Answers Mid-term exam SCI 211, October 30, 2003

Problem 1 The velocity vector is equal to $(1-\cos t,\, \sin t)$, of which the square of the length is equal to

$$(1-\cos t)^2 + (\sin t)^2 = 1 - 2\cos t + (\cos t)^2 + (\sin t)^2 = 2 - 2\cos t,$$

because $\cos^2 + \sin^2 = 1$. Therefore the length of the curve is equal to

$$\int_0^{2\pi} \sqrt{2-2\cos t}\ \mathrm{d}t = 2\int_0^{\pi} \sqrt{2-2\,(1-2(\sin s)^2)}\ \mathrm{d}s = 4\int_0^{\pi} \sin s\ \mathrm{d}s = 4\,(-\cos\pi - (-\cos 0)) = 8,$$

where in the second identity we have used the substitution of variables $t = 2s$ in the integral.

Problem 2 Assume in the sequal that not both $x = 0$ and $y = 0$, which implies that $x^2 + y^2 \neq 0$. Using the chain rule for differentiation we obtain $\partial V(x,\, y)/\partial x = \left(x^2+y^2\right)^{-1}\, 2x$ and differentiating once more with respect ot x:

$$\frac{\partial^2 V(x,\, y)}{\partial x^2} = -\left(x^2+y^2\right)^{-2}\,(2x)^2 + 2\left(x^2+y^2\right)^{-1}.$$

Similarly

$$\frac{\partial^2 V(x,\, y)}{\partial y^2} = -\left(x^2+y^2\right)^{-2}\,(2y)^2 + 2\left(x^2+y^2\right)^{-1},$$

and therefore

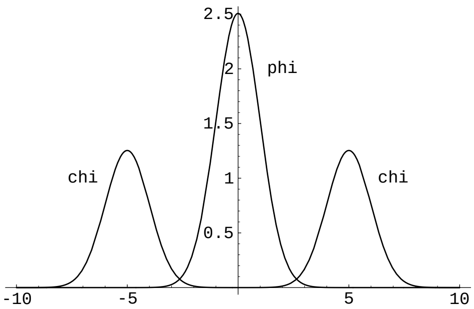
$$(\Delta V)(x,\, y) = \frac{\partial^2 V(x,\, y)}{\partial x^2} + \frac{\partial^2 V(x,\, y)}{\partial y^2} = -4\left(x^2+y^2\right)^{-2}\left(x^2+y^2\right) + 4\left(x^2+y^2\right)^{-1} = 0.$$

Problem 3

a) Formula (4.7) in the Guide Book yields $\chi(\omega) = \left(\phi(\omega-5) + \phi(\omega+5)\right)/2$. Acoording to formula (4.4) in the Guide Book we have $\phi(\omega) = \sqrt{2\pi}\ \mathrm{e}^{-\omega^2/2}$, and therefore

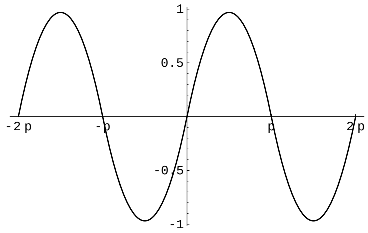
$$\chi(\omega) = \frac{1}{2}\sqrt{2\pi}\left(\mathrm{e}^{-(\omega-5)^2/2} + \mathrm{e}^{-(\omega+5)^2/2}\right).$$

b) The sketch should look like



Problem 4

a) The graph looks like



How it has been obtained can be read off from the Mathematica instruction

f[x_] := (Pi/8) x (Pi - x);
ParametricPlot[{{x, f[x]}, {- x, - f[x]}, {x - 2 Pi, f[x]}, {- x + 2 Pi, - f[x]}}, {x, 0, Pi}]

b) Because $f(x)$ is odd and 2π-periodic, we conclude from Theorem 1.2 in the Guide Book that its Fourier series is a sine series, with coefficients

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^\pi f(x)\,\sin(k\,x)\;\mathrm{d}x = \frac{2}{\pi}\int_0^\pi \frac{\pi}{8}\,x(\pi-x)\,\sin(k\,x)\;\mathrm{d}x \\ &= \frac{1}{4}\int_0^\pi \frac{\mathrm{d}\,x(\pi-x)}{\mathrm{d}x}\,\frac{1}{k}\cos(k\,x)\;\mathrm{d}x = -\frac{1}{4}\int_0^\pi \frac{\mathrm{d}^2\,x(\pi-x)}{\mathrm{d}x^2}\,\frac{1}{k^2}\sin(k\,x)\;\mathrm{d}x \\ &= \frac{1}{2k^2}\int_0^\pi \sin(k\,x)\;\mathrm{d}x = \frac{1}{2k^3}\,[-\cos(k\,\pi)-(-\cos 0)] = \frac{1}{2k^3}\,\left(-(-1)^k+1\right). \end{aligned}$$

Here we have used a partial integration in the third and in the fourth identity. The boundary terms in the third identity vanish because $f(x)=0$ when $x=0$ and when $x=\phi$, whereas the boundary terms in the fourth identity vanish because $\sin(k\,x)=0$ when $x=0$ and when $x=\pi$. The desired formula now follows from the obervation that $-(-1)^k+1=0$ when k is even and $-(-1)^k+1=2$ when k is odd.

c) It follows from b) that

$$f(x)-\sin x = \sum_{l=1}^{\infty}\frac{1}{(2l+1)^3}\,\sin((2l+1)\,x), \quad x\in\mathbf{R},$$

which is a sine series with $b_1=0$, $b_k=0$ when k is even and $b_k=1/k^3$ when k is odd and $k\ge 3$. The desired identity therefore follows from Parseval's identity (2.21) in the Guide Book, with $p=2\pi$ and $a=-\pi$.

d) **(bonus)** It follows from section 2.5 in the Guide book that the function $\sin x$ is orthogonal to all the functions $\sin((2l+1)\,x)$, $l\ge 1$, which appear in the sine series of the function $f(x)-\sin x$, Therefore $\langle f-\sin,\,\sin\rangle = 0$, hence $\langle f,\,\sin\rangle = \langle\sin,\,\sin\rangle$, and therefore

$$\langle f-\sin,\,f-\sin\rangle = \langle f,\,f\rangle - 2\langle f,\,\sin\rangle + \langle\sin,\,\sin\rangle = \langle f,\,f\rangle - \langle\sin,\,\sin\rangle.$$

Now using the symmetry of $f(x)^2$ the integral over $[-\pi,\,\pi]$ is twice the integral over $[0,\,\pi]$, hence

$$\begin{aligned} \langle f,\,f\rangle &= \frac{1}{\pi}\int_0^\pi \left(\frac{\pi}{8}\,x\,(\pi-x)\right)^2\;\mathrm{d}x = \frac{\pi}{64}\int_0^\pi x^2\;\left(\pi^2-2\pi\,x+x^2\right)\;\mathrm{d}x \\ &= \frac{\pi}{64}\left(\pi^2\,\frac{\pi^3}{3}-2\pi\,\frac{\pi^4}{4}+\frac{\pi^5}{5}\right) = \frac{\pi^6}{64}\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right) = \frac{\pi^6}{64}\,\frac{10-15+6}{30} = \frac{\pi^6}{1920}. \end{aligned}$$

On the other hand

$$\langle\sin,\,\sin\rangle = \frac{1}{\pi}\int_0^\pi (\sin x)^2\;\mathrm{d}x = \frac{1}{\pi}\int_0^\pi \frac{1-\cos(2x)}{2}\;\mathrm{d}x = \frac{1}{2},$$

because $\sin 2x = 0$ when $x=\pi$ and when $x=0$.