Mid-term exam UCU SCI211, October 31, 2001

1 Consider the function $f(x)$, $x \in \mathbb{R}$, which is defined by the following properties:

i) $f(x)$ is periodic with period equal to 2π.

ii) If $0 \le x \le \pi$, then $f(x) = \frac{\pi^2}{4} - \left(x - \frac{\pi}{2}\right)^2$.

iii) $f(-x) = -f(x)$ for every $x \in \mathbb{R}$.

a) Sketch the graph of $f(x)$, for $-3\pi/2 \le x \le 3\pi/2$.

b) Prove that

$$f(x) = \sum_{k=1}^{\infty} b_k \, \sin(k\,x),$$

where

$$\begin{aligned} b_k &= \frac{2}{\pi} \int_0^{\pi} f(x)\, \sin(k\,x)\, dx \\ &= \frac{2}{\pi\,k} \int_0^{\pi} f'(x)\, \cos(k\,x)\, dx \\ &= -\frac{2}{\pi\,k^2} \int_0^{\pi} f''(x)\, \sin(k\,x)\, dx. \end{aligned}$$

Give your arguments for each of the identities!

c) Compute the coefficients b_k and show that

$$\frac{8}{\pi} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^3} \sin\left((2l+1)\,x\right) = \frac{\pi^2}{4} - \left(x - \frac{\pi}{2}\right)^2, \quad 0 \le x \le \pi.$$

2 For the function $f(t)$, $t \in \mathbb{R}$, we define

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-i\omega\,t}\, dt$$

as its Fourier transform. Prove, using the Euler formula for the cosine, that the Fourier transform of the function $g(t) = f(t)\,\cos(bt)$ is given by

$$\widehat{g}(\omega) = \frac{1}{2}\left(\widehat{f}(\omega - b) + \widehat{f}(\omega + b)\right).$$

Let $f(t) = \mathrm{e}^{-a\,t^2}$. Then it is known that $\widehat{f}(\omega) = \sqrt{\pi/a}\; \mathrm{e}^{-\omega^2/4a}$. Make a sketch of the graph of $\widehat{g}(\omega)$, for $-2 < \omega < 2$, in the case that $a = 0.01$ and $b = 1$.

3 Let D be the set of all $(x,\, y) \in \mathbb{R}^2$ such that $(x,\, y) \ne (0,\, 0)$. Prove that the differential form

$$\omega := \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy,$$

defined in D, is closed.
Let γ be the closed curve in D which is defined by $\gamma(t) = (\cos t,\, \sin t)$, $0 \le t \le 2\pi$. Compute $\oint_\gamma \omega$. Prove that ω is not exact in D.