Principal Fiber Bundles

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For some of the notations we use, see Section 14. Some textbooks dealing with material of this course are [20], [23], [32].

Contents

1 The Theorem of Frobenius

Let M be a finite-dimensional smooth manifold and let H be smooth vector subbundle of $\mathrm{T}M$.

If α^i, $1 \leq i \leq l$, are smooth one-forms on M which at each point are linearly independent, then the

$$H_m := \bigcap_{i=1}^{l} \ker \alpha^i \tag{1.1}$$

define a smooth codimension l vector subbundle of M, and every smooth codimension k vector subbundle H of $\mathrm{T}M$ is locally of this form. In the 19th century terminology, one-forms were called *Pfaffian forms* and the vector subbundle H in (1.1) was called the *Pfaffian system* $\alpha^i = 0$, $i \leq i \leq l$.

If v is a smooth vector field on M, then we will write $v \subset H$ if $v_m \in H_m$ for every $m \in M$. This notation is literally correct if we view the section v of $\mathrm{T}M$ as a submanifold of $\mathrm{T}M$, i.e. we identify $v : M \to \mathrm{T}M$ with $v(M) \subset \mathrm{T}M$.

If $u, v \in \mathcal{X}(M)$, $u \subset H$ and $v \subset H$ then we have that for every point $m \in M$ the element $[u, v](m) + H_m \in \mathrm{T}_m M/H_m$ depends only on the values $u(m)$ and $v(m)$ of u and v at the point m, respectively. This is a little surprising because $[u, v](m)$ definitely depends also on the first order derivatives of u and v at the point m. In order to prove the Lie brackets modulo H_m only depend on $u(m)$ and $v(m)$, we can use the formula

$$\mathrm{i}_{[u,\, v]}\, \alpha = \mathcal{L}_u \left(\mathrm{i}_v\, \alpha\right) - \mathrm{i}_v \left(\mathcal{L}_u \alpha\right) = \mathcal{L}_u \left(\mathrm{i}_v\, \alpha\right) - \mathrm{i}_v \left(\mathrm{d}\left(\mathrm{i}_u\, \alpha\right)\right) - \mathrm{i}_v \left(\mathrm{i}_u \left(\mathrm{d}\, \alpha\right)\right),$$

which holds for any $u, v \in \mathcal{X}(M)$ and $\alpha \in \Omega^1(M)$. If $u \subset H$, $v \subset H$ and $\alpha = 0$ on H, then $\mathrm{i}_u\, \alpha = 0$, $\mathrm{i}_v\, \alpha = 0$ and we conclude that

$$\alpha([u, v]) = -(\mathrm{d}\alpha)(u, v). \tag{1.2}$$

Because at a given point $m \in M$ the right hand side of (1.2) only depends on $u(m)$ and $v(m)$, and the left hand side at the point m measures $[u, v](m)$ modulo H_m if we let α run through the α^i of the local Pfaffian system, the statement is proved.

The statement means that, for every $m \in M$, there is a mapping $a, b \mapsto [a, b]_{/H_m}$ from $H_m \times H_m$ to $\mathrm{T}_m M/H_m$, called the *Lie brackets modulo* H_m *on* $H_m \times H_m$, such that

$$[u, v](m) + H_m = [u(m), v(m)]_{/H_m} \tag{1.3}$$

for every pair of smooth vector fields u and v on M which are contained in H. It follows from (1.3) that the Lie brackets modulo H_m define an antisymmetric bilinear mapping from $H_m \times H_m$ to $\mathrm{T}_m M/H_m$, which moreover depends smoothly on $m \in M$.

An *integral manifold of* H is a smooth submanifold I of M such that $\mathrm{T}_i\, I = H_i$ for every $i \in I$. H is called *integrable* if for every $m_0 \in M$ there is an integral manifold I such that $m_0 \in I$.

Theorem 1.1 (Frobenius) *H is integrable if and only if $[u, v] \subset H$ for every pair $u, v \in \mathcal{X}(M)$ such that $u \subset H$ and $v \subset H$. In other words, if and only if, for every $m \in M$, the Lie brackets modulo H_m on $H_m \times H_m$ vanish. If H is integrable then for every $m_0 \in M$ there exists a diffeomorphism Φ from an open neighborhood M_0 of m_0 in M onto an open subset of $\mathbf{R}^n \times \mathbf{R}^l$, such that, for each $m \in M_0$, $(\mathrm{T}_m\, \Phi)(H_m) = \mathbf{R}^n \times \{0\}$.*

Proof If I is an integral manifold of H and $u \subset H$, $v \subset H$, then $u|_I \subset \mathrm{T}I$ and $v|_I \subset \mathrm{T}I$, which implies that $[u, v]|_I = [u|_I, v|_I] \subset \mathrm{T}I \subset H$. This proves that if H is integrable, then $u \subset H$ and $v \subset H$ implies that $[u, v] \subset H$.

In order to prove the converse, let $m_0 \in M$. If $n = \dim H_{m_0}$, we can find an open neighborhood U of m_0 in M and functions $x^j \in \mathcal{F}(U)$, $1 \leq j \leq n$, such that the restrictions to H_{m_0} of the linear forms $(\mathrm{d}x^j)(m_0)$, $1 \leq j \leq n$, are linearly independent. It follows that the $m \in U$, such that the $(\mathrm{d}x^j)(m)\big|_{H_m}$ are linearly independent, form an open neighborhood V of m_0 in U. Write $\delta_i^j = 0$ when $i \neq j$ and $\delta_i^j = 1$ when $i = j$. Then, for each $m \in V$, there are unique vectors $v_i(m) \in H_m$ such that

$$(\mathrm{d}x^j)(m)\, v_i(m) = \delta_i^j, \quad 1 \leq i,\, j \leq n, \tag{1.4}$$

and $v_i(m)$ depends smoothly on m. In other words, $v_i \in \mathcal{X}(V)$, $v_i \subset H$ and $v_i\, x^j = \delta_i^j$.

Because the functions δ_i^j are constant, we have $[v_i,\, v_j]\, x^k = v_i\, (v_j\, x^k) - v_j\, (v_i\, x^k) = v_i\, \delta_j^k - v_j\, \delta_i^k = 0$. Now assume that H satisfies the integrability condition that $u,\, v \subset H \Longrightarrow [u,\, v] \subset H$. It follows from (1.4) that the vectors $v_i(m)$ are linearly independent, hence they form a basis of H_m. Therefore there exist functions c_{ij}^k on V such that $[v_i,\, v_j] = \sum_{k=1}^n c_{ij}^k\, v_k$ on V. This implies that, for every k, $0 = [v_i,\, v_j]\, x^k = \sum_{h=1}^n c_{ij}^h\, v_h\, x^k = \sum_{h=1}^n c_{ij}^h\, \delta_h^k = c_{ij}^k$, and therefore $[v_i,\, v_j] = 0$ on V.

If $\dim M = n + l$, then we can find a smooth mapping ψ from an open subset Y of $\mathbf{R}^l$ to M , and $y_0 \in Y$ such that $\psi(y_0) = m_0$ and $\mathrm{T}_{m_0} M = H_{m_0} + \mathrm{T}_{y_0}\, \psi(\mathbf{R}^l)$. Then

$$\Psi(t^1,\, \ldots,\, t^n,\, y) := \mathrm{e}^{t^n\, v_n} \circ \ldots \circ \mathrm{e}^{t^1\, v_1}(\psi(y))$$

defines a smooth mapping Ψ from an open neighborhood of $\{0\} \times Y$ in $\mathbf{R}^n \times Y$ to M such that $\mathrm{T}_{(0,\, y_0)}\, \Psi$ is invertible. In view of the inverse mapping theorem we obtain an open neighborhood Z of $(0,\, y_0)$ in $\mathbf{R}^n \times Y$ such that $\Psi|_Z$ is a diffeomorphism from Z onto an open neighborhood M_0 of m_0 in M.

Because the vector fields v_i commute, the flows $\mathrm{e}^{t^i\, v_i}$ commute. Using this, we obtain for every $1 \leq i \leq n$ that $\partial\Psi(t,\, y)/\partial t^i = v_i(\Psi(t,\, y))$. This proves the last statement of the theorem if we take $\Phi = \Psi^{-1}$. □

Remark 1.1 Theorem 1.1 is named after Frobenius because of the article [19], in which he reviewed previous proofs and gave a proof of his own. His oldest reference is to Deahna [13], which paper I did not understand when I read it. According to Samelson [30], Deahna's necessary and sufficient condition, "pushed a little", is that if H is given by (1.1) and if $v \in \mathcal{X}(M)$, $v \subset H$, then, at each point x and for each i, $\mathcal{L}_v \alpha^i$ is a linear combination of the α^j. (The paper [30] was pointed out to me by Joop Kolk.) Frobenius also referred to Clebsch [11], who reduced the problem to the case that H has a basis of commuting vector fields, which case had been solved by Jacobi [22, pp. 257–263]. However, Jacobi did not use the commuting flows of the commuting vector fields, this idea had to wait for Lie [26]. In our proof we presented Clebsch's construction of a basis of commuting vector fields in H. ⊘

2 Connections

Let $\pi : M \to X$ be a fibration of the smooth $n+l$-dimensional manifold M over the smooth l-dimensional "base" manifold X, with smooth n-dimensional fibers $M_x := \pi^{-1}(\{x\}$, $x \in X$. An *(infinitesimal) connection* for π is defined as a codimension l smooth vector subbundle H of $\mathrm{T}M$ with the property that, for each $m \in M$, H_m is complementary to the tangent space $\ker(\mathrm{T}_m\,\pi)$ of the fiber $M_{\pi(m)}$ through the point m. In formula:

$$\mathrm{T}_m\,M = H_m \oplus \ker(\mathrm{T}_m\,\pi), \quad m \in M. \tag{2.1}$$

Because the fibers usually are thought of as vertical, H_m is called the *horizontal subspace* of $\mathrm{T}_m\,M$ at the point m.

If H would be integrable, then the integral manifolds would at least locally be sections of the bundle, intersecting each fiber at precisely one point. Mapping the point m in the fiber M_x to the intersection point with M_y of the integral manifold through m, we would obtain an identification of M_x with M_y, and if everything would work globally, we would in this way have connected every fiber M_x to every other fiber M_y in a unique fashion. However, in general H will not be integrable and it is for this reason that H is called "only an infinitesimal connection". For a somewhat more specific explanation of the names "infinitesimal connection" and "connection", see Remark 4.1 below. It is a consequence of (2.1) that the restriction to H_m of $\mathrm{T}_m\,\pi$ is a linear isomorphism from H_m onto $\mathrm{T}_{\pi(m)}\,M$. It follows that for each $m \in M$ there is a unique vector $v_{\mathrm{hor}}(m) \in H_m$ such that

$$\mathrm{T}_m\,\pi\,(v_{\mathrm{hor}}(m)) = v(\pi(m)). \tag{2.2}$$

The vector field v_{hor} on M is called the *horizontal lift* of v with respect to the connection H. It is smooth when v is smooth and $v \mapsto v_{\mathrm{hor}}$ is a continuous linear mapping from $\mathcal{X}(X)$ to $\mathcal{X}(M)$. Moreover, if γ is a solution curve in M of the differential equation $\mathrm{d}m/\,\mathrm{d}t = v_{\mathrm{hor}}(m)$, then $\delta := \pi\circ\gamma$ is a solution curve in X of the differential equation $\mathrm{d}x/\,\mathrm{d}t = v(x)$, which implies that

$$\pi \circ \mathrm{e}^{t\,v_{\mathrm{hor}}} = \mathrm{e}^{t\,v} \circ \pi. \tag{2.3}$$

This implies in turn that

$$\mathrm{T}_m\,\pi\,([u_{\mathrm{hor}},\, v_{\mathrm{hor}}]\,(m)) = [u,\, v](\pi(m)), \quad u,\, v \in \mathcal{X}(M), \quad m \in M. \tag{2.4}$$

It follows that, for every $x \in X$, $a,\, b \in \mathrm{T}_x\,X$ and $m \in M_x$, there is a unique vector $R_x(a,\, b)(m) \in \mathrm{T}_m(M_x)$, such that, for every $u,\, v \in \mathcal{X}(X)$,

$$-R_x(u(x),\, v(x))(m) = [u_{\mathrm{hor}},\, v_{\mathrm{hor}}](m) - [u,\, v]_{\mathrm{hor}}(m). \tag{2.5}$$

Note that, for every $x \in X$ and $a,\, b \in \mathrm{T}_x\,X$, $R_x(a,\, b) : m \mapsto R_x(a,\, b)(m)$ is a smooth vector field on the fiber M_x over x, and it depends in an antisymmetric bilinear way on $a,\, b \in \mathrm{T}_x\,X$. R is called the *curvature* of the connection H, where the minus sign in (2.5) has been introduced in order to get agreement with formula (3.7) below. It is clear from the definition that $R = 0$ if and only if H is an integrable vector subbundle of $\mathrm{T}M$, in which case the connection H is called *flat*.

Remark 2.1 In the proof of Theorem 1.1, we introduced a local fibration $(x^1,\, \dots,\, x^n) : M \to \mathbf{R}^n$ such that H is a connection in this fiber bundle. In this terminology the vector fields v_i are equal to the horizontal lifts $v_i = (e_i)_{\mathrm{hor}}$ of the constant vector fields e_i in $\mathbf{R}^n$, in which e_i, $1 \leq i \leq n$, denotes the standard basis of $\mathbf{R}^n$. If H is integrable, then $[u_{\mathrm{hor}},\, v_{\mathrm{hor}}] = [u,\, v]_{\mathrm{hor}}$ for any $u,\, v \in \mathcal{X}(M)$. Because $[e_i,\, e_j] = 0$, this shows in another way that $[v_i,\, v_j] = 0$. ⊘

3 Covariant Differentiation

Let $\pi : M \to X$ be a *vector bundle*, meaning that the fibers M_x, $x \in X$ are l-dimensional vector spaces. More precisely, the local trivializations of M are diffeomorphisms $\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{R}^l$, in which the U_α form an open covering of M and, for any α and β, the retrivialization $\tau_\beta \circ \tau_\alpha$ is a diffeomorphism from $(U_\alpha \cap U_\beta) \times \mathbf{R}^l$ onto itself of the form

$$(x,\, y) \mapsto (x,\, (\tau_{\beta\,\alpha}(x))(y)), \tag{3.1}$$

in which $\tau_{\beta\,\alpha}(x)$ is an invertible linear transformation of $\mathbf{R}^l$ which depends smoothly on $x \in U_\alpha \cap U_\beta$.

Now assume that H is a given connection in M. Let s is a smooth section of the vector bundle $\pi : M \to X$, meaning that s is a smooth mapping from X to M such that $\pi \circ s$ is equal to the identity in X. Then, for every $x, \in X$ and $v \in \mathrm{T}_x X$, we define the *covariant derivative* $\nabla_v s(x)$ of s at x and in the direction of the vector v, as follows. Because

$$\mathrm{T}_{s(x)}\, \pi \circ \mathrm{T}_x\, s\,(v) = \mathrm{T}_x(\pi \circ s)\,(v) = v,$$

and $(\mathrm{T}_{s(x)}\, \pi)(v_{\mathrm{hor}}(s(x))) = v$, it follows that $(\mathrm{T}_x\, s)\,(v) - v_{\mathrm{hor}}(s(x))$ belongs to $\ker \mathrm{T}_{s(x)}\, \pi = \mathrm{T}_{s(x)}\, M_x$, the tangent space at $s(x)$ of the fiber M_x over the point x. However, because M_x is a vector space, we can use the natural identification of all tangent spaces of M_x with M_x, and define

$$\nabla_v s(x) := (\mathrm{T}_x\, s)\,(v(x)) - v_{\mathrm{hor}}(s(x)) \in M_x. \tag{3.2}$$

In particular, if $v \in \mathcal{X}(X)$, then $\nabla_v s : x \mapsto \nabla_{v(x)} s(x)$ is a smooth section of M again, which is called the *covariant derivative of s with respect to the vector field v*. If $\Gamma(M)$ denotes the space of all smooth sections of $\pi : M \to X$, then, for every $v \in \mathcal{X}(X)$, the covariant derivative $\nabla_v : s \mapsto \nabla_v s$ with respect to v is a first order partial differential operator acting on $\Gamma(M)$.

The fact that M is a vector bundle makes the space of sections $\Gamma(M)$ into a vector space. The connection H is called a *linear connection* if all the covariant derivatives are linear operators. In order to investigate when this holds, we use a local trivialization of M and local coordinates in X in order to pass to the situation that X is an open subset of $\mathbf{R}^n$, $M = X \times \mathbf{R}^l$ and $\pi : (x,\, y) \mapsto y$. In this situation, we write the connection H as

$$H_{(x,\,y)} = \{(v,\, A(x,\, y)\, v) \mid v \in \mathbf{R}^n\} \tag{3.3}$$

for unique linear mappings $A(x,\, y) : \mathbf{R}^n \to \mathbf{R}^l$ depending smoothly on $(x,\, y)$. It follows that $v_{\mathrm{hor}}(s(x)) = (v,\, A(x,\, s(x))\, v)$. This leads to the formula

$$(\nabla_v s)^i(x) := \sum_{j=1}^n \frac{\partial s^i(x)}{\partial x_j}\, v^j - \sum_{j=1}^n A^i_j(x,\, s(x))\, v^j$$

for the covariant derivative. It is clear that the covariant derivatives are linear operators if and only if the matrices $A^i_j(x,\, y)$ depend linearly on y, i.e.

$$A^i_j(x,\, y) = -\sum_{k=1}^l \Gamma^i_{jk}(x)\, y^k \tag{3.4}$$

for some coefficients $\Gamma^i_{jh}(x)$ which depend smoothly on x, and which are called the *Christoffel symbols* of the connection H, with respect to the given local trivialization of M and local coordinates in X, respectively. The minus sign in (3.4) is chosen in order get a plus sign in the formula

$$(\nabla_v s)^i(x) := \sum_{j=1}^{n} \frac{\partial s^i(x)}{\partial x_j}\, v^j + \sum_{j=1}^{n}\sum_{k=1}^{l} \Gamma^i_{jk}(x)\, v^j\, s^k(x) \tag{3.5}$$

for the covariant derivative.

The Christoffel symbols depend sensitively on the choice of the local trivialization of the bundle and the local coordinates in the base manifold. The Christoffel symbols do not even transform as a tensor, cf. Lemma 7.1 below.

It follows from (3.5) that the covariant derivative with respect to a linear connection satisfies the *Leibniz rule for covariant differentiation*

$$\nabla_v(f\, s) = (v\, f)\, s + f\, \nabla_v s, \quad v \in \mathcal{X}(X), \quad f \in \mathcal{F}(X), \quad s \in \Gamma(M), \tag{3.6}$$

in which $v\, f$ denotes the derivative of the function f in the direction of the vector field v

In general, an operator $\nabla : (v,\, s) \mapsto \nabla_v s : \mathcal{X}(X) \times \Gamma(M) \to \Gamma(M)$ is called a *covariant derivative* if, for every $v \in \mathcal{X}(M)$, $s \mapsto \nabla_v s$ is a linear mapping from $\Gamma(M)$ to $\Gamma(M)$ which satisfies the Leibniz rule (3.6) and, for every $s \in \Gamma(M)$, $v \mapsto \nabla_v s$ is a linear mapping from $\mathcal{X}(X)$ to $\Gamma(M)$ such that $\nabla_{f\, v} s = f\, \nabla_v s$ for every $f \in \mathcal{F}(X)$, $v \in \mathcal{X}(X)$ and $s \in \Gamma(M)$. It is easily verified that for every covariant derivative ∇ there is a unique linear connection H in M such that ∇ is equal to the covariant derivative defined by H.

Another direct calculation in terms of Christoffel symbols shows that, for every $u,\, v \in \mathcal{X}(X)$, $s \in \Gamma(M)$ and $x \in X$,

$$\left(\nabla_u(\nabla_v s) - \nabla_v(\nabla_u s) - \nabla_{[u,\, v]} s\right)(x) = R_x(u(x),\, v(x))(s(x)), \tag{3.7}$$

in which R is the curvature of the connection introduced in (2.5). The operator acting on sections s of M which appears in the left hand side,

$$R(u,\, v) := [\nabla_u,\, \nabla_v] - \nabla_{[u,\, v]}, \tag{3.8}$$

is called the *curvature operator* R. A priori it is a second order linear partial differential operator, but the formula (3.7) shows that it actually is of order zero. Often (3.7) is presented as the definition of the curvature R of the connection H.

Remark 3.1 The idea of a connection and of parallel transport, cf. Section 4 below, has been introduced by Levi-Civita [25], for the Levi-Civita connection in a (pseudo-)Riemannian manifold $(X,\, \beta)$. See Section 4 and 7 for the definition of parallel transport and Levi-Civita connection, respectively.

After various generalizations by, among others, Hermann Weyl and Élie Cartan, the general definition which we have presented here was given by Ehresmannn [18]. $\oslash$

4 Parallel Transport

Let $\pi : M \to X$ be a smooth bundle with smooth connection H and let $\gamma : I \to X$ be a smooth curve in X, in which I is an interval in $\mathbf{R}$ which contains at least two points. A smooth curve $\delta : I \to M$ will be called a *horizontal lift of* γ if

i) $\pi \circ \delta = \gamma$, i.e. $\delta(t) \in M_{\gamma(t)}$ for every $t \in I$, and

ii) For every $t \in I$ we have that $\delta'(t) \in H_{\delta(t)}$.

In a local trivialization $M \simeq U \times Y$,

$$H_{(x,\,y)} = \{(v,\, A(x,\, y)\, v) \mid v \in \mathrm{T}_x\, X\}$$

for a unique linear mapping $A(x,\, y)$ from $\mathrm{T}_x\, X$ to $\mathrm{T}_y\, Y$, depending smoothly on $(x,\, y) \in U \times Y$, cf. (3.3). The condition i) means that $\delta(t) = (\gamma(t),\, \epsilon(t))$ for a smooth curve ϵ in Y, whereas the condition ii) is equivalent to

$$\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} = A(\gamma(t),\, \epsilon(t))\, \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t},$$

which is a time-dependent ordinary differential equation for the curve $\epsilon : I \to Y$ of the form

$$\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} = f(t,\, \epsilon(t)), \tag{4.1}$$

if we write

$$f(t,\, y) := A(\gamma(t),\, y)\, \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}.$$

It follows from the theory of ordinary differential equations, cf. [12], that for every $a \in I$ and $y \in Y$ the equation (4.1) has a unique maximal solution $\epsilon(t) = \epsilon_y(t)$, defined for t in an open interval I_y in I containing a, such that $\epsilon(a) = y$. Moreover, $U := \{(t,\, y) \mid y \in Y,\ t \in I_y\}$ is an open subset of $I \times Y$ containing $\{a\} \times Y$ and $(t,\, y) \mapsto \epsilon_y(t)$ is a smooth mapping from U to Y. Finally, if $I_y \neq I$, for instance $T = \sup I_y \in I \setminus I_y$, then for every compact subset K of Y there exists an $b < T$ such that $\epsilon(t) \in Y \setminus K$ for every $b < t < T$. In other words, the horizontal lift is not defined on the whole interval I if and only if it runs to infinity in the sense that it runs out of every compact subset of the fiber.

In the bundle M, this implies that for every $a \in I$ and $y \in M_{\gamma(a)}$ the curve γ has a unique maximal horizontal lift δ_y which is defined on an open interval I_y in I. The set

$$U := \{(t,\, y) \mid y \in M_{\gamma(a)},\ t \in I_y\}$$

is an open subset of $I \times M_{\gamma(a)}$, $(t,\, y) \mapsto \delta_y(t)$ is a smooth mapping from U to M, and $I_y \neq I$ if and only if $\delta_y(t)$ runs out of every compact subset of M before t reaches the boundary of I. Note that for every compact subset J of I we have that the $\pi(\delta_y(t)) = \gamma(t)$, $t \in J$, stay in the compact subset $\gamma(J)$ of X, which means that this running to inifinity has to take place "in the direction of the fibers of M".

We say that H *allows lifting* if for every smooth curve $\gamma : I \to X$, $a \in I$ and $y \in M_{\gamma(a)}$ there exists a lift $\delta = \delta_y : I \to M$ of γ such that $\delta(a) = y$.

Lemma 4.1 *Let H be a connection in the fiber bundle M over X. Then H allows lifting if for every smooth curve $\gamma : I \to X$, in which I is an interval in $\mathbf{R}$, the following condition is satisfied. For every $b \in I$ there exists an open interval J_b in I such that $b \in J_b$ and such that for every $z \in M_{\gamma(b)}$ there exists a horizontal lift $\delta_{b,\,z} : J_b \to M$ of $\gamma|_{J_b}$ such that $\delta_{b,\,z}(b) = z$.*

Proof Let $a \in I$, $y \in M_{\gamma(a)}$ and let $\delta : I_y \to M$ denote the maximal horizontal lift such that $\delta(a) = y$. We have to prove that $I_y = I$. Suppose that $T = \sup I_y \in I \setminus I_y$, we will prove that this leads to a contradiction. The case that $T = \inf I_y \in I \setminus I_y$ can be treated similarly.

There exists a $b \in I_y \cap J_T$. We have $\delta_{b,\,\delta(b)}(b) = \delta(b)$ and it follows from the uniqueness of solutions of ordinary differential equations with the same initial values that $\delta_{b,\,\delta(b)}(t) = \delta(t)$ for all $t \in I_y \cap J_T$. But this implies that $\delta_{b,\,\delta(b)}$ and δ have a common extension $\widetilde{\delta}$ to $I_y \cup J_T$, which is a horizontal lift of γ such that $\widetilde{\delta}(a) = y$. It follows that $I_y \cup J_T \subset I_y$, in contradiction with the definition of T. $\square$

If the connection allows lifting, then the lifting is unique and $(t,\, y) \mapsto \delta_y(t)$ is a smooth mapping from $I \times M_{\gamma(a)}$ to M. Moreover, $\delta_y(t) \in M_{\gamma(t)}$ for every $t \in I$. The mapping $h_\gamma^{a,\,t} : y \mapsto \delta_y(t)$ is a smooth mapping from $M_{\gamma(a)}$ to $M_{\gamma(t)}$, which is called the *parallel transport from the fiber $H_{\gamma(a)}$ of H over $\gamma(a)$ to the fiber $H_{\gamma(t)}$ of H over $\gamma(t)$, along the curve γ in X*. Because obviously $h_\gamma^{t,\,a}$ is a two-sided inverse of $h_\gamma^{a,\,t}$, $h_\gamma^{a,\,t}$ is a diffeomorphism from $H_{\gamma(a)}$ onto $H_{\gamma(t)}$.

Remark 4.1 One says that parallel transport establishes a *connection of $M_{\gamma(a)}$ with $M_{\gamma(t)}$, which is obtained by integrating the infinitesimal connection H along the curve γ.* $\oslash$

The lifting can be extended to piecewise smooth continuous curves in X. It is clear that the parallel transport along the concatenation "γ_2 after γ_1" of the curves γ_1 and γ_1, in which the endpoint of γ_1 is equal to the initial point of γ_2, is equal to the composition $h_{\gamma_2} \circ h_{\gamma_1}$ of the parallel transports. If γ is a *loop based at* x, a closed piecewise smooth continuous curve $\gamma : [a,\, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = x$, then $h_\gamma^{a,\,b}$ is a diffeomorphism from H_x onto itself. The set $\mathrm{Loop}(X,\, x)$ of all loops based at x is a group with the concatenation of loops as the product structure, and $h : \gamma \mapsto h_\gamma^{a,\,b}$ is a homomorphism from $\mathrm{Loop}(X,\, x)$ to the group $\mathrm{Diffeo}(M_x)$ of all diffeomorphisms of M_x, with the composition of mappings as the product. This homomorphism h is called the *holonomy representation* of $\mathrm{Loop}(X,\, x)$ in $\mathrm{Diffeo}(M_x)$. The image $h(\mathrm{Loop}(X,\, x))$ is a subgroup of $\mathrm{Diffeo}(M_x)$, called the *holonomy group* in M_x. Of course, everything being defined by the connection H in M which allows lifting.

Proposition 4.2 *Every linear connection H in a vector bundle M over a manifold X allows lifting. For every $x \in X$, the holonomy group is a subgroup of $\mathrm{GL}(M_x)$, the group of all linear transformations in M_x.*

Proof The differential equation (4.1) takes the form

$$\frac{\mathrm{d}\epsilon^i(t)}{\mathrm{d}t} = \sum_{k=1}^{l} f_k^i(t)\, \epsilon^k(t)),$$

in which

$$f_k^i(t) := -\sum_{j=1}^{n} \Gamma_{jk}^i(\gamma(t))\, \frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t},$$

cf. (3.4). For a linear system of ordinary differential equations with coefficients depending continuously on t we have local solutions. Moreover, if e_i, $1 \leq i \leq l$, form a basis of $M_{\gamma(b)}$, then the intersection I_0 of the intervals of definition of the horizontal lifts δ_i such that $\delta_i(b) = e_i$ is an open interval in I which contains b. Using the linearity of the system of differential equations for horizontal curves, we have the superposition principle which says that for any $z = \sum_{i=1}^{l} c_i\, e_i$ the curve $\delta : t \mapsto \sum_{i=1}^{l} c_i\, \delta_i(t)$ defines a horizontal curve δ on I_0 such that $\delta(b) = z$. Using Lemma 4.1 we may therefore conclude that H allows lifting.

The second statement follows because the aforemntioned superposition principle implies that in the local trivializations the parallel transports are given by linear transformations in $\mathbf{R}^l$. $\square$

5 The Frame Bundle

Let M be a smooth vector bundle over X. If f_i, $1 \leq i \leq l$, is a basis of M_x, then

$$f : c \mapsto \sum_{i=1}^{l} c_i \, f_i$$

is a linear isomorphism from $\mathbf{R}^l$ onto M_x, which we will call a *frame* in M_x. If e_i, $1 \leq i \leq l$, denotes the standard basis of $\mathbf{R}^l$, then $f_i = f(e_i)$, which shows that we have a bijection between the set of all bases in M_x and the set $\mathrm{F}M_x$ of linear isomorpisms from $\mathbf{R}^l$ to M_x.

The $f \in \mathrm{F}M_x$ will be called the *frames in* M_x. The group $G = \mathrm{GL}(l, \mathbf{R})$ acts on $P_x := \mathrm{F}M_x$, by means of

$$g_P : f \mapsto f \circ g^{-1}, \tag{5.1}$$

and this action is proper, free, and transitive. The $\mathrm{F}M_x$, $x \in X$ together form a smooth fiber bundle $\mathrm{F}M$ over X, called the *frame bundle* of M. On $\mathrm{F}M$ the aforementioned action of G is proper and free. Because the G-orbits are the fibers $\mathrm{F}M_x$, $x \in X$ of the bundle $\pi : \mathrm{F}M \to X$, it follows that $P := \mathrm{F}M$ is a principal $\mathrm{GL}(l, \mathbf{R})$-bundle of which the orbit space is identified with X.

If $f \in \mathrm{F}M_x$, then $\epsilon_i(f) := f(e_i) \in M_x$, and

$$\epsilon : f \mapsto (\epsilon_1(f), \, \ldots, \, \epsilon_l(f))$$

defines a diffeomorphism from $\mathrm{F}M$ onto an open subset V of the fiberwise l-fold Cartesian power

$$M^{(l)} = \left\{(y_1, \, \ldots, \, y_l) \in M^l \mid \pi_i(y_i) = \pi_j(y_j) \quad \text{for all } i \text{ and } j\right\}$$

of M. More explicitly, $M^{(l)}$ is the vector bundle over X such that, for every $x \in X$, $M_x^{(l)} = (M_x)^l$. The subset V of $M^{(l)}$ is defined by the property that, for every $x \in X$, $(y_1, \, \ldots, \, y_l) \in V \cap M_x^{(l)}$ if and only if the y_i, $1 \leq i \leq l$, are linearly independent in M_x, which is an open subset of $M^{(l)}$.

If H is a connection in M, then we define a corresponding connection H in $\mathrm{F}M$, which we denote by the same letter, by means of

$$H_f := \{w \in \mathrm{T}_f(\mathrm{F}M) \mid \mathrm{T}_f \, \epsilon_i(w) \in H_{f_i} \quad \text{for every} \quad 1 \leq i \leq l\}, \quad f \in \mathrm{F}M. \tag{5.2}$$

As the intersection of the codimension l linear subspaces $(\mathrm{T}_f \, \epsilon_i)^{-1}(H_{f_i})$ of $\mathrm{T}_f(\mathrm{F}M)$, H_f is a linear subspace of $\mathrm{T}_f(\mathrm{F}M)$ of codimension at most l^2, and because $\dim \mathrm{F}M = n + l^2$, it follows that $\dim H_f \geq n = \dim X$.

Furthermore, we have for every $1 \leq i \leq l$ that $\pi_{\mathrm{F}M} = \pi_M \circ \epsilon_i$. Therefore, if $w \in H_f$ and $\mathrm{T}_f \, \pi(w) = 0$, then we have for every $1 \leq i \leq l$ that $\mathrm{T}_{f_i} \, \pi_M \circ \mathrm{T}_f \, \epsilon_i(w) = 0$, which implies that $\mathrm{T}_f \, \epsilon_i(w) = 0$ because $\mathrm{T}_f \, \epsilon_i(w) \in H_{f_i}^M$ and the restriction to H_{f_i} of $\mathrm{T}_{f_i} \, \pi_M$ is injective. In turn $\mathrm{T}_f \, \epsilon_i(w) = 0$ for every i implies that $\mathrm{T}_f \, \epsilon(w) = 0$, and therefore $w = 0$ because ϵ is a diffeomorphism. The conclusion is that the restriction to H_f of $\mathrm{T}_f \, \pi_{\mathrm{F}M}$ is injective, and because $\dim H_f \geq \dim X$, it follows that this restriction is bijective and H_f is complementary to the tangent space $\ker \mathrm{T}_f \, \pi_{\mathrm{F}M}$ at f of the fiber. This completes the proof that (5.2) defines a connection in $\mathrm{F}M$.

A smooth local section f of $\mathrm{F}M$ is a smooth mapping $f : U \to \mathrm{F}M$ from an open subset U of X to $\mathrm{F}M$ such that, for each $x \in U$, $f(x) \in \mathrm{F}M_x$. For each $1 \leq i \leq n$ we will write $f_i(x) = f(x)(e_i) = \epsilon_i(f(x))$, this defines a smooth local section f_i of the vector bundle M, where we have the property that for each $x \in U$ the $f_i(x)$ form a basis of M_x. A smooth local section of $\mathrm{F}M$ was called a *moving frame* by Élie Cartan.

Let f be a moving frame, $x \in X$ and $v \in \mathrm{T}_x X$. Then, according to, (5.2), $\mathrm{T}_x f(v) \in H_f$ if and only if, for every $1 \leq i \leq l$, $\mathrm{T}_x f_i(v) \in H_{f_i(x)}$, which in turn is equivalent to the condition that $(\nabla_v f_i)(x) = 0$ for every $1 \leq i \leq l$. This property characterizes the connection in the frame bundle as the one for which, for every $x \in X$ and $v \in \mathrm{T}_x X$, we have that $\nabla_v f(x) = 0$ if and only if $(\nabla_v f_i)(x) = 0$ for every $1 \leq i \leq l$.

If $g \in G$ then

$$g^{-1}(e_i) = \sum_{k=1}^{l} A_i^k \,(e_k)$$

for some constant matrix A_i^k, and therefore

$$(f \circ g^{-1})_i = \sum_{k=1}^{l} A_i^k \, f_k.$$

If the connection H in M is linear, then $(\nabla_v f_i(x) = 0$ for all $1 \leq i \leq n$ implies that $(\nabla_v (f \circ g^{-1})_i)(x) = 0$ for all $1 \leq i \leq l$. Therefore *the linearity of the connection in M implies that the connection H in the principal G-bundle P is G-invariant*, in the sense that

$$(\mathrm{T}_p \, g_P)(H_p) = H_{g_P(p)}, \quad p \in P, \quad g \in G. \tag{5.3}$$

The mapping $(f,\, c) \mapsto f(c)$ from $M_x \times \mathbf{R}^l$ to M_x is surjective. Moreover, we have $f(c) = f'(c')$ if and only if $c' = g(c)$, in which $g := (f')^{-1} \circ f \in \mathrm{GL}(l,\, \mathbf{R}) = G$. Because $g := (f')^{-1} \circ f$ is equivalent to $f' = f \circ g^{-1} = g_P(f)$, we obtain that the fibers of the mapping $(f,\, c) \mapsto f(c)$ are equal to the orbits of G on $\mathrm{F}M_x \times \mathbf{R}^l$, where $g \in G$ acts on $\mathrm{F}M_x \times \mathbf{R}^l$ by means of $(f,\, c) \mapsto (g_P(f),\, g(c))$.

In this way we obtain a smooth mapping from $\mathrm{F}M \times \mathbf{R}^l$ onto M, which actually is a fibration with fibers equal to the $\mathrm{GL}(l,\, \mathbf{R})$-orbits in $\mathrm{F}M \times \mathbf{R}^l$, and this mapping leads to the identification

$$\mathrm{F}M \times_{\mathrm{GL}(l,\, \mathbf{R})} \mathbf{R}^l \xrightarrow{\sim} M \tag{5.4}$$

of M with the *associated vector bundle* $P \times_G Y$, in which $Y := \mathbf{R}^l$.

Because the action of G on the fiber $P_x = \mathrm{F}M_x$ of P is free and transitive, the orbit

$$\{g_P(f),\, g_V(c)) \mid g \in G\}$$

is equal to the graph of a mapping $S : P_x \to Y$, which is uniquely determined by the conditions that $S(f) = c$ and

$$S(g_P(f)) = g_V(S(f)), \quad g \in G. \tag{5.5}$$

One says that S *intertwines* the action of G on P_x with the action of G on Y, or that the mapping $S : P_x \to Y$ is *equivariant* with respect to the action of G on P_x and Y, respectively.

A section s of M is a mapping which assigns to each $x \in X$ and element $s(x)$ of M_x, whereas now in turn $s(x)$ is identified with a mapping from P_x to V. Because P_x is the fiber over x of P, this leads to an identification of s with a smooth mapping $S : P \to Y$ which is G-equivariant in the sense of (5.5). If we denote by $\mathcal{F}(P,\, Y)^G$ the space of smooth mappings $S : P \to Y$ which are which are equivariant in the sense of (5.5), then we obtain an identification of the space $\Gamma(M)$

of smooth sections of M with the linear subspace $\mathcal{F}(P,\,Y)^G$ of the space $\mathcal{F}(P,\,Y)$ of all smooth mappings from P to the fixed vector space V.

In order to express the covariant derivative of a section s of M in terms of the Y-valued function S on P, We choose, for a given point $x_0 \in X$. a local section f of the frame bundle $\mathrm{F}M$ which is horizontal at the point x_0, which means that

$$\mathrm{T}_{x_0} f\,(\mathrm{T}_{x_0} X) = H_{f(x_0)}.$$

Because $S(f) = f^{-1}\,s(\pi(f))$ for every $f \in \mathrm{F}\,M$, we have

$$s(x) = f(x)\,S(f(x)) = f(x)\left[\sum_{i=1}^{l} S_i(f(x))(e_i)\right] = \sum_{i=1}^{l} S_i(f(x))\,f(x)(e_i) = \sum_{i=1}^{l} S_i(f(x))\,f_i(x)$$

for every x in the domain of definition of f. Here $f_i : x \mapsto f(x)(e_i)$ is a local section of M. In view of (3.6) we have for any $v \in \mathcal{X}(X)$ that

$$(\nabla_v s)(x) = \sum_{i=1}^{l} \left[(\mathrm{d}S_i(f(x))\ \mathrm{T}_x\,f(v(x)))\,f_i(x) + S_i(f(x))\,(\nabla_v f_i)(x)\right].$$

Because f is horizontal at the point x_0 we have that $\mathrm{T}_x\,f(v(x_0)) = v_{\mathrm{hor}}(f(x_0))$, whereas the definition of the connection in $\mathrm{F}M$ and again the fact that f is horizontal at the point x_0 implies that $(\nabla_v f_i)(x_0) = 0$ for every $1 \le i \le l$. We therefore have

$$(\nabla_v s)(x_0) = \sum_{i=1}^{l} (v_{\mathrm{hor}} S_i)(x_0)\,f_i(x_0) = f(x_0)\,(v_{\mathrm{hor}} S)(x_0).$$

Therefore, if we denote the section of the vector bundle M and the corresponding element of $\mathcal{F}(P,\,Y)^G$ with the same letter, we arrive at the conclusion that

$$\nabla_v s = v_{\mathrm{hor}}\,s, \tag{5.6}$$

in which in the left and right hand side s is viewed as a section of M and an element of $\mathcal{F}(P,\,Y)^G$, respectively. In other words, covariant differentiation is equal to the ordinary differentiation in the direction of the horizontal lift of the vector field in the frame bundle.

Conversely, for any G-invariant connection in P, the right hand side in (5.6) defines a covariant derivative on sections of M, and therefore defines a unique linear connection in M, which induces the connection in P. This leads to a canonical bijective correspondence

$$\text{linear connections in } M \leftrightarrow G\text{-invariant connections in } P$$

6 Orthogonal and Unitary Frame Bundles

Now let β_x be an inner product in the fiber M_x of M, depending smoothly on $x \in X$. An orthonormal basis of M_x with respect β_x corresponds to an orthogonal linear mapping f from $\mathbf{R}^l$ provided with its standard inner product to M_x provided with the inner product β_x. The space of all orhtogonal linear mappings $f : \mathbf{R}^l \to M_x$ will be denoted by $\mathrm{OF}M_x$.

Let $\mathrm{O}(l)$ denote the standard orthogonal group in $\mathbf{R}^l$. If $g \in \mathrm{O}(l)$ and $f \in \mathrm{OF}M_x$, then $f \circ g^{-1} \in \mathrm{OF}M_x$. This defines a proper, free and transitive action of $\mathrm{O}(l)$ on $\mathrm{OF}M_x$. The $\mathrm{OF}M_x$, $x \in X$ form a subbundle $\mathrm{OF}M$ of the frame bundle $\mathrm{F}M$, which is a principal $\mathrm{O}(l)$-bundle, called the *orthogonal frame bundle* of M. For many purposes it is advantageous to have a smaller structure group, and especially the fact that the orthogonal group $\mathrm{O}(l)$ is compact (where $\mathrm{GL}(n, \mathbf{R})$ isn't) is often of great help. In the same way as in (5.4), we have an identification

$$\mathrm{OF}M \times_{\mathrm{O}(l)} \mathbf{R}^l \xrightarrow{\sim} M \tag{6.1}$$

of M with the associated vector bundle $\mathrm{OF}M \times_{\mathrm{O}(l)} \mathbf{R}^l$ with structure group $\mathrm{O}(n)$.

If H is a linear connection in M, then we have the induced $\mathrm{GL}(l, \mathbf{R})$-invariant connection H in $\mathrm{F}M$, and if its restriction of H to $\mathrm{OF}M$ would be tangent to $\mathrm{OF}M$, then it would automatically define an $\mathrm{O}(l)$-invariant connection H in $\mathrm{OF}M$, for which all the observations in Subsection 5 would hold.

The restriction to $\mathrm{OF}M$ of the connection H in $\mathcal{F}M$ is tangent to $\mathrm{OF}M$, if an only if, at every point $x_0 \in X$, every β_{x_0}-orthonormal basis f_0 in H_{x_0} can be extended to a local section of $\mathrm{OF}M$, of which the tangents at x_0 are horizontal. For this, it is clearly necessary that, whenever $r, s \in \Gamma(M)$, $v \in \mathcal{X}(X)$ $(\nabla_v r)(x_0) = 0$ and $(\nabla_v s)(x_0) = 0$, we have that $(v(\beta(r, s)))(x_0) = 0$. We say that β is *covariantly constant with respect to the connection H* if the latter condition holds.

We will now prove that if β is covariantly constant, then the restriction to $\mathrm{OF}M$ of the connection H in $\mathcal{F}M$ is tangent to $\mathrm{OF}M$.

Proof (suggested to me by Erik van den Ban)

Let $f \in \mathrm{OF}M_x$ and $w \in H_f$. Write $v = \mathrm{T}_f\, \pi(w) \in \mathrm{T}_x\, X$. Choose a smooth curve $\gamma : I \to X$ in X, such that $0 \in I$, $\gamma(0) = x$ and $\gamma'(0) = v$. For each $1 \leq i \leq l$, let δ_i be the horizontal lift of γ in $\mathrm{F}M$ such that $\delta_i(0) = f(e_i)$.

Because $\delta_i'(t) \in H^M_{\delta(t)}$ and β is covariantly constant, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \beta_{\gamma(t)}\, (\delta_i(t),\, \delta_j(t)) = 0, \quad 1 \leq i,\, j \leq l, \quad t \in I.$$

It follows that $\beta_{\gamma(t)}\, (\delta_i(t),\, \delta_j(t))$ is constant as a function of t, hence equal to its value at $t = 0$, which is equal to 0 when $i \neq j$ and equal to 1 when $i = j$. In other words, if $f(t)$ denotes the frame in $M_{\gamma(t)}$ such that $f(t)(e_i) = \delta_i(t)$, then we have for every $t \in I$ that $f(t) \in \mathrm{OF}M$, which implies that $f'(t) \in \mathrm{T}_{f(t)}(\mathrm{OF}M)$. Because $f(0) = f$ and $f'(0)$ is equal to the horizontal lift $w = v_{\mathrm{hor}}$ of v in $\mathrm{T}_f(\mathrm{F}M)$, we conclude that $w \in \mathrm{T}_f(\mathrm{OF}M)$. □

Under the identification of the sections of M with the $\mathrm{O}(n)$-equivariant mappings from $\mathrm{OF}M$ to $\mathbf{R}^l$, the inner product takes the standard form

$$\beta(r,\, s) = (r,\, s) \tag{6.2}$$

where in the left hand side r and s are viewed as sections of M, and in the right hand side we have taken the standard inner product of the $\mathbf{R}^l$-valued functions r and s on $\mathrm{OF}M$.

Clearly (6.2) defines inner products β_x on the fibers M_x of M which are covariantly constant with respect to the connection in M defined by (5.6). It follows that the the restriction to $\mathrm{OF}M$ of the connection in $\mathrm{F}M$ is tangent to $\mathrm{OF}M$, if and only if the inner products on the fibers of M are covariantly constant.

Another important version occurs if M is a complex vector bundle over provided with a Hermitian structure h. This means that, for each $x \in X$, H_x is a complex vector space of dimension, say, l, and h_x is a Hermitian inner product on H_x which depends smoothly on x. We then introduce the set $\mathrm{UF}M_x$ of all complex linear mappings $f : \mathbf{C}^l \to M_x$ which map the standard basis of $\mathbf{C}^l$ to an h-orthonorml basis. The $\mathrm{UF}M_x$, $x \in M$ form a smooth bundle $\mathrm{UF}M$ over X which is a principal $\mathrm{U}(l)$-bundle, where $\mathrm{U}(l)$ dentes the group of unitary transformations in $\mathbf{C}^l$. Furthermore, as in (5.4) we have the identification

$$\mathrm{UF}M \times_{\mathrm{U}(l)} \mathbf{C}^l \xrightarrow{\sim} M \tag{6.3}$$

of M with the associated vector bundle $\mathrm{UF}M \times_{\mathrm{U}(l)} \mathbf{C}^l$ with structure group $\mathrm{U}(n)$.

If H is a linear connection in M which is *complex* in the sense that $s \mapsto \nabla_v s$ is a complex linear operator, and if the Hermitian structure h is covariantly constant, then we obtain a $\mathrm{U}(n)$-invariant connection in $\mathrm{UF}M$, in terms of which the covariant differentiation satisfies (5.6) (because it does so in the bigger frame bundle). We also have the analogon

$$h(r,\, s) = (r,\, s) \tag{6.4}$$

of (6.2) where in the left hand side r and s are viewed as sections of M, and in the right hand side we have taken the standard inner product of the $\mathbf{C}^l$-valued functions r and s on $\mathrm{UF}M$.

Note that $\mathbf{C}^l \simeq \mathbf{R}^{2l}$ and that $\mathrm{U}(l)$ is a subgroup of $\mathrm{O}(2l)$ of quite smaller dimension: $\dim \mathrm{U}(l) = l^2$ and $\dim \mathrm{O}(2l) = l(2l+1)$.

7 The Levi-Civita Connection

If $M = \mathrm{T}X$ is equal to the tangent bundle of X, then for every $x \in X$ the fiber $M_x = \mathrm{T}_x$ is equal to the tangent space of X at x, and with the identification of all the tangent spaces of a vector space with the vector space itself, also each tangent space $\mathrm{T}_v M_x$, $v \in M_x$, is identified with $\mathrm{T}_x X$.

Let H be a linear connection in $\mathrm{T}X$, which sometimes also is referred to as an *affine connection in* X. if $v, w \in \mathcal{X}(X)$, then $\nabla_v w \in \mathcal{X}$, and one can define the *torsion* T of the connection H by means of the formula

$$T(v, w) := \nabla_v w - \nabla_w v - [v, w]. \tag{7.1}$$

On the other hand, any system of local coordinates $(x_1, \ldots, x_n)$, which is a diffeomorphism from an open subset U of X onto an open subset V of $\mathbf{R}^n$, induces a local trivialization of $\mathrm{T}X$, and the covariant derivative is expressed in terms of the Christoffel symbols by means of the formula (3.5), in which $s = w$ and $l = n$. It follows that $T(v, w)(x)$ only depends on $v(x)$ and $w(x)$ and is given by the antisymmetric bilinear mapping $T(x) : \mathrm{T}_x X \times \mathrm{T}_x X \to \mathrm{T}_x X$ which in local coordinates is given by

$$T(x)(v, w) = \sum_{j=1}^{n} \sum_{k=1}^{l} \Gamma^i_{jk}(x) \left(v^j w^k - w^j v^k \right) = \sum_{j=1}^{n} \sum_{k=1}^{l} \left(\Gamma^i_{jk}(x) - \Gamma^i_{kj}(x) \right) v^j w^k, \tag{7.2}$$

where the second equation is obtained by interchanging the summation indices j and k in the terms with the minus signs. The antisymmetric bilinear mapping $T(x) : \mathrm{T}_x X \times \mathrm{T}_x X \to \mathrm{T}_x X$ is called the *torsion of the connection* H *at the point* x.

Lemma 7.1 *Let H be a linear connection in $\mathrm{T}X$ and let $x \in X$. Then the following statements are equivalent.*

i) *There exists a system of local coordinates in a neighborhood of x such that $\Gamma^i_{jk}(x) = 0$ for all indices $1 \leq i, j, k \leq n$.*

ii) $T(x) = 0$.

iii) *In any system of local coordinates in a neighborhood of x, we have $\Gamma^i_{jk}(x) = \Gamma^i_{kj}(x)$ for all indices $1 \leq i, j, k \leq n$.*

iv) *In some system of local coordinates in a neighborhood of x, we have $\Gamma^i_{jk}(x) = \Gamma^i_{kj}(x)$ for all indices $1 \leq i, j, k \leq n$.*

Proof i) $\Longrightarrow$ ii) and ii) $\Longrightarrow$ iii) follow from (7.2), whereas iii) $\Longrightarrow$ iv) is obvious.

A diffeomorphism ϕ from an open subset U of $\mathbf{R}^n$ to an open subset V of $\mathbf{R}^n$ induces the transformation $\mathrm{T}\phi$ from $\mathrm{T}U = U \times \mathbf{R}^n$ onto $\mathrm{T}V = V \times \mathbf{R}^n$ which is given by

$$(x, v) \mapsto (\phi(x), (\mathrm{D}\phi(x))\, v),$$

in which

$$((\mathrm{D}\phi(x))\, v)^i = \sum_{k=1}^{n} \frac{\partial \phi^i(x)}{\partial x^k}\, v^k$$

is given by the Jacobi matrix of ϕ.

It follows that the image of $\mathbf{R}^n \times \{0\}$ under $\mathrm{T}_{(x,\,v)}(\mathrm{T}\phi)$ is equal to the space of all vecors $((\mathrm{D}\phi(x))\,u,\; Q(u,\,v))$ in $\mathbf{R}^n \times \mathbf{R}^n$, in which

$$Q(u,\,v)^i := \sum_{j,\,k=1}^{n} \frac{\partial^2 \phi^i(x)}{\partial x^j\,\partial x^k}\, u^j\, v^k$$

and u ranges over $u \in \mathbf{R}^n$. Now choose ϕ such that $\phi(0) = 0$ and $\mathrm{D}\phi(0) = I$. Then the Christoffel symbols in the y-coordinates at 0 of the connection which is equal to the trivial connection $\mathbf{R}^n \times \{0\}$ in the x-coordinates are given by

$$\Gamma(0)^i_{jk} = -\left.\frac{\partial^2 \phi^i(x)}{\partial x^j\,\partial x^k}\right|_{x=0}.$$

Because every symmetric matrix is equal to the Hessian (= second order partial derivative matrix) of suitable smooth function, we obtain iv) $\Longrightarrow$ i). $\square$

Remark 7.1 It is trivial that for any smooth manifold M, $m \in M$ and linear subspace H_m of $\mathrm{T}_m\, M$, there exists a smooth submanifold S of M such that $\mathrm{T}_m\, S = H_m$. If M is a smooth fiber bundle over a smooth manifold X and H is a connection in M, then this implies that for any $x \in X$, $y \in M_x$ there is a smooth section s defined in an open neighborhood of x in M, such that $s(x) = y$ and $(\nabla_v s)(x) = 0$ for every $v \in \mathrm{T}_x\, X$. If we apply this to the frame bundle of a vector bundle with a linear connection, then it follows that one can always find a local trivialization (= local section of the frame bundle = moving frame) such that with respect to any local system of coordinates all the Christoffel symbols vanish at the chosen point $x \in X$.

If $\mathrm{T}X$, then a local trivialization (= moving frame) is usually not induced by a local system of coordinates. For instance, if $T(x) \neq 0$, then it follows from Lemma 7.1 that there exists no system of local coordinates in a neighborhood of x such that all the Christoffel symbols vanish at the point x, whereas there always is a local trivialization of $\mathrm{T}X$ such that with respect to any local system of coordinates all the Christoffel symbols vanish at x.

It is the extra freedom of allowing moving frames which are not constant in any system of local coordinates which makes the moving frames of Élie Cartan such a flexible tool in differential geometry. $\oslash$

A linear connection H in $\mathrm{T}X$ (= affine connection in X) is called *torsion-free* if, for every $x \in X$, any of the equivalent conditions i) — iv) in Lemma (7.1) holds.

Now assumme that β is a *pseudo-Riemannina structure on* X, i.e., β is a smooth mapping which assigns to every $x \in X$ a nondegenerate symmetric bilinear form on $\mathrm{T}_x\, X$. If, for each $x \in X$, β_x is positive definite, i.e. β_x is an inner product on $\mathrm{T}_x\, X$, then β is called a *Riemannian structure* on X.

Remark 7.2 If $\dim X = 4$ and the signature of β_x is $-1,\,-1,\,-1,\,1$, then β is a *Lorentz metric* on a space-time manifold X of relativity theory. The $v \in \mathrm{T}_x\, X$ such that $\beta_x(v,\,v) > 0$ and $\beta_x(v,\,v) < 0$ are called *time-like* and *space-like*, respectively, whereas the set of $v \in \mathrm{T}_x\, X$ such that $\beta_x(v,\,v) = 0$ form the *light cone.* Particles with positive mass describe time-like curves, whereas particles which travel with the speed of light, which have zero mass, describe curves which are tangent to the light cone. $\oslash$

In local coordinates β is given by

$$\beta_x(u,\, v) = \sum_{i,\, j=1}^{n} \beta_{ij}(x)\, u^i\, v^j, \tag{7.3}$$

in which the $\beta_{ij}(x)$, $1 \leq i,\, j \leq n$, form a nondegenerate symmetric $n \times n$-matrix, depending smoothly on x. It is a natural question to ask whether, for the given pseudo-Riemannian structure β, and for any given point x, on can find a system of local coordinates such that all first order partial derivatives *at the point* x of all the coefficients $\beta_{ij}(x)$ vanish, i.e.

$$\frac{\partial \beta_{ij}(x)}{\partial x_k} = 0, \quad 1 \leq i,\, j,\, k \leq n. \tag{7.4}$$

Suppose that H is a torsion-free connection in $\mathrm{T}X$ such that β is covariantly constant with respect to H. According to Lemma 7.1 there exists a system of local coordinates such that all the Christoffel symbols of H vanish at the given point x. The condition that β is covariantly constant at the point x then is equivalent to (7.4). It therefore follows from Theorem 7.2 below that for the any pseudo-Riemannian structure β, and for any given point x, on can find a system of local coordinates such that all first order partial derivatives at the point x of all the coefficients $\beta_{ij}(x)$ vanish.

Theorem 7.2 *Let β be a pseudo-Riemannian structure on X. Then there exists a unique torsion-free connection H in $\mathrm{T}X$ such that β is covariantly constant with respect to H.*

Proof We work in local coordinates around $x = 0$. If e_i, $1 \leq i \leq n$, denotes the standard basis in $\mathbf{R}^n$, then, for any $a \in \mathbf{R}^n$, the vector field v_a defined by

$$v_a^i(x) = a^i - \sum_{=1}^{n} \Gamma^i_{jk}\, x^j\, a^k$$

is horizontal at the point $x = 0$. Here $\Gamma^i_{jk} := \Gamma^i_{jk}(0)$ denote the Christoffel symbols of H at the point $x = 0$. It follows that β is covariantly constant at $x = 0$ with respect to H if and only if, for any $a,\, b \in \mathbf{R}^n$ and any $1 \leq k \leq n$,

$$\left.\frac{\partial \beta_x(v_a(x),\, v_b(x))}{\partial x_k}\right|_{x=0} = 0.$$

Writing out these equations for $a = e_i$, $b = e_j$, in which the e_i, $1 \leq i \leq n$, denote the standard basis of $\mathbf{R}^n$, we arrive at the equations

$$\beta_{ij,\, k} = \sum_{p=1}^{n} \beta_{pj}\, \Gamma^p_{ki} + \sum_{p=1}^{n} \beta_{ip}\, \Gamma^p_{kj}, \tag{7.5}$$

in which we have written $\beta_{ij} = \beta_{ij}(0)$ and $\beta_{ij,\, k} := \left.\frac{\partial \beta_{ij}(x)}{\partial x^k}\right|_{x=0}$. It follows from (7.5) that

$$\beta_{jk,\, i} = \sum_{p=1}^{n} \beta_{pk}\, \Gamma^p_{ij} + \sum_{p=1}^{n} \beta_{jp}\, \Gamma^p_{ik} \quad \text{and} \quad \beta_{ki,\, j} = \sum_{p=1}^{n} \beta_{pi}\, \Gamma^p_{jk} + \sum_{p=1}^{n} \beta_{kp}\, \Gamma^p_{ji}. \tag{7.6}$$

Using the symmetries $\beta_{ij} = \beta_{ji}$ and $\Gamma^i_{jk} = \Gamma^i_{kj}$, we obtain from (7.5) and (7.6) that

$$\beta_{ij,\,k} - \beta_{jk,\,i} + \beta_{ki,\,j} = 2 \sum_{p=1}^{n} \beta_{pj}\, \Gamma^p_{ki}. \tag{7.7}$$

If $\beta^{ij}(x)$ denotes the inverse matrix of the matrix $\beta_{ij}(x)$, and if we rename the indices in order to obtain an expression for Γ^i_{jk}, then this leads to

$$\Gamma^i_{jk}(x) = \frac{1}{2} \sum_{p=1}^{n} \beta^{ip}(x) \left[\frac{\partial \beta_{kp}(x)}{\partial x^j} - \frac{\partial \beta_{pj}(x)}{\partial x^k} + \frac{\partial \beta_{jk}(x)}{\partial x^p} \right], \quad 1 \leq i,\, j,\, k \leq n, \tag{7.8}$$

at $x = 0$. This proves the uniqueness in Theorem 7.2.

In order to prove that the system (7.5) of linear equations (for all $1 \leq i,\, j\, k \leq n$) for the Christoffel symbols has a solution, we observe that the right hand side, viewed as a linear function of the Christoffel symbols, defines a linear mapping L from $\mathrm{Sym}^2(\mathbf{R}^n)^* \otimes \mathbf{R}^n$ to $\mathrm{Sym}^2(\mathbf{R}^n)^* \otimes (\mathbf{R}^n)^*$. Because of the aformentioned uniqueness, the linear mapping L is injective, and therefore bijective, because the source space and the target space have the same dimension. □

The unique torsion-free connection H in $\mathrm{T}X$ such that β is covariantly constant with respect to H, of which the Christoffel symbols in local coordinates are given by (7.8), is called the *Levi-Civita connection* of the (pseudo-)Riemannian structure β. The curvature of this connection is called the *Riemannian curvature tensor of* $(X,\, \beta)$.

Remark 7.3 Riemann introduced the concept of what nowadays is called a Riemannian structure in his Habilitationsvorlesung in 1854 in Göttingen, cf. [29, XIII]. In it, he discussed the problem of finding coordinates in which the Riemannian stucture β is the standard one of $\mathbf{R}^n$, and indicates curvature as the obstruction to this, but without an formula for the curvature tensor.

In [29, XXII] Riemann proved that the first order derivatives of β can be made equal to zero at a given point using suitable local coordinates, and introduced the curvature tensor R as the obstruction for making the second order derivatives of β equal to zero at the given point. There he also states, without further proof, that if R vanishes identically, then the Riemannian structure is the standard one in a suitable coordinate system.

However, he had written this down in an article which had submitted in 1861 to the Academy in Paris as his answer to a prize question, which had been posed in 1858 and withdrawn in 1868, without an award. Riemann did not get the prize because the proofs were not sufficiently complete. Due to health problems, Riemann never worked out the details in the way he had planned.

Knowing Riemann's Habiliationsvorlesung but not Riemann's prize memoir, Christoffel [10] and Lipschitz [27] independently proved that the first order derivatives of the metric tensor can be made equal to zero at a given point by choosing suitable coordinates, and that the curvature tensor R is the obstruction to doing the same with the second order derivatives. Christoffel [10] also proved that, for any k, the covariant derivatives up to the order k of R at x are the obstructions to making all derivatives up to the order $k + 2$ equal to zero at x.

Levi-Civita [25] introduced the parallel transport inthe tangent bundle $\mathrm{T}X$ with respect to the infinesimal connection with the Christoffel symbols (7.8), and which since then is called the Levi-Civita connection in the (pseudo-)Riemannian manifold $(X,\, \beta)$. ⊘

Remark 7.4 The covariant derivative can be used in order to define all kinds of linear partial differential operators acting on sections of vector bundles, in terms of given differential geometric structures on the bundle and the base manifold.

For instance, let $(X,\, \beta)$ be an n-diemsnional pseudo-Riemannian manifold and M is a vector bundle over X with connection H. Let U be an opene subset of X and f_i, $1 \leq i \leq n$, a local frame of vector fields in U. Define the dual frame of one-forms ϕ^j, $1 \leq j \leq n$, by $\phi^j(f_i) = \delta_i^j$. Then

$$\Delta := \sum_{i,\, j=1}^{n} \beta^{-1}(\phi_i,\, \phi_j) \left(\nabla_{f_i} \circ \nabla_{f_j} - \nabla_{\nabla^{\beta}_{f_i} f_j} \right) \tag{7.9}$$

defines a second order linear partial differential operator action on sections of M (over U). Here $\nabla_v : s \mapsto \nabla_v s$ denotes the covariant derivative with in the direction of the vector field v, regarded as a first order linear partial differential operator acting on sections of M. The vector field $v = \nabla^{\beta}_{f_i} f_j$ denotes the covariant derivative of the vector field f_j in the direction of the vector field f_i, with respect to the Levi-Civita connection in $\mathrm{T}X$ defined by the pseudo-Riemannian structure β.

The point of substracting the covariant derivative in the direction of v is that it makes the right hand side in (7.9) *independent of the choice of the local frame* f_i, $1 \leq i \leq n$, and therefore (7.9) leads to a globally defined second order linear partial differential operator $\Delta : \Gamma(M) \to \Gamma(M)$, acting on sections of the vector bundle M. If β is a Riemannian structure on X, then the operator Δ is a *Laplace operator*, an elliptic second order linear partial differential operator, whereas Δ is a *wave operator* if β has a Lorentz signature.

A first order linear partial differential operator D acting on $\Gamma(M)$ is obtained if we write

$$D = \sum_{i=1}^{n} c(\phi_i)\, \nabla_{f_i}, \tag{7.10}$$

in which, for each $x \in X$, c_x denotes a linear mapping from $(\mathrm{T}_x\, X)^*$ to the space $\mathrm{Lin}(M_x,\, M_x)$ of linear mappings from M_x to itself. In other words, c_x makes M_x into a $(\mathrm{T}_x\, X)^*$-module. Here it is even simpler to verify that the right hand side in (7.10) does not depend on the choice of the local frame. This is particularly interesting if, for any $\xi \in (\mathrm{T}_x\, X)^*$, we have that

$$c_x(\xi)^2 = \beta_x^{-1}(\xi,\, \xi)\, I,$$

in which I denotes the identity in M_x. In this case D^2 has the same second order part as Δ, and D is called a *Dirac operator* acting on $\Gamma(M)$. An example is the *spin-c Dirac operator*, which has applications in symplectic geomtery, cf. [16]. The spin-c Dirac operator occurs also in the Seiberg--Witten theory, which led to a revolution in the theory of smooth four-dimensional manifolds, cf. Morgan [28]. $\oslash$

8 Connections in a Principal Fiber Bundle

Now let G be any Lie group with Lie algebra $\mathfrak{g}$, and let $\pi : P \to X$ a principal G-bundle over X. (See the course of Joop Kolk for the definition.) The fact that on each fiber P_x of P the proper action of G is free and transitive, implies that it is infinitesimally free and transitive, which means that for every $p \in P_x$ the linear mapping

$$i(p) : X \mapsto X_P(p) : \mathfrak{g} \to \mathrm{T}_p\, P_x \tag{8.1}$$

has zero kernel and is surjective, respectively, which together imply that this mapping is a linear isomorphism. The mappings $i(p) : \mathfrak{g} \xrightarrow{\sim} \mathrm{T}_p\, P_x = \ker \mathrm{T}_p\, \pi$ are used to identify all the tangent spaces of the fibers of P with one and the same Lie algebra $\mathfrak{g}$ of G.

Now assume that H is a connection in P, i.e. for each $p \in P$

$$\mathrm{T}_p\, P = H_p \oplus \ker \mathrm{T}_p\, \pi \xleftarrow{\sim} H_p \oplus \mathfrak{g}.$$

Élie Cartan introduced the *connection form* of H as the $\mathfrak{g}$-valued one-form θ such that, for every $p \in P$, θ_p is equal to zero on H_p and equal to the identity on the tangent space of the fiber, identified as above with $\mathfrak{g}$. In formula,

$$\ker \theta_p = H_p \quad \text{and} \tag{8.2}$$
$$\theta_p(X_P(p)) = X, \quad X \in \mathfrak{g}. \tag{8.3}$$

In view of (8.2), the connection H is G-invariant if and only if

$$\ker \theta_{g_P(p)} = \mathrm{T}_p\, g_P\, (\ker \theta_p)\,, \quad g \in G,\ p \in P.$$

One might think that this condition is equivalen tot the condition that θ is G-invariant in the sense that $g_P^*\, \theta = \theta$ for every $g \in G$, but the situation is a bit subtler (and nicer) than that. It follows from

$$g_P \circ \left(\mathrm{e}^{t\, X}\right)_P \circ g_P^{-1} = \left(g\ \mathrm{e}^{t\, X}\ g^{-1}\right)_P = \left(\mathrm{e}^{t\ \mathrm{Ad}\, g(X)}\right)_P$$

that

$$g_P \circ \left(\mathrm{e}^{t\, X}\right)_P = \left(\mathrm{e}^{t\, (\mathrm{Ad}\, g)(X)}\right)_P \circ g_P.$$

Evaluating this at $p \in P$ and differentiating the left and right hand side with respect to t at $t = 0$, we obtain that

$$\mathrm{T}_p\, g_P\, X_P(p) = (\mathrm{Ad}\, g)(X)_P\, (g_P(p)), \quad g \in G, \quad p \in P. \tag{8.4}$$

In view of (8.3) we therefore arrive at the conclusion that H is G-invariant if and only if the connection form θ is G-*equivariant* in the sense that

$$((g_P)^*\theta)_p = (\mathrm{Ad}\, g)(\theta_p), \quad g \in G, \quad p \in P. \tag{8.5}$$

Conversely, any smooth $\mathfrak{g}$-valued one-form θ on P which satisfies (8.3) and (8.5) is called *a connection form* on the principal G-bundle P.

Lemma 8.1 *Every principal G-bundle over a smooth manifold X admits a connection form.*

Proof This is obvious if the G-bundle is trivial. X. For any locally finite covering of X with open subsets U_α such that $\pi^{-1}(U_\alpha$ is a trivial G-bundle over U_α, we have a connection form θ_α on $\pi^{-1}(U_\alpha)$. Let χ_α be a smooth partition of unity on X which is subordinate to the covering U_α, i.e. for every α we have that $\chi_\alpha \in \mathcal{F}(X)$ has a compact support which is contained in U_α, and $\sum_\alpha \chi_\alpha = 1$ on X. Then $\theta := \sum_\alpha \chi_\alpha \, \theta_\alpha$ defines a connection from on P. $\square$

If θ is a connection form on P, then (8.2) defines an invariant smooth connection in P, of which θ is the connection form. It therefore follows from Lemma 8.1 that every principal fiber bundle admits an invariant smooth connection.

If in (8.5) we substitute $g = \mathrm{e}^{t\,X}$, $t \in \mathbf{R}$, $X \in \mathfrak{g}$ and we differentiate the left and right hand side with respect to t at $t = 0$, then we obtain the *infinitesimal equivariance* $\mathcal{L}_{X_P}\theta = (\mathrm{ad}\, X)\,(\theta)$. We have

$$\mathcal{L}_{X_P}\theta = \mathrm{d}(\mathrm{i}_{X_P}\,\theta) + \mathrm{i}_{X_P}(\mathrm{d}\theta) = \mathrm{d}\, X + \mathrm{i}_{X_P}(\mathrm{d}\theta) = \mathrm{i}_{X_P}(\mathrm{d}\theta),$$

where in the first, second and third identity we used the homotopy identity for the Lie derivative, cf. Section 14, the formula (8.3, and the fact that X is constant, repectively. Therefore the infinitesimal equivariance of θ takes the form

$$\mathrm{i}_{X_P}(\mathrm{d}\theta) = (\mathrm{ad}\, X)\,(\theta), \quad X \in \mathfrak{g}. \tag{8.6}$$

If we apply θ_p to the left and right hand side of (2.5), we obtain in view of (8.2) and (8.3) that

$$\theta_p R_x(u(x),\, v(x))(p) = \, -\theta_p([u_{\mathrm{hor}}(p),\, v_{\mathrm{hor}}(p)]) = (\mathrm{d}\theta)_p(u_{\mathrm{hor}}(p),\, v_{\mathrm{hor}}(p)),$$

where in the second identity we have used (1.2). In other words, if we use θ_p in order to identify the vertical space $V_p := \mathrm{T}_p\, P_x$ with $\mathfrak{g}$, then the curvature applied to $u,\, v \in \mathcal{X}(X)$ is given by the $\mathfrak{g}$-valued function $\mathrm{d}\theta(u_{\mathrm{hor}},\, v_{\mathrm{hor}})$ on P.

A q-form β on $\mathrm{T}_p\, P$ is called *horizontal* if $\mathrm{i}_v\, \beta = 0$ for every $v \in V_p$. If α is any q-form on H_p, then there is a unique horizontal p-form β on $\mathrm{T}_p\, P$ which agrees with α on $(H_p)^q$. This form β is called the *horizontal part* α_{hor} *of* α. Indeed, each $v \in \mathrm{T}_p\, P$ has a unique decomposition $v = v_H + v_V$ with $v_H \in H_p$ and $v_V \in V_p$, which each depend linearly on v, and $\alpha_{\mathrm{hor}}(v_1,\, \ldots v_q) := \alpha(v_{1,\,H},\, \ldots,\, v_{q,\,H})$ defines the unique horizontal extension β of $\alpha|_{H_p \times H_p}$ to $(\mathrm{T}_p\, P)^q$.

Consider the $\mathfrak{g}$-valued two-form Ω on P which is defined by

$$\Omega(u,\, v) = (\mathrm{d}\theta)(u,\, v) - [\theta(u),\, \theta(v)], \quad u,\, v \in \mathcal{X}(P). \tag{8.7}$$

Then, for each $X \in \mathfrak{g}$,

$$(\mathrm{i}_{X_P}\, \Omega)(v) = (\mathrm{i}_{X_P}(\mathrm{d}\theta))(v) - [\theta(X_P),\, \theta(v)] = (\mathrm{ad}\, X)\,(\theta(v)) - [X,\, \theta(v)] = 0,$$

where in the second equation we have used (8.6), and in the last equation that $(\mathrm{ad}\, X)(Y) := [X,\, Y]$ for any $X,\, Y \in \mathfrak{g}$. Because Ω agrees with $\mathrm{d}\theta$ on $H_p \times H_p$ in view of (8.2), we conclude that $\Omega = (\mathrm{d}\theta)_{\mathrm{hor}}$, the horizontal part of the exterior derivative of θ.

It follows also from (8.7), (8.5) and the fact that the exterior derivative commutes with pull-backs, that the $\mathfrak{g}$-valued two-form Ω on P is G-equivariant in the sense that

$$(g_P)^*\Omega = (\mathrm{Ad}\, g)\,\Omega, \quad g \in G. \tag{8.8}$$

The G-equivariant horizontal $\mathfrak{g}$-valued two-form Ω is called the *curvature form* of the connection H in P (or of the connection form θ on P), and we have the identity

$$\theta_p R_x(u(x),\, v(x))(p) = \Omega_p(u_{\mathrm{hor}}(p),\, v_{\mathrm{hor}}(p)), \quad u,\, v \in \mathcal{X}(X). \tag{8.9}$$

Remark 8.1 The connection form and curvature form have been introduced by Élie Cartan [7, pp. 383–390]. $\oslash$

Remark 8.2 According to the "Wu and Yang dictionary" between field theory in physics and differential geometry, cf. [31], a "gauge type" is a principal G-bundle P, a "gauge potential" is an invariant connection in P, given by an equivariant connection form $\theta \in (\omega^1(P) \otimes \mathfrak{g})^G$, and the "field strength" is the curvature form Ω of the connection. The "gauge group" is the group $\mathcal{G}$ if diffeomorphisms $\Phi : P \to P$ such that $\pi \circ \Phi = \pi$ and $\Phi \circ g_{_P} = g_{_P} \circ \Phi$ for every $g \in G$. The Yang-Mills equations are invariant under $\mathcal{G}$, and the relevant fields are the $\mathcal{G}$-orbits of solutions of the Yang-Mills equations.

For base manifolds X which are compact and four-dimensional, Donaldson used the so-called anti-self-dual connections modulo $\mathcal{G}$ in his study of the geometry of X, cf. [14]. $\oslash$

9 Associated Vector Bundles

Now suppose that Y is a finite-dimensional vector space and ρ is a representation of G in Y, i.e. ρ is a homomorphism of Lie groups from G to $\mathrm{GL}(Y)$. In $P \times Y$ we have that action $(p,\, y) \to (g_P(p),\, (\rho(g))(y))$ of $g \in G$. This action is proper and free, because the action of G on P is proper and free. We denote by $M := P \times GY$ the space of all the orbits

$$G\,(p,\, y) = \{(g_P(p),\, (\rho(g))(y)) \mid g \in G\} \tag{9.1}$$

of this action of G in $P \times Y$.

We have $G\,(p,\, y) = G\,(p',\, y')$ if and only if there exists a $g \in G$ such that $p' = g_P(p)$ and $y' = (\rho(g))(y)$, which implies that $\pi(p') = \pi(p)$. therefore we have a well-defined mapping $\pi_M : M \to X$ such that $\pi_M(G\,(p,\, y)) = \pi(p)$, which is surjective, and in fact exhibits M as a fiber bundle over X. In the sequel we will write π instead of π_M.

If $x \in X$, then the fact that G acts freely and transitively on P_x implies that, for any chosen $p \in P_x$, the mapping $g \mapsto g_P(p)$ is a diffeomorphism from G onto P_x. If, for given $y \in Y$, we compose the inverse of this mapping with the mapping $g \mapsto (\rho(g))(y)$ from G to Y, then we obtain a smooth mapping s from P_x to Y of which the orbit $G\,(p,\, y)$ is the graph. This mapping is G-*equivariant* in the sense that

$$s(g_P(p)) = (\rho(g))(s(p)), \quad g \in G, \quad p \in P_x, \tag{9.2}$$

i.e. s intertwines the action of G on P with the representation ρ of G in Y.

Identifying a mapping with its graph, which actually is the set-theoretic definition of a mapping, we identify the fiber M_x over $x \in X$ of the bundle M with the space of all mappings $a : P_x \to Y$ which satisfy (9.2). Because Y is a vector space, the space Y^{P_x} of all mappings $s : P_x \to Y$ is a vector space with the pointwise addition and scalar multiplication. Because, for each $g \in G$, $\rho(g)$ is a linear mapping from Y to Y, the space M_x of Y-valued function s on P which satisfy (9.2) is a linear subspace of Y^{P_x}, and for each $p \in P_x$ the mapping $\mathrm{ev}_p : s \mapsto s(p)$ (= "evaluation at the point p") defines a linear isomorphism from M_x onto Y. This exhibits M as a vector bundle over X with fibers isomorphic to Y. Each smooth local section $p : U \to P$ of the bundle P, in which U is an open subset of X, induces a local trivialization $\tau : \pi_M^{-1}(U) \to U \times Y$ which is defined by $\tau(G\,(p(x), y)) = (x,\, y)$ when $x \in U$ and $y \in Y$. These local trivializations exhibits M as a smooth vector bundle over X.

Because a section of M is a mapping which assigns to each $x \in X$ an element $s(x)$ of M_x, the sections of M are identified with the mappings $s : P \to Y$ which satisfy the G-equivariance condition

$$s(g_P(p)) = (\rho(g))(s(p)), \quad g \in G, \quad p \in P. \tag{9.3}$$

In this way the vector space $\Gamma(M)$ of all sections of M is identified with the vector space $(\mathcal{F}(P) \otimes Y)^G$ of all G-equivariant Y-valued functions on P.

If H is a G-invariant connection in P and $v \in \mathcal{X}(X)$, then v_{hor} is a G-invariant vector field on P. If $s \in (\mathcal{F}(P) \otimes Y)^G$ we therefore have that

$$(g_P)^*(v_{\mathrm{hor}}\, s) = (g_P)^*(v_{\mathrm{hor}})\, (g_P)^*(s) = v_{\mathrm{hor}}\, (\rho(g)\, s) = \rho(g)\, (v_{\mathrm{hor}}\, s),$$

where in the last identity we have used that the linear transformation $\rho(g)$ does not depend on the point in P. In other words, the differentiation of Y-valued functions on P in the direction of the vector field v_{hor} maps $\Gamma(M) = (\mathcal{F}(P) \otimes Y)^G$ to itself, and this defines a covariant diffentiation

$$\nabla_v s := v_{\mathrm{hor}}\, s, \quad v \in \mathcal{X}(X), \quad s \in \Gamma(M) \tag{9.4}$$

in the space of sections of M, since the Leibniz rule (3.6) is obviously satisfied by the right hand side in (9.4). In this way the G-invariant connection $H = H^P$ in P induces a linear connection $H = H^M$ in M, for which it can be verified that

$$H^M_{G\,(p,y)} = \mathrm{T}_{(p,\,y)}\,\psi\left(H^P_p \times \{0\}\right), \quad (p,\,y) \in P \times Y, \tag{9.5}$$

if $\psi : P \times Y \to M = P \times_G Y$ is the projection defined by $\psi(p,\,y) = G\,(p,\,y)$.

Remark 9.1 The formula (9.4) is the same as the formula (5.6) when $P = \mathrm{F}\,M$. However, in Section 5 we started out with a linear connection in M, which we used to define a G-invariant connection in $P = \mathrm{F}\,M$ for which the identity (5.6) holds. Here the procedure has been the other way around: starting from a G-invariant connection on a general principal G-bundle P, we *define* the linear connection on the associated vector bundle $M = G \times_G Y$ by means of the formula (9.4). $\oslash$

In order to compute the curvature of the connection H^M in M, we write, for any $s \in \Gamma(M)$ and $u,\,v \in \mathcal{X}(X)$,

$$R^M(u,\,v)\,s = [\nabla_u,\,\nabla_v]\,s - \nabla_{[u,\,v]}\,s = (\mathrm{d}s)\left([u_{\mathrm{hor}},\,v_{\mathrm{hor}}] - [u,\,v]_{\mathrm{hor}}\right),$$

where in the first and second identity we have used (3.8) and (9.4), respectively.

Now the vector field $w = [u_{\mathrm{hor}},\,v_{\mathrm{hor}}] - [u,\,v]_{\mathrm{hor}}$ is vertical, which implies that we can write, for any $p \in P$,

$$w(p) = X_P(p), \quad X = \theta_p(w(p)) = \theta_p\left([u_{\mathrm{hor}},\,v_{\mathrm{hor}}]\right) = -\,\mathrm{d}\theta\left(u_{\mathrm{hor}}(p),\,v_{\mathrm{hor}}(p)\right),$$

cf. (8.3) and (1.2). On the other hand the G-equivariance (9.3) implies that

$$\mathrm{i}_{X_P}\,\mathrm{d}s = \mathcal{L}_{X_P}\,s = \rho'(X)\,s,$$

if

$$\rho'(X) = \frac{\mathrm{d}}{\mathrm{d}t}\rho\left(\mathrm{e}^{t\,X}\right)\Big|_{t=0}$$

denotes the infinitesimal representation of $X \in \mathfrak{g}$ in Y. (The mapping ρ' is a homomorphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra $\mathrm{Lin}(Y,\,Y)$ of linear mappings from Y to Y.) If we combine our formulas with (8.7), we arrive at the formula

$$R^M(u,\,v)\,s = -\rho'(\Omega(u_{\mathrm{hor}},\,v_{\mathrm{hor}}))\,s, \tag{9.6}$$

valid for any $u,\,v \in \mathcal{X}(X)$ and $s \in \Gamma(M) = (\mathcal{F}(P) \otimes Y)^G$, which expresses the curvature of H^M in terms of the curvature form of the G-invariant connection in P.

If we have a $\rho(G)$-invariant inner product on Y, then (6.2) defines a covariantly constant inner product β on the fibers of the accociated vector bundle $M = P \times_G Y$. Similarly, if Y is a complex vector space with Hermitian form h and $\rho(G) \subset \mathrm{U}(Y,\,h)$, then $M = P \times_G Y$ is a complex vector bundle over X and (6.4) defines a covariantly constant Hermitian structure on the fibers of M.

Remark 9.2 There are quite a number of sign issues in differential geometry. To begin with, if G is a Lie group acting smoothly on a manifold M then the infinitesimal action of an element X of the Lie algebra $\mathfrak{g}$ of G is a smooth vector field X_M on M. For any $X, Y \in \mathfrak{g}$ we have

$$[X, Y]_M = -[X_M, Y_M], \tag{9.7}$$

which means that the mapping $X \mapsto X_M : \mathfrak{g} \to \mathcal{X}(M)$ is not a homomomorphism of Lie algebras, but an *anti-homomorphism.* This is caused by the fact that $X_M f$ is equal to the derivative with respect to t at $t = 0$ of the pullback of the function f by means of the time t flow of the vector field X_M, and the mapping which assigns to $g \in G$ the pullback operator $(g_M)^*$ is not a homomorphism from G to the group of linear transformations in $\mathcal{F}(M)$, but an anti-homomorphism.

In the literature one often denotes the action of $g \in G = \mathrm{GL}(l, \mathbf{R})$ on the frame bundle $P = \mathrm{F}M$ not by g_P as in (5.1), but by $\widetilde{g_P} : f \mapsto f\, g$. This lead to an anti-homomorphism $g \mapsto \widetilde{g_P}$ from G to the group of diffeomorphisms of P. Such an anti-homorphism is also called a *right action* of G on P, whereas a homomorphism from G to $\mathrm{Diffeo}(P)$ is called a *left action* of G on P.

If one has a right action $g \mapsto \widetilde{g_P}$ on P, then the connection form θ in (8.3) is replaced by the form $\widetilde{\theta}$ which has kernel equal to H and is equal to X when applied to $\widetilde{X_P}$. Because of the anti-homomorphism property, the equivariance (8.5) is replaced by

$$(\widetilde{g_P})^* \widetilde{\theta} = (\mathrm{Ad}\, g)^{-1}\, \widetilde{\theta}, \quad g \in G, \tag{9.8}$$

hence

$$\mathrm{i}_{\widetilde{X_P}}\, \mathrm{d}\widetilde{\theta} = -(\mathrm{ad\ X})\, \widetilde{\theta}, \quad X \in \mathfrak{g}. \tag{9.9}$$

It follows that the curvature form, the horizontal extension of $\mathrm{d}\widetilde{\theta}$ on $H \times H$, is given by

$$\widetilde{\Omega}(u, v) = (\mathrm{d}\widetilde{\theta})(u, v) + [\widetilde{\theta}(u), \widetilde{\theta}(v)], \tag{9.10}$$

instead of (8.7).

In a similar fahsion, the equivariance (9.3) for sections s of M is replaced by

$$s(\widetilde{g_P}(p)) = \rho(g)^{-1}(s(p)), \quad g \in G, \quad p \in P, \tag{9.11}$$

which makes that ρ' in (9.6) has to be replaced by $-\rho'$, which leads to the formula

$$R^M(u, v) = \rho'(\widetilde{\Omega}(u_{\mathrm{hor}}, v_{\mathrm{hor}})) \tag{9.12}$$

for the curvature in M in terms of the curvature form $\widetilde{\Omega}$ on P.

Therefore the use of right actions leads to the disappearance of the minus sign in (9.6). On the other hand it leads to less natural formulas for the equivariance of the connection form, the curvature form, and sections of the associated vector bundle. $\oslash$

10 Principal Circle Bundles

The horizontality of the curvature form Ω implies that for every $x \in X$ and $p \in P_x$, there is a unique two-form ω_p on $\mathrm{T}_x X$, such that $\Omega_p = (\mathrm{T}_p \pi)^* \omega_p$. However, this only will define a two-form ω on X if $\omega_p = \omega_x$ does not depend on the choice of $p \in P_x$. This will be the case if and only if, for every $g \in G$, $g_P^* \Omega = \Omega$, which in view of the equivariance property (8.8) of Ω is equivalent to the condition that $(\mathrm{Ad}\, g)\, \Omega = \Omega$ for every $g \in G$.

It follows that if the adjoint representation of G on $\mathfrak{g}$ is trivial, which happens in particular when G is commutative, then the is a unique $\omega \in \Omega^2(X) \otimes \mathfrak{g}$ such that $\Omega = \pi^* \omega$. Note that in this case the formula (8.7) also simplifies to $\Omega = \mathrm{d}\theta$. It follows that $\pi^*(\mathrm{d}\omega) = \mathrm{d}(\pi^*\omega) = \mathrm{d}\Omega = \mathrm{d}(\mathrm{d}\theta) = 0$, which implies that $\mathrm{d}\omega = 0$, i.e. the two-form ω on X is closed, and therefore it defines a de Rham cohomology class

$$[\omega] \in \mathrm{H}^2_{\text{de Rham}}(X) \otimes \mathfrak{g}.$$

Furthermore, if θ' is another invariant connection form, then $\theta' = \theta + \pi^* \lambda$ for some $\lambda \in \Omega^(X) \otimes \mathfrak{g}$. Therefore $\Omega' = \mathrm{d}\theta' = \mathrm{d}\theta + \mathrm{d}(\pi^*\lambda) = \pi^*\omega + \pi^*(\mathrm{d}\lambda) = \pi^*(\omega + \mathrm{d}\lambda)$, which shows that the de Rham cohomology class of ω does not depend on the choice of the invariant connection.

The circle group

$$G = \mathrm{U}(1) := \{z \in \mathbf{C} \mid \ |z| = 1\},$$

with the multiplication of complex numbers as the group structure, is commutative and has Lie algebra equal to $\mathfrak{u}(1) = \mathrm{i}\,\mathbf{R}$, the purely imaginary axis in $\mathbf{C}$.

Theorem 10.1 *Let P be a principal* U(1)*-bundle over X and let Ω be the curvature of an invariant connection in P. Let $\sigma \in \Omega^2(X)$ be the closed two-form on X such that*

$$-\frac{1}{2\pi\,\mathrm{i}}\,\Omega = \pi^*\sigma. \tag{10.1}$$

Then the cohomology class $[\sigma]$ is integral, in the sense that it belongs to the image of the natural homomorphism

$$i : \mathrm{H}^2_{\text{Čech}}(X,\, \mathbf{Z}) \to \mathrm{H}^2_{\text{Čech}}(X,\, \mathbf{R}) \xrightarrow{\sim} \mathrm{H}^2_{\text{de Rham}}(X)$$

from the Čech cohomology group $\mathrm{H}^2_{\text{Čech}}(X,\, \mathbf{Z})$ to the de Rham cohomology group $\mathrm{H}^2_{\text{de Rham}}(X)$.

Conversely, for every closed two-form $\sigma \in \Omega^2(X)$ such that $[\sigma] = i(c)$, $c \in \mathrm{H}^2_{\text{Čech}}(X,\, \mathbf{Z})$, there is a U(1)*-bundle over X with an invariant connection and corresponding curvature form Ω, such that* (10.1) *holds.*

Proof For the Čech cohomology which we use here, we refer to [21, §2–4]. In the proof we will spell out what the homomorphism i from $\mathrm{H}^2_{\text{Čech}}(X,\, \mathbf{Z})$ to $\mathrm{H}^2_{\text{de Rham}}(X)$ means.

We start with the closed two-form σ on X such that $[\sigma] = i(c)$, $c \in \mathrm{H}^2_{\text{Čech}}(X,\, \mathbf{Z})$. Let U_α be an open covering of X such that every non-empty intersection of finitely many U_α's is contractible. (This can be arranged by taking the U_α's equal to sufficiently small balls with respect to a Riemannian metric in X.)

Using the Poincaré lemma in U_α, we can write

$$\sigma = \mathrm{d}\tau_\alpha \quad \text{on} \quad U_\alpha, \tag{10.2}$$

for some $\tau_\alpha \in \Omega^1(U_\alpha)$.

If $U\alpha \cap U_\beta \neq \emptyset$, then we have $\mathrm{d}(\tau_\alpha - \tau_\beta) = \mathrm{d}\tau_\alpha - \mathrm{d}\tau_\beta = \sigma - \sigma = 0$ on $U_\alpha \cap U_\beta$, and applying the Poincaré lemma on $U_\alpha \cap U_\beta$, we find $f_{\alpha\beta} \in \mathcal{F}(U_\alpha \cap U_\beta)$, such that

$$\tau_\alpha - \tau_\beta = \mathrm{d} f_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{10.3}$$

By replacing $f_{\alpha\,\beta}$ by $(f_{\alpha\beta} - f_{\beta\alpha})/2$, we still have(10.3), but in addtion we have the antisymmetry $f_{\beta\alpha} = -f_{\alpha\beta}$.

If $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, we consider

$$g_{\alpha\beta\gamma} := f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma. \tag{10.4}$$

It follows from (10.3) that $\mathrm{d}g_{\alpha\beta\gamma} = (\tau_\alpha - \tau_\beta) + (\tau_\beta - \tau_\gamma) + (\tau_\gamma - \tau_\alpha) = 0$, which means that the $g_{\alpha\beta\gamma} \in \mathbf{R}$ are real constants.

The $g_{\alpha\beta\gamma}$ therefore define a two-cochain with values in $\mathbf{R}$, and the condition that $[\sigma] = i(c)$ for some $c \in \mathrm{H}^2_{\text{Čech}}(X,\, \mathbf{Z})$ now means that there exists a two-cochain $c_{\alpha\beta\gamma} \in \mathbf{Z}$ with values in $\mathbf{Z}$ and a one-cochain $d_{\alpha\beta} \in \mathbf{R}$ with values in $\mathbf{R}$, such that

$$g_{\alpha\beta\gamma} := c_{\alpha\beta\gamma} + d_{\alpha\beta} + d_{\beta\gamma} + d_{\gamma\alpha} \quad \text{whenever} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset. \tag{10.5}$$

It follows from (10.6) and (10.5) that, if we replace $f_{\alpha\beta}$ by $f_{\alpha\beta} - d_{\alpha\beta}$, then still (10.3) holds, whereas (10.4) is replaced by

$$c_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} \in \mathbf{Z}. \tag{10.6}$$

Now define

$$\phi_{\alpha\beta} := \mathrm{e}^{2\pi\,\mathrm{i}\,f_{\alpha\beta}} \in \mathrm{U}(1) \quad \text{on} \quad U_\alpha \cap U_\beta \tag{10.7}$$

when $U_\alpha \cap U_\beta \neq \emptyset$. It then follows from $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} \in \mathbf{Z}$ that

$$\phi_{\alpha\beta}\, \phi_{\beta\gamma}\, \phi_{\gamma\alpha} = 1 \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset. \tag{10.8}$$

This implies that the trivial U(1)-bundles $U_\alpha \times \mathrm{U}(1)$ can be glued together to a smooth U(1)-bundle P over X, by identifying $(x,\, z_\alpha) \in U_\alpha \times \mathrm{U}(1)$ with $(x,\, z_\beta) \in U_\beta \times \mathrm{U}(1)$ if $x \in U_\alpha \cap U_\beta$ and

$$z_\alpha = (\pi^* \phi_{\alpha\beta})\, z_\beta \quad \text{on} \quad \pi^{-1}(U_\alpha \cap U_\beta). \tag{10.9}$$

Here z_α is viewed as the smooth mapping from the open subset $\pi^{-1}(U_\alpha)$ of P to U(1), such that $p \mapsto (\pi(p),\, z_\alpha(p))$ is the corresponding local trivialization over U_α of the U(1)-bundle P.

Let θ be a U(1)-invariant connection form on P. On $\pi^{-1}(U_\alpha)$ we have that $g_P^* z_\alpha = g\, z_\alpha$ for every $g \in \mathrm{U}(1)$, hence $\mathrm{i}_{X_P}\, \mathrm{d}z_\alpha = X\, z_\alpha$ for every $X \in \mathfrak{u}(1)$, which means that the logarithmic derivative $\mathrm{d}(\ln z_\alpha) = z_\alpha^{-1}\, \mathrm{d}z_\alpha$ of z_α is a connection form in $\pi^{-1}(U_\alpha)$. It follows from (8.3) that the one-form $\theta - z_\alpha^{-1}\, \mathrm{d}z_\alpha$ is horizontal, and because it also is U(1)-invariant, there is a unique $\theta_\alpha \in \Omega^1(U_\alpha)$ such that

$$\theta - z_\alpha^{-1}\, \mathrm{d}z_\alpha = \pi^* \theta_\alpha \quad \text{on} \quad \pi^{-1}(U_\alpha). \tag{10.10}$$

If we take the logarithmic derivative of both sides in (10.10, then we obtain

$$z_\alpha^{-1}\, \mathrm{d}z_\alpha = (\pi^* \phi_{\alpha\beta})^{-1}\, \mathrm{d}(\pi^* \phi_{\alpha\beta}) + z_\beta^{-1}\, \mathrm{d}z_\beta = \pi^* \left(\phi_{\alpha\beta}^{-1}\, \mathrm{d}\phi_{\alpha\beta}\right) + z_\beta^{-1}\, \mathrm{d}z_\beta,$$

where in the second identity we have used that d commutes with π^* and that π^* is an algebra homomorphism. If we insert this in (10.10), then we see that the $z_\alpha^{-1}\, \mathrm{d}z_\alpha + \pi^* \theta_\alpha$ piece togeher to a global connection form θ in P, if and only if $\pi^* \left(\phi_{\alpha\beta}^{-1} \, \mathrm{d}\phi_{\alpha\beta} \right) + \pi^* \theta_\alpha = \pi^* \theta_\beta$ on $\pi^{-1}(U_\alpha \cap U_\beta)$, which is equivalent to

$$\phi_{\alpha\beta}^{-1} \, \mathrm{d}\phi_{\alpha\beta} + \theta_\alpha = \theta_\beta \quad \text{on} \quad U_\alpha \cap U_\beta \tag{10.11}$$

when $U_\alpha \cap U_\beta \neq \emptyset$.

It follows from (10.7) and (10.3) that

$$\phi_{\alpha\beta}^{-1} \, \mathrm{d}\phi_{\alpha\beta} = 2\pi \, \mathrm{i} \, \mathrm{d}f_{\alpha\beta} = 2\pi \, \mathrm{i} \, (\tau_\alpha - \tau_\beta),$$

and therefore (10.11) is equivalent to $2\pi \, \mathrm{i} \, \tau_\alpha + \theta_\alpha = 2\pi \, \mathrm{i} \, \tau_\beta + \theta_\beta$ on $U_\alpha \cap U_\beta$, which means that there is a globally defined one-form $\lambda \in \Omega^1(X)$ on X such that

$$\lambda = 2\pi \, \mathrm{i} \, \tau_\alpha + \theta_\alpha \quad \text{on} \quad U_\alpha. \tag{10.12}$$

This can for instance be arranged with $\lambda = 0$ by taking $\theta_\alpha := -2\pi \, \mathrm{i} \, \tau_\alpha$.

If we take the exterior derivative of both sides of (10.12) and (10.10), then we obtain in view of (10.2) and the fact that the exterior derivive of the logarithmic derivative of z_α is equal to zero, that $\mathrm{d}\lambda = 2\pi \, \mathrm{i} \, \sigma + \mathrm{d}\theta_\alpha$ and $\Omega = \mathrm{d}\theta = \mathrm{d}(\pi^* \theta_\alpha)$, respectively. Combining these two equations, we arrive at the conclusion that

$$\Omega = \pi^*(-2\pi \, \mathrm{i} \, \sigma + \mathrm{d}\lambda). \tag{10.13}$$

This completes the proof of the second statement in Theorem 10.1. For the first statement, one starts with the retrivializations (10.7), for which (10.6) defines the Čech cohomology class $c \in \mathrm{H}^2(X, \mathbf{Z})$. The discussion following (10.9) then proves (10.1), whereas $[\sigma] = i(c)$, follows from (10.3) and (10.2). □

Let L be the complex line bundle which is associated to the principal $\mathrm{U}(1)$-bundle P by means of the standard representation of $\mathrm{U}(1)$ in $\mathbf{C}$, where $\rho(g)$ is equal to the multiplication by the complex number $g \in \mathrm{U}(1)$. In other words, $P = \mathrm{UF}\, L$, cf. (6.3). Then the cohomology class $c \in \mathrm{H}^2(X, \mathbf{Z})$ which is assigned to $P = \mathrm{UF}(L)$ by means of Theorem 10.1 is called the *Chern class* $\mathrm{c}(L)$ of L.

Note that the standard Hermitian structure on $\mathbf{C}$ induces a Hermitian structure on L which is covariantly constant with respect to the connection in L. Conversely every complex line bundle L over X with Hermitian structure h and connection such that h is covariantly constant arises in this way for a unique closed two-form $\sigma \in \Omega^2(X)$, which is called the *Chern form defined by the Hermitian connection in* L.

In view of (10.1) and (9.6) with $\rho(g) = g$, the Chern class corresponds to $\frac{1}{2\pi \, \mathrm{i}}$ times the curvature of the connection in L. Note that $\mathrm{c}(L_1 \otimes L_2) = \mathrm{c}(L_1) + \mathrm{c}(L_2)$.

A general complex representation ρ of $\mathrm{U}(1)$ in a complex vector space Y splits as a direct sum of complex one-dimensional representations. For each one-dimensional representation there is an integer m such that, for each $g \in \mathrm{U}(1)$, $\rho(g)$ is equal to multiplication by the complex number g^m. This means that the associated vector bundle $P \times_{\mathrm{U}(1)} Y$ is a direct sum of complex line bundles of the form $L^{\otimes m}$, with Chern class equal to $m \, \mathrm{c}(P)$.

Remark 10.1 If σ is a symplectic form on X, then the symplectic manifold (X, σ) can be viewed as the phase space of a classical mechanical system. If $[\sigma] = i(c)$ with $c \in \mathrm{H}^2(X, \mathbf{Z})$, then we have the complex line bundle L with Chern class equal to c. A suitable subspace of the space of sections of L may be proposed as the Hilbert space on which the linear operators of quantum mechanics act. This is the the program of *geometric quantization* of Kostant [24]. ⊘

11 Equivariant Cohomology

Equivariant cohomology is a structure which is attached to any smooth action of a Lie group G on a smooth manifold P. In the theory of Henri Cartan [8], it is a variation of de Rham cohomology, in which the algebra $\Omega^*(P)$ of smooth differential forms on P is replaced by the algebra

$$A := (\mathbf{C}[\mathfrak{g}] \otimes \Omega^*(M))^G \tag{11.1}$$

of polynomial mappings $\alpha : \mathfrak{g} \to \Omega^*(P)$ from the Lie algebra $\mathfrak{g}$ of G to $\Omega^*(P)$, which are equivariant in the sense that

$$\alpha((\mathrm{Ad}\, g)\, X) = (g_P^*)^{-1}(\alpha(X)), \quad g \in G, \quad X \in \mathfrak{g}. \tag{11.2}$$

(Because it is convenient to allow complex valued differential forms, all algebras will be over $\mathbf{C}$.)

The algebra A has a *double grading*: as a vector space it is equal to the direct sum of the $A^{k,\,l}$, in which $\alpha \in A^{k,\,l}$ if $\alpha \in A$ and α is a homogenous polynomial of degree k with values in the space $\Omega^l(P)$ of differential forms of degree l. $A^{k,\,0}$ is the space of homogeneous $\mathbf{C}$-valued polynomials of degree k on $\mathfrak{g}$, whereas $A^{0,\,l} = \Omega^l(P)^G$ is the space of G-invariant differential forms of degree l on P, constant as a polynomial on $\mathfrak{g}$. The *total degree* of $\alpha \in A^{k,\,l}$ is the number $m = 2k + l$, this prepares for the substution of $X \in \mathfrak{g}$ by components of the curvature form in Theorem 11.1 below. The space of equivariant forms of total degree m will be denoted by

$$A^m := \bigoplus_{k,\,l \mid 2k+l=m} A^{k,\,l}. \tag{11.3}$$

The *equivariant exterior differentiation* $\mathrm{d}_{\mathfrak{g}}$ in the algebra A is defined as the following combination of the exterior differentiation of differential forms and the inner product with the infinitesimal action of $\mathfrak{g}$ on P:

$$(\mathrm{d}_{\mathfrak{g}}\, \alpha)(X) = \mathrm{d}(\alpha(X)) - \mathrm{i}_{X_P}(\alpha(X)), \quad \alpha \in A, \quad X \in \mathfrak{g}. \tag{11.4}$$

We say that α is *equivariantly closed* if $\mathrm{d}_{\mathfrak{g}}\, \alpha = 0$, and *equivariantly exact* if there exist a $\beta \in A$ such that $\alpha = \mathrm{d}_{\mathfrak{g}}\, \beta$. If $\alpha \in A^{k,\,l}$, then the first and the second term in the right hand side of (11.4) belong to $A^{k,\,l+1}$ and $A^{k+1,\,l-1}$, respectively, and it follows that $\mathrm{d}_{\mathfrak{g}}$ maps A^m to A^{m+1}.

If in (11.2) we replace g by $\mathrm{e}^{-t\,X}$, with $t \in \mathbf{R}$ and $X \in \mathfrak{g}$, and differentiate with respect to t at $t = 0$, then we get in view of $(\mathrm{Ad}\, g)\, X = X$ that $0 = \mathcal{L}_{X_P}\alpha(X) = \left\{\mathrm{d} \circ \mathrm{i}_{X_P} + \mathrm{i}_{X_P} \circ \mathrm{d}\right\} \alpha(X)$, where in the second identity we used the homotopy identity for the Lie derivative. Using this, we obtain that $\mathrm{d}_{\mathfrak{g}} \circ \mathrm{d}_{\mathfrak{g}} = 0$, or $\mathrm{d}_{\mathfrak{g}}(A) \subset \ker \mathrm{d}_{\mathfrak{g}}$. Therefore we can define the *equivariant cohomology* of the G-action on P as

$$\mathrm{H}^*_G(P) := \ker \mathrm{d}_{\mathfrak{g}} \,/\, \mathrm{d}_{\mathfrak{g}}(A). \tag{11.5}$$

The equivariant cohomology is graded, i.e. $\mathrm{H}^*_G(P)$ is equal to the direct sum over all $m \in \mathbf{Z}_{\geq 0}$ of the equivariant cohomology groups $\mathrm{H}^m_G(P)$ of degree m, which are defined by

$$\mathrm{H}^m_G(P) := (\ker \mathrm{d}_{\mathfrak{g}} \cap A^m) /\, \mathrm{d}_{\mathfrak{g}}(A^{m-1}). \tag{11.6}$$

The equivariant form $\alpha \in A$ is called *basic* if $\alpha \in A \cap \Omega^*(P) = \Omega^*(P)^G$, i.e. α is a G-invariant differential form on P, constant as a function on $\mathfrak{g}$, and moreover is *horizontal* in the sense that $\mathrm{i}_{X_P}\, \alpha = 0$ for every $X \in \mathfrak{g}$. The basic forms form a subalgebra of A which will be denoted by $\Omega^*_{fopbas}(P)$, and $\mathrm{d}_{\mathfrak{g}} = \mathrm{d}$ on $\Omega^*_{fopbas}(P)$. We have the *basic cohomology group*

$$\mathrm{H}^*_{\mathrm{bas}}(P) := (\ker \mathrm{d} \cap \Omega^*_{\mathrm{bas}}(P)) /\, \mathrm{d}(\Omega^*_{\mathrm{bas}}(P)), \tag{11.7}$$

and the mapping $\alpha + \mathrm{d}(\Omega^*_{\mathrm{bas}}(P)) \mapsto \alpha + \mathrm{d}_{\mathfrak{g}}(A)$ leads to a homomorphism

$$i_{\mathrm{bas}} : \mathrm{H}^*_{\mathrm{bas}}(P) \to \mathrm{H}^*_G(P) \tag{11.8}$$

from the basic cohomology to the equivariant cohomology of P.

If $\pi : P \to X$ is a principal G-bundle over X then the pullback $\pi^* : \Omega^*(X) \to \Omega^*_{\mathrm{bas}}(P)$ by means of the projection π is an isomorphism which commutes with the exterior derivative, and we have the corresponding isomorphism

$$\pi^* : \mathrm{H}^*(X) \xrightarrow{\sim} \mathrm{H}^*_{\mathrm{bas}}(P) \tag{11.9}$$

between the (ordinary) de Rham cohomology of X and the basic cohomology of P. The theorem of Henri Cartan, Theorem 11.1 below, implies that if $\pi : P \to X$ is a principal G-bundle, then the mapping $i_{\mathrm{bas}} \circ \pi^* : \mathrm{H}^*(X) \to \mathrm{H}^*_G(P)$ is an isomorphism from the de Rham cohomology of X onto the equivariant cohomology of P. If X is compact, then the dimension of $\mathrm{H}^*(X)$, and it follows that the dimension of $\mathrm{H}^*_G(P)$ is finite.

In contrast, if there exist $p \in P$ and $X \in \mathfrak{g}$ such that $X \neq 0$ and $X_P(p) = 0$, then the equivariant cohomology $\mathrm{H}^*_G(P)$ of P is infinite-dimensional, even if both P and G are compact. This makes equivariant cohomology a very sensitive tool in the investigation of actions which are not proper and free. An expression of this property is the *localization formula* of Berline-Vergne [6] and Atiyah-Bott [4]. For a survey, see ([15]).

Let us return to the case that the G-action on P is proper and free, or slightly more generally, that *the G-action admits a connection form*, a $\mathfrak{g}$-valued smooth one-form θ on P which satisfies (8.3) and (8.4). (Such a generalization is useful in the category of orbifolds.)

Choose a basis in $\mathfrak{g}$ and denote the coordinates of $X \in \mathfrak{g}$ with respect to this basis by X_i. Then, for any $\alpha \in A$, we can write $\alpha(X)$ as a sum of monomials

$$\alpha(X) = \sum_{\mu} \left(\prod_i {X_i}^{\mu_i} \right) \alpha_\mu, \tag{11.10}$$

in which the μ are multi-indices, and the "monomial coefficients" $a_\mu \in \Omega^*(P)$ are uniquely defined differential forms on P. Then the differential form

$$\alpha(\Omega) = \sum_{\mu} \left(\prod_i {\Omega_i}^{\mu_i} \right) \alpha_\mu \in \Omega^*(P) \tag{11.11}$$

on P, obtained by "substituting X by Ω in $\alpha(X)$", is independent of the choice of the basis in $\mathfrak{g}$. Note that the order of the factors Ω_i does not matter, because two-forms commute with forms of any degreee.

Every $\alpha \in A$ can be written in a unique way as a sum of β and elements of the form

$$\alpha = \beta + \sum_{j=1}^{\dim \mathfrak{g}} \sum_{i_1 < \ldots < i_j} \theta_{i_1} \wedge \ldots \wedge \theta_{i_j} \wedge \beta_{i_1, \ldots, i_j}, \tag{11.12}$$

in which β and $\beta_{i_1, \ldots, i_j}$ are horizontal elements of A. $\alpha_{\mathrm{hor}} := \beta$ is called the *horizontal part* of α with respect tot the connection form θ. For ordinary differential forms on P this agrees with the horizontal part defined in Section 8.

Theorem 11.1 *If the action of G on P admits a connection from θ, then the homomorphism* (11.8)*, from the basic cohomlogy of P to the equivariant cohomology of P, is an isomorphism. if P is a principal G-bundle over X, then $i_{\mathrm{bas}} \circ \pi^* : \mathrm{H}^*(X) \to \mathrm{H}^*_G(P)$, where π^* is defined in* (11.9)*, is an isomorphism from the de Rham cohomology of X onto the equivariant cohomology of P.*

More specifically, if α is an equivariantly closed form, then $(\alpha_{\mathrm{hor}})(\Omega) = (\alpha(\Omega))_{\mathrm{hor}}$, and this form β is basic, closed, and equivariantly cohomologous to α. Here $\alpha(\Omega)$ is obtained from α by substituting the variable $X \in \mathfrak{g}$ in $\alpha(X)$ by the curvature form Ω, as in (11.11).

A workout of the proof of Henri Cartan [8] can be found in [16, pp. 229–234].

If we take α equal to an $\mathrm{Ad}\, G$-invariant polynomial f on $\mathfrak{g}$, which is an equivariantly closed form of degree zero as a differential form on P, then Theorem 11.1 implies the following theorem of André Weil [33]:

Corollary 11.2 *Let P is a principal G-bundle over X with connection form θ and corresponding curvature form Ω. If f is an $\mathrm{Ad}\, G$-invariant homogeneous polynomial f of degree j on $\mathfrak{g}$, then $f(\Omega) = \pi^*\omega$ for a uniquely determined closed differential form $\omega \in \Omega^{2j}(X)$ of degree $2j$ on X. If the connection form θ is replaced by another connection form $\widetilde{\theta}$, with curvature form $\widetilde{\Omega}$, then ω is replaced by a closed differential form $\widetilde{\omega} \in \Omega^{2j}(X)$ which is cohomologous to ω. In other words, the de Rham cohomology class $\mathrm{w}(f) := [\omega] \in \mathrm{H}^{2j}(X)$ does not depend on the choice of the connection form θ. In this way one obtains a homomorphism $\mathrm{w} : \mathbf{C}[\mathfrak{g}]^{\mathrm{Ad}\, G} \to \mathrm{H}^{\mathrm{even}}(X)$ from the algebra of $\mathrm{Ad}\, G$-invariant polynomials on $\mathfrak{g}$ to the algebra of de Rham cohomolgy classes of even degree on X.*

Note that the fact that $f(\Omega)$ and $f(\widetilde{\Omega})$ both are equivariantly cohomologous to f implies that $\widetilde{\omega}$ is cohomologous to ω. The homomorphism w is called the *Weil homomorphism* and the cohomology classes $\mathrm{w}(f)$, $f \in \mathbf{C}[\mathfrak{g}]^{\mathrm{Ad}\, G}$ are called the *characteristic classes* of the principal G-bundle P over X. If P is a frame bundle of a vector bundle M, then the characteristic classes of P are also called the *characteristic classes of* M. The forms $\omega \in \Omega^{2j}(X)$, defined in terms of the connection form θ and curvature form Ω on P, are called the *characteristic forms* of the principal G-bundle (or vector bundle) over X.

For $G = \mathrm{U}(1)$ and $\mathfrak{g} \simeq \mathrm{i}\,\mathbf{R}$, the Chern form is the characteristic form which is assigned to the polynomial $X \mapsto -(2\pi\,\mathrm{i})^{-1}X : \mathfrak{g} \to \mathbf{R}$, which is invariant because G is commutative. As we have seen in Section 10, the Chern form classifies complex line bundles with Hermitian linear connections Although in general the characteristic forms do not classify vector bundles with linear connections, they have numerous applications.

Remark 11.1 Let E and F be vector bundles over a compact smooth manifold X and let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic elliptic linear partial differential operator, mapping sections of E to sections of F. Then the dimension of the kernel of P and the codimension of the range of P are finite numbers, and the difference $\dim(\ker P) - \mathrm{codim}(\mathrm{range}\, P)$ is called the *index* of P. The *Atiyah-Singer index formula* [5, Thm. (2.12)] expresses the index of P as an integral over the base manifold of a characteristic form of a vector bundle defined by the so-called *principal symbol* of P.

A special case is $P = d + d^*$, viewed as an operator which maps even differential forms to odd differential forms, where the Hodge adjoint d^* is taken with respect to a suitable Riemannina structure on X. in this case the index of P is equal to *Euler number* $\chi(X)$ of X. The integral formula for $\chi(X)$ in terms of a characteristic form of (the orthogonal frame bundle of) $\mathrm{T}X$ is the generalization to arbitrary dimensions of the classical *Gauss-Bonnet theorem*, which states that the Euler number of a compact oriented surface is equal to $1/4\pi$ times the integral of the scalar curvature. In the case when X is a submanifold of a Euclidean space, this generalization had been found by Allendoerfer [1], then was generalized to so-called “Riemannian polyhedra” by Allendoerfer and Weil, [2], and proved intrinsically, for “abstract” Riemannian manifolds, by Chern [9]. Weil did not mention this application of characteristic classes in [33]. $\oslash$

12 The Ambrose-Singer Theorem

Although this section has not been discussed in the course, it is included as a natural sequel to Section 4.

Lemma 12.1 *Any invariant connection H in a principal G-bundle P over X admits lifting.*

Proof Let $\gamma : I \to X$ be a smooth curve in X, where I is an interval in $\mathbf{R}$, and let $b \in I$. Write $x = \gamma(b)$ and choose $p \in P_x$. Then there exists an open interval J_b around b in I, such that $\gamma|_{J_b}$ has a horizontal lift $\delta : J_b \to M$ with $\delta(b) = p$. Because the connection H is G-invariant, it follows that, for every $g \in G$, the curve $\delta_g : J_b \to P$ defined by $\delta_g(t) := g_{_P}(\delta(t),\, t \in J_b$ is a horizontal lift δ_g of γ such that $\delta_g(b) = g_{_P}(p)$. Because P_x is equal to the set of all $g_{_P}(p)$ such that $g \in G$, the conclusion of the lemma now follows from Lemma 4.1. □

We assume in the sequel that H is an invariant connection in P. As in Section 4, the parallel transport along a piecewise smooth continuous curve $\gamma : [a,\, b] \to X$ defines a diffeomorphism $h_\gamma : P_{\gamma(a)} \to P_{\gamma(b)}$ from the fiber over $\gamma(a)$ onto the fiber over $\gamma(b)$. If $\gamma(a) = x = \gamma(b)$, H_γ is a diffeomorphism of P_x and $\gamma \mapsto h_\gamma$ is a homomorphism from the group of loops starting and ending at x to the group of diffeomorphisms of P_x. The image is a subgroup $\mathcal{H}_x$ of $\mathrm{Diffeo}(P_x)$, called the *holonomy group* of transformations of P_x defined by the connection H in P.

As already observed in the proof of Lemma 12.1, we have that, for any $g \in G$, $g_{_P} \circ \delta$ is a horizontal lift of γ if δ is a horizontal lift of γ, and therefore

$$g_{_P} \circ h_\gamma = h_\gamma \circ g_{_P}, \quad g \in G. \tag{12.1}$$

Let $x \in X$ and $p \in P_x$. Because G acts freely and transitively on P_x, we obtain that for each $h \in \mathcal{H}_x$ there is a unique $g = \phi_p(h) \in G$ such that $h(p) = g_{_P}(p)$, and it follows from (12.1) that

$$\phi_p(h_1 \circ h_2)_{_P}(p) = h_2 \circ h_1(p)) = h_2 \circ \phi_p(h_1)_{_P}(p) = \phi_p(h_1)_{_P} \circ h_2(p) = \phi_p(h_1)_{_P} \circ \phi_p(h_2)_{_P}(p),$$

which implies that $\phi_p : h \mapsto \phi_p(h)$ defines an *anti-homomorphism* from the group $\mathcal{H}_x$ to the group G. It follows that the image $\widehat{H}_p := \phi_p(\mathcal{H}_x)$ is a subgoup of G, which is called the *holonomy sugroup of G based at the point $p \in P$.*

Assume in the sequel also that the basis manifold X is connected. If $p,\, q \in P$, then there exists a (piecewise) smooth curve $\gamma : [a,\, b] \to X$ such that $\gamma(a) = x := \pi(p)$ and $\gamma(b) = y := \pi(q)$. Because $h_\gamma(p)$ and q both belong to $P_{\gamma(b)}$, there is a unique $g \in G$ such that $q = g_{_P} \circ h_\gamma(p)$. For any $h \in \mathcal{H}_y$ we have that $h' := h_\gamma^{-1} \circ h \circ h_\gamma \in \mathcal{H}_p$, and therefore

$$\begin{aligned}\phi_q(h)_{_P}(q) &= h(q) = h \circ g_{_P} \circ h_\gamma(p) = g_{_P} \circ h_\gamma \circ h'(p) = g_{_P} \circ h_\gamma \circ \phi_p(h')_{_P}(p) = g_{_P} \circ \phi_p(h')_{_P} \circ h_\gamma(p)\\ &= g_{_P} \circ \phi_p(h')_{_P} \circ g_{_P}^{-1}(q),\end{aligned}$$

which implies that $\phi_q(h) = g\, \phi_p(h')\, g^{-1}$. It follows that $\widehat{H}_q = g\, \widehat{H}_p\, g^{-1}$, which implies that all the holonomy subgroups of G are conjugate by elements of G. Note also that $\widehat{H}_q = \widehat{H}_p$ if $q = h_\gamma(p)$.

Let $p \in P$ and denote by P_p the subset of all $q \in Q$ which can be connected with p by means of a horizontal curve. Let $\mathfrak{h}_p$ be the Lie subalgebra of $\mathfrak{g}$ which is generated by all $\Omega_q(u,\, v) = (\mathrm{d}\theta)_q(u,\, v)$, in which $q \in P_p$ and $u,\, v \in H_q$. Then the theorem of Ambrose and Singer [3] states:

Theorem 12.2 *Assume that P is a principal G-bundle over a connected and paracompact manifold X. Assume that H is an invariant connection in P. Then, for any $p \in P$, the holonomy subgroup $\widehat{H}_p$ of G is an immersed Lie subgroup of G with Lie algebra equal to $\mathfrak{h}_p$. The connected component $\widehat{H}_p^o$ in $\widehat{H}_p$ of the identity element is equal to the set of $g \in G$ such that $g_P(p) = h_\gamma(p)$, in which γ ranges over the piecewise smooth curves in X, starting and ending at $x = \pi(p)$, which are homotopic to the constant curve at the point x.*

It follows that $[\gamma] \mapsto \phi_p(h_\gamma)\, \widehat{H}_p^o$ is an anti-homomorphism from the fundamental group $\pi_1(X,\, x)$ of X (which is countable) onto the component group $\widehat{H}_p/\widehat{H}_p^o$ of $\widehat{H}_p$. This homomorphism is sometimes called the *monodromy representation* of the fundamental group of X.

Another consequence is that P_p is an immersed smooth submanifold of P which is a principal $\widehat{H}_p$-bundle over X. Moreover, for every $q \in P_p$ we have that $H_q \subset \mathrm{T}_q\, P_p$, which means that the H_q, $q \in P_p$ define a $\widehat{H}_p$-invariant connection in P_p. Conversely, if $\widehat{H}$ is an immersed Lie subgroup of G and Q is a smoothly immersed submanifold of P which at the same time is a principal $\widetilde{H}$-bundle over X such that $H_q \subset \mathrm{T}_q\, Q$ for every $q \in Q$, then we have, for every $p \in Q$, that $\widehat{H}_p \subset \widehat{H}$ and $P_p \subset Q$. In other words, P_p is the smallest principal subbundle Q over X containing p and $\widehat{H}_p$ is the smallest structure group $\widehat{H}$, such that the restriction of H to Q is tangent to Q.

13 Exercises

Exercise 13.1 Let $\alpha^i \in \Omega^1(M)$, $1 \leq i \leq l$, be linearly independent at every point and let the vector subbundle H of $\mathrm{T}M$ be defined by (1.1). Prove that H is integrable if and only if there exist $\beta^i_j \in \Omega^1(M)$ such that

$$\mathrm{d}\alpha^i = \sum_{j=1}^{l} \beta^i_j \wedge \alpha^j, \quad 1 \leq i \leq l.$$

This is Élie Cartan's form of the integrability condition. $\oslash$

Exercise 13.2 Prove that if $v \in \mathcal{X}(X)$, then the horizontal lift of the curve $t \mapsto \mathrm{e}^{t\,v}(x)$ which starts at $m \in M_x$ is equal to $t \mapsto \mathrm{e}^{t\,v_{\mathrm{hor}}}(m)$.

Now let $u,\, v \in \mathcal{X}(X)$ and $[u,\, v] = 0$. Let γ_h be the closed curve in X which is composed of $\mathrm{e}^{t\,u}(x)$, $0 \leq t \leq h$, followed by $\mathrm{e}^{t\,v} \circ \mathrm{e}^{h\,u}(x)$, $0 \leq t \leq h$, followed by $\mathrm{e}^{-t\,u} \circ \mathrm{e}^{h\,v} \circ \mathrm{e}^{h\,u}(x)$, $0 \leq t \leq h$, and concluded by $\mathrm{e}^{-t\,v} \circ \mathrm{e}^{-h\,u} \circ \mathrm{e}^{h\,v} \circ \mathrm{e}^{h\,u}(x)$, $0 \leq t \leq h$. Let δ_h be the horizontal lift of γ_h which starts at the point $m \in M_x$, and let $m(h)$ denote the endpoint of δ_h.

Prove that $m(h) \in M_x$. Prove that $h^{-2}\,(m(h) - m)$ converges as $h \downarrow 0$, and express the limit in terms of the curvature. $\oslash$

Exercise 13.3 Verify the Leibniz rule (3.6) for covariant differentiation. $\oslash$

Exercise 13.4 Prove that in any bundle with connection, there exist through each point of the bundle a local section of which the tangent space is horizontal. Verify that a local section of the frame bundle of a vector bundle corresponds to a local trivialization of the bundle. Prove that if the local section has horizontal tangent space at a given point, then at the corresponding base point the Christoffel symbols vanish. $\oslash$

Exercise 13.5 Verify the formula (3.8) by expressing the left and right hand side (which is defined in (2.5)) in terms of Christoffel symbols. The computations are simplified if one chooses the local trivialization of the bundle such that, at the special point, the Christoffel symbols vanish. $\oslash$

Exercise 13.6 Let H be a linear connection in the vector bundle M over X and let β_x be an inner product on H_x, depending smoothly on $x \in X$. Prove that β is covariantly constant with respect to the connection H if and only if, for every $r,\, s \in \Gamma(M)$, and $v \in \mathcal{X}(X)$, we have that $v(\beta(r,\, s)) = \beta(\nabla_v r,\, s) + \beta(r,\, \nabla_v s)$. $\oslash$

Exercise 13.7 Let $(X,\, \beta)$ be a Riemannian manifold and R the Riemannian curvature tensor of $(X,\, beta)$. Let e_i denote a β_x-orthonormal basis of $\mathrm{T}_x\, X$. Prove that the curvature coefficents

$$R_{ijkl} := \beta_x\,(R(x)(e_k,\, e_l)\, e_j,\, e_i)$$

satisfy the antisymmetry conditions that $R_{ijkl} = -R_{ijlk}$ and $R_{ijkl} = -R_{jikl}$.

Prove that if $n = 2$, then the scalar $K := R_{1221}$, called the *Gaussian curvature*, is independent of the choice of the orthonormal basis, and R is completely determined by K. $\oslash$

Exercise 13.8 Assume that G is a closed Lie subgroup of a Lie group F, where $\mathfrak{g}$ and $\mathfrak{f}$ denotes the Lie algebra of G and F, respectively. On F we have the right action $(g,\, f) \mapsto f\, g^{-1} : G \times F \to F$ of G, which exhibits F as a principal G-bundle over F/G. On F we also have the left action $(\phi,\, f) \mapsto \phi\, f : F \times F \to F$ of F. Prove that for every linear subspace $\mathfrak{h}$ of $\mathfrak{f}$ such that $\mathfrak{f} = \mathfrak{h} \oplus \mathfrak{g}$ there is a unique connection H in $F \to F/G$ which is invariant under the left action of F on F, and such that $H_1 = \mathfrak{h}$. Prove that H is also invariant under the right action of G on F, which is what we always require for a connection in a principal G-bundle, if and only if $(\mathrm{Ad}\, g)(\mathfrak{h}) = \mathfrak{h}$ for every $g \in G$. Prove that the connection form θ and the curvature form Ω are invariant under the left action of F on F, and describe Ω_1 in terms of the Lie brackets of elements in $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$. Prove that the connection is flat if and only if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{f}$, which then automatically is an ideal in $\mathfrak{f}$.

Comments Such a principal G-bundle is called a *homogeneous principal fiber bundle.* (Usually F and G are denoted in the literature by G and H, respectively.) If ρ is a linear representation of G in a vector space Y, then F acts (from the left) on the space $(\mathcal{F}(F) \otimes Y)^G$ of sections of the associated vector bundle $F \times_G Y$. These is the *induced representation* of F, as used in the course of Erik van den Ban. $\oslash$

Exercise 13.9 Prove that (9.5) defines a connection in M, which moreover satisfies (9.4). $\oslash$

Exercise 13.10 Let $(M,\, \sigma)$ be a symplectic manifold and G a connected Lie group with Lie algebra $\mathfrak{g}$, which acts in a Hamiltoninan fashion on $(M,\, \sigma)$, with momentum mapping $\mu : P \to \mathfrak{g}^*$, cf. the course on Symplectic Geometry. Define $\alpha(X) := \langle X,\, \mu\rangle - \sigma \in \mathcal{F}(P) \oplus \Omega^2(M)$, $X \in \mathfrak{g}$. Prove that $\alpha \in A = (\mathbf{C}[\mathfrak{g}] \otimes \Omega^*(M))^G$, and that $\mathrm{d}_{\mathfrak{g}}\, \alpha = 0$.

Now assume that M is a principal G-bundle over a manifold Q, with projection $\pi : M \to Q$, and let θ be a connection form in P. Prove that there exists a unique closed two-form $\gamma \in \Omega^2(Q)$ on Q such that $\langle \Omega,\, \mu\rangle - \sigma_{\mathrm{hor}} = \pi^*\gamma$.

(When G is a torus, this conclusion is closely related to the proof of [17, Thm. 1.1].) $\oslash$

14 Appendix: Some Notations

If M is a smooth manifold, and $m \in M$, then $\mathrm{T}_m M$ denotes the tangent space of M at m, which is the fiber over m of the tangent bundle $\mathrm{T}M$ of M. If $\Phi : M \to N$ is a smooth mapping from M to a smooth manifold N, then for each $m \in M$ the tangent mapping of Φ at m is denoted by $\mathrm{T}_m \Phi$, it is a linear mapping from $\mathrm{T}_m M$ to $\mathrm{T}_{\Phi(m)} N$.

$\mathcal{F}(M)$ denotes the space of smooth functions on a smooth manifold M, with pointwise addition and multiplication these form a commutative algebra.

$\mathcal{X}(M)$ denotes the space of smooth vector fields on M, it is a Lie algebra with respect to the Lie brackets $[u,\, v]$ of the vector fields, which are given by the commutatator $[u,\, v]\, f = u\,(v\, f) - v\,(u\, f)$ if the vector fields are viewed as derivations on the algebra $\mathcal{F}(M)$.

$\Omega^p(M)$ denotes the space of smooth differential forms of degree p on M. One has the exterior derivative $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ and the wedge product $\alpha \wedge \beta \in \Omega^{p+q}(M)$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$. The sum

$$\Omega^*(M) := \bigoplus_{p=0}^{\dim M} \Omega^p(M)$$

is an associative algebra with respect to the wedge product, supercommutative in the sense that $\alpha \wedge \beta = (-1)^{p\,q} \beta \wedge \alpha$ if $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$.

If $\Phi : M \to N$ and $\alpha \in \Omega^p(N)$, then the pull-back $\Phi^*\alpha \in \Omega^p(M)$ is defined by

$$\Phi^*\alpha_m(v_1,\, \ldots,\, v_p) = \alpha_{\Phi(m)}(\mathrm{T}_m \Phi\, v_1,\, \ldots,\, \mathrm{T}_m \Phi\, v_p).$$

If v is a smooth vector field on M then $\mathrm{e}^{t\, v}$ denotes its flow after time t, defined by $\mathrm{e}^{t\, v}(m) = \gamma(t)$ if $\mathrm{d}\, \gamma(t)/\, \mathrm{d}t = v(\gamma(t))$ and $\gamma(0) = m$. In the terminology of Lie [26], $t \mapsto \mathrm{e}^{t\, v}$ is the one-parameter group of transformations in M which is generated by the "infinitesimal transformation" v.

The *inner product* $\mathrm{i}_v\, \alpha \in \Omega^{p-1}(M)$ of $\alpha \in \Omega^p(M)$ with v is defined by

$$(\mathrm{i}_v\, \alpha)_m(v_2,\, \ldots,\, v_p) = \alpha_m(v(m),\, v_2,\, \ldots,\, v_p)$$

The Lie derivative $\mathcal{L}_v\alpha \in \Omega^p(M)$ satisfies

$$\mathcal{L}_v\alpha := \frac{\mathrm{d}}{\mathrm{d}t}\, \left(\mathrm{e}^{t\, v}\right)^*\, \alpha\big|_{t=0} = \mathrm{d}(\mathrm{i}_v\, \alpha) + \mathrm{i}_v(\mathrm{d}\alpha),$$

in which the second identity is called the *homotopy identity.*

References

[1] C.B. Allendoerfer: The Euler number of a Riemannian manifold. *Amer. J. Math.* **62** (1940) 243–248.

[2] C.B. Allendoerfer and A. Weil: The Gauss-Bonnet theorem for Riemannian polyhedra. *Trans. A.M.S.* **53** (1943) 101–129.

[3] W. Ambrose and I.M. Singer: A theorem on holonomy. *Trans. Amer. Math. Soc.* **75** (1953) 428–443.

[4] M.F. Atiyah and R. Bott: The moment map and equivariant cohomology. *Topology* **23** (1984) 1–28.

[5] M.F. Atiyah and I.M. Singer: The index of elliptic operators, III. *Annals of Mathematics* **87** (1968) 546–604.

[6] N. Berline et M. Vergne: Classes caractéristiques invariantes. Formules de localisation et cohomologie équivariante. *C.R. Acad. Sci. Paris* **295** (1982) 539–541.

[7] É. Cartan: Sur les variétés a connexion affine et la théorie de la relativité généralisée. *Ann. Éc. Norm. Sup.* **40** (1923) 325–412 = Œuvres Complètes, Partie III, vol. 1, pp. 659–746.

[8] H. Cartan: La transgression dans un groupe de Lie et dans un fibré principal. pp. 15–27 in: *Colloque de Topologie Bruxelles (Espaces Fibrés) 1950.* G. Thone, Liège; Masson et Cie, Paris, 1951.

[9] S.-s. Chern: A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. *Annals of mathematics* **45** (1944) 747–752.

[10] E.B. Christoffel: Ueber die Transformationen der homogenen Differentialausdrücke zweiten Grades. *Journal für die reine und angewandte Mathematik* **70** (1869) 46–70, 241–245.

[11] A. Clebsch: Ueber die simultane Integration linearer partieller Differentialgleichungen. *Journal für die reine und angewandte Mathematik* **65** (1866) 257–268.

[12] E.A. Coddington and N. Levinson: *Thoery of Ordinary Differential Equations.* McGraw-Hill, New York, Toronto, London, 1955.

[13] F. Deahna: Ueber die Bedingungen der Integrabilität lineärer Differentialgleichungen erster Ordnung zwischen einer beliebigen Anzahl veranderlichen Grössen. *Journal für die reine und angewandte Mathematik* **20** (1840) 340–349.

[14] S.K. Donaldson and P.B. Kronheimer: *The Geometry of Four-Manifolds.* Oxford University Press, New York, 1990.

[15] J.J. Duistermaat: Equivariant cohomology and stationary phase. *Contemporary Mathematics* **179** (1994) 45–62.

[16] J.J. Duistermaat: *The Heat Kernel Lefschetz Fixed Point Formula for the Spin-c Dirac Operator.* Birkhäuser, Boston, Basel, Berlin, 1996.

[17] J.J. Duistermaat and G.J. Heckman: On the variation in the cohomology of the symplectic of the reduced phase space. *Invent. math.* **69** (1982) 259–268 and **72** (1983) 153–158.

[18] C. Ehresmann: Les connexions infinitésimales dans un espace fibré differentiable. pp. 29–55 in: *Colloque de Topologie Bruxelles (Espaces Fibrés) 1950.* G. Thone, Liège; Masson et Cie, Paris, 1951.

[19] G. Frobenius: Über das Pfaffsche Problem. *Journal für die reine und angewandte Mathematik* **82** (1877) 230–315 = *Gesammelte Abhandlungen*, Springer-Verlag, Berlin, 1966.

[20] W.H. Greub, S. Halperin, R. Vanstone: Connections, Curvature and Cohomology. I De Rham cohomology of manifolds and vector bundles. II Lie groups, principal bundles, and characteristic classes. III Cohomology of principal bundles and homogeneous spaces. Academic Press, New York, 1972–76.

[21] R.C. Gunning: *Lectures on Riemann Surfaces.* Princeton University Press, Princeton, New Jersey, 1966.

[22] C.G.J. Jacobi: *Vorlesungen über Dynamik.* gehalten an der Universität Königsberg im Wintersemester 1842–43 und nach einem von C.W. Borchardt ausgearbeiteten hefte. Verlag G. Reimer, Berlin 1881. Reprint Chelsea Publ. Co., New York, N.Y., 1969.

[23] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry, I, II.* Interscience, New York, London, 1963, 1969.

[24] B. Kostant: Quantization and unitary representations. pp. 87–208 in: ed. C.T. Taam : *Lectures in Modern Analysis and Applications III.* Lecture Notes in Mathematics 170, Springer-Verlag, Berlin, 1970.

[25] T. Levi-Civita: Nozione di parallelismo in una varietà qualunque e consequente specificazione geometrica della curvatura Riemanniana. *Rend. Circ. Mat. Palermo* **42** (1917) 173–205.

[26] S.Lie und F. Engel: *Theorie der Transformationsgruppen I.* Teubner Verlag, Leipzig und Berlin, 1888. Reprinted in 1930.

[27] R. Lipschitz: Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen. *Journal für die reine und angewandte Mathematik* **70** (1869) 71–102.

[28] J.W. Morgan: *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds.* Princeton University Press, Princeton, New Jersey, 1996.

[29] B. Riemann: *Gesammelte Mathemathische Werke.* Teubner Verlag, Leipzig, 1876, 1892. Reprint of the second edition, Dover, New York, 1953.

[30] H. Samelson: Differential forms, the early days; or the story of deahna's theorem and of Volterra's theorem. *Amer. Math. Monthly* **108** (2001) 522–530.

[31] I.M. Singer: Some problems in the quantization of gauge theories and string theories. pp. 199–216 in: ed. R.O. Wells, Jr.: *The Mathematical Heritage of Hermann Weyl.* Proc. Symp. Pure Math. **48**, Amer. math. Soc., Providence, Rhode Island, 1988.

[32] M. Spivak: *A Comprehensive Introduction to Differential Geometry, I–V*. Publish or Perish, Berkeley, 1979.

[33] A. Weil: (1949) Géométrie différentielle des espaces fibrés. pp. 422–436 in: *Œuvres Scientifiques (Collected Papers), vol. I*. Springer-Verlag, New York, Heidelberg, Berlin, 1980.