathrm{d}t = 2x'(t)\,x(t) - 2y'(t)\,y(t) = 2y(t)\,x(t) - 2x(t)\,y(t) = 0.$$

This implies that $x(t)^2 - y(t)^2$ is constant as a function of t and therefore equal to $x(0)^2 - y(0)^2 = 1^2 - 0^2 = 1$.

b) The Euler approximation for this system of ordinary differential equations is

$$x_{n+1} = x_n + h\,y_n, \quad y_{n+1} = y_n + h\,x_n, \quad x_0 = 1, \quad y_0 = 0.$$

The identity for $(x_n)^2 - (y_n)^2$ obviously holds when $n = 0$. Now assume that it holds for some given n. Then

$$\begin{aligned}(x_{n+1})^2 - (y_{n+1})^2 &= (x_n + h\,y_n)^2 - (y_n + h\,x_n)^2 = \big(1 - h^2\big)\,\Big((x_n)^2 - (y_n)^2\Big)\\ &= \big(1 - h^2\big)\,\big(1 - h^2\big)^n = \big(1 - h^2\big)^{n+1},\end{aligned}$$

which is the identity for $(x_n)^2 - (y_n)^2$ with everywhere n replaced by $n+1$. Here we have used the Euler procedure in the first equality, a straightforward calculation in the second equality, and the induction hypothesis in the third equality. This completes the proof of the identity for $(x_n)^2 - (y_n)^2$ for every $n \geq 0$.

c) **(bonus)** The first identity is obtained by the substitution of $n = N$, $h = t/N$, where we subsequently write

$$\left(1 - \frac{t^2}{N^2}\right)^N = \mathrm{e}^{N\,\ln\left(1 - \frac{t^2}{N^2}\right)}.$$

For the second and the third identity we use the first order Taylor expansion which says that for any differentiable function $f(x)$ we have $f(x) = f(a) + f'(y)\,(x - a)$, in which y is some point between a and x. If we do this for the function $f(x) = \ln x$, with $a = 1$, $x = 1 - t^2/N^2$ and $y = p$, we obtain the second identity. We obtain the third identity if we do this for $f(x) = \mathrm{e}^x$, with $a = 0$,

$$x = -\frac{1}{p}\,\frac{t^2}{N}$$

and $y = q$.

The last statement follows if we write $\theta_N = \mathrm{e}^q\,/p$ and observe that the inequalities for p and q imply that $p \to 1$ and $q \to 0$ as $N \to \infty$.