Dynamical Systems with Symmetry

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Contents

In these notes, we collect some basic general facts about dynamical systems with symmetry. Most of the results are known in the literature or even standard, but some are less well-known than they perhaps should be. An example is the quasi-periodicity of relative periodic solutions when the shift element is elliptic, cf. the first statement in Proposition 8.5.

1 The Dynamical System

We assume throughout these notes that v is a smooth vector field on a smooth manifold M.

Here “smooth” $= \mathrm{C}^\infty = \mathrm{C}^k = k$ times continuously differentiable for arbitrary k. All statement can also be phrased in a C^k setting with finite k, but then one has to keep track of the orders of differentiablity of the various objects. In the other direction, all statements remain true without any change in the proofs if C^∞ is replaced by “real analytic”, with the exception of some global statements in which smooth partitions of unity have been used.

We recall the following general facts about the flow induced by the vector field v, cf. Coddington and Levinson [5]. For each $m \in M$ there is a unique solution $\gamma = \gamma_m : I_m \to M$, defined on a maximal open interval I_m around 0 in $\mathbf{R}$, of the differential equation

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = v(\gamma(t)), \quad t \in I_m, \tag{1.1}$$

which satisfies the initial condition

$$\gamma(0) = m.$$

The set

$$D := \{(t,\, m) \in \mathbf{R} \times M \mid t \in I_m\}$$

is open in $\mathbf{R} \times M$ and the mapping

$$\Phi : (t,\, m) \mapsto \gamma_m(t),$$

called the *flow* of the vector field v, is a smooth mapping from D to M.

Because the domain of defintion D of the flow Φ is an open subset of $\mathbf{R} \times M$, we have for each $t \in \mathbf{R}$ that

$$D_t := \{m \in M \mid (t,\, m) \in D\} = \{m \in M \mid t \in I_m\}$$

is an open subset of M, and the smoothness of Φ implies that

$$\mathrm{e}^{t\,v} : m \mapsto \Phi(t,\, m) = \gamma_m(t),$$

called the *flow after time* t, is a smooth mapping from D_t to M. The exponential notation reflects the defining equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{t\,v}(m) = v\left(\mathrm{e}^{t\,v}(m)\right), \quad \mathrm{e}^{0\,v}(m) = m.$$

Obviously $D_0 = M$ and $\mathrm{e}^{0\,v}$ is equal to the identity in M.

The flow satisfies the *group property* that if $s,\, t \in \mathbf{R}$ and $m \in D_s$ and $\mathrm{e}^{s\,v}(m) \in D_t$, then $m \in D_{s+t}$ and

$$\mathrm{e}^{t\,v}\left(\mathrm{e}^{s\,v}(m)\right) = \mathrm{e}^{(s+t)\,v}(m). \tag{1.2}$$

It follows that $\mathrm{e}^{t\,v}$ is a diffeomorphism from D_t onto D_{-t}, with inverse equal to $\mathrm{e}^{-t\,v}$.

One has $s := \sup I_m < \infty$, if and only if for each compact subset K of M there exists an $\epsilon > 0$ such that $\mathrm{e}^{t\,v}(m) \notin K$ for every $t \in \,]s-\epsilon,\, s[\, \cap I_m$. In other words, the solution of (1.1) which starts at m *runs out of every compact subset of* M *in a finite time.* Similarly $i := \inf I_m > -\infty$, if and only if for each compact subset K of M there exists an $\epsilon > 0$ such that $\mathrm{e}^{t\,v}(m) \notin K$ for every $t \in \,]i,\, i+\epsilon,\, s[\, \cap I_m$.

Definition 1.1 The vector field v is called *complete* if, for every $m \in M$, we have that $I_m = \mathbf{R}$. In other words, if no solution runs out of every compact subset of M in a finite time. It follows that if M is compact, then every smooth vector field on M is complete. $\oslash$

The vector field v is complete if and only if, for every $t \in \mathbf{R}$, we have that $D_t = M$, and the flow after time t is a diffeomorphism from M onto M. The group property (1.2) then implies that $t \mapsto \mathrm{e}^{t\,v}$ is a smooth homomorphism from the additive group of the real numbers $(\mathbf{R},\, +)$ to the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M.

2 The Group Action

Definition 2.1 A *smooth action* of a Lie group G on the manifold M is a smooth mapping $A : G \times M \to M$ such that

$$A(g,\, A(h,\, m)) = A(g\,h,\, m), \quad m \in M, \quad g,\, h \in G. \tag{2.1}$$

For each $g \in G$ we will denote its action $m \mapsto A(g,\, m) : M \to M$ by g_M. The equation (2.1) means that $g \mapsto g_M$ is a smooth homomorphism from G to $\mathrm{Diff}(M)$. Note that if v is a complete vector field on M, then its flow is a smooth action on M of $(\mathbf{R},\, +)$.

If there is no danger of confusion, then we we will use the shorthand notation

$$g \cdot m := g_M(m) := A(g,\, m), \quad g \in G,\; m \in M.$$

$\oslash$

Definition 2.2 Let $\mathfrak{g} = \mathrm{T}_1 G$ denote the Lie algebra of G. For every $X \in \mathfrak{g}$ and $m \in M$, we write

$$X \cdot m := X_M(m) := \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{t\,X}\right)_M (m)\Big|_{t=0} \in \mathrm{T}_m\, M$$

for the *infinitesimal action* of X on the element m of M. Then $X_M : m \mapsto X_M(m)$ is a smooth vector field on M, it is complete and we have

$$\mathrm{e}^{t\,X_M} = \left(\mathrm{e}^{t\,X}\right)_M, \quad t \in \mathbf{R}.$$

Actually, the infinitesimal action $\alpha : X \mapsto X_M$ is a homomorphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra $\mathcal{X}^\infty(M)$ of all smooth vector fields on M, if the latter is provided with the opposite of the usual Lie brackets of vector fields. We will write α_m for the linear mapping $X \mapsto X_M(m)$ from $\mathfrak{g}$ to $\mathrm{T}_m\, M$. $\oslash$

Definition 2.3 For any $m \in M$, the *stabilizer* or *isotropy group* of m in G is defined as

$$G_m := \{g \in G \mid g \cdot m = m\}\,. \tag{2.2}$$

G_m is a closed Lie subgroup of G, with Lie algebra equal to

$$\mathfrak{g}_m = \{X \in \mathfrak{g} \mid X \cdot m = 0\} = \ker \alpha_m. \tag{2.3}$$

m is called a *fixed point* for the G-action, if $G_m = G$, i.e. if $g \cdot m = m$ for every $g \in G$. $\oslash$

We have $h \in G_{g\cdot m}$ if and only if $h \cdot (g \cdot m) = g \cdot m$ if and only if $(g^{-1}\,h\,g) \cdot m = m$ if and only if $g^{-1}\,h\,g \in G_m$. This shows that $G_{g\cdot m}$ is conjugate in G to G_m by means of the element $g \in G$, in the sense that

$$G_{g\cdot m} = g\,G_m\,g^{-1}. \tag{2.4}$$

Definition 2.4 The set

$$G \cdot m := \{g \cdot m \mid g \in G\}$$

is called the *orbit through m*.

The mapping $A_m : g \mapsto A(g,\, m) : G \to M$ induces a mapping

$$\widetilde{A}_m : g\,G_m \mapsto A(g,\, m) : G/G_m \to M, \tag{2.5}$$

which is bijective from G/G_m onto $G \cdot m$. It intertwines the transitive action of G on G/G_m defined by left multiplications with the transitive action of G on the orbit through m. Here we use the terminology that a mapping $f : X \to Y$ *intertwines* the action A of G on X with the action B of G on Y if $f(A(g,\, x)) = B(g,\, f(x))$ for every $g \in G$ and $x \in X$. One also says in this case that the mapping f is *equivariant* with respect to the actions of G on X and Y. $\oslash$

The mapping $\widetilde{A}_m$ is a smooth immersion from G/G_m into M, which exhibits the orbit through m as a smoothly immersed submanifold of M, with tangent space at m equal to

$$\mathrm{T}_m(G \cdot m) = \alpha_m(\mathfrak{g}). \tag{2.6}$$

Definition 2.5 Two orbits are either equal to each other or disjoint, which implies that M is partitioned into orbits. The set

$$G\backslash M := \{G \cdot m \mid m \in M\}$$

of all orbits in M is called the *orbit space* for the group action A. The mapping

$$\pi : m \mapsto G \cdot m : M \to G\backslash M$$

is called the *canonical projection* from M onto $G\backslash M$. The fibers of π are the G-orbits in M. $\oslash$

Until now the orbit space is just defined as a set, without any further structure. At some points, especially when we want to consider nearby orbits, we will need more structure on the orbit space. For instance, it is natural to provide the orbit space $G\backslash M$ with the strongest topology for which the projection $\pi : M \to G\backslash M$ is continuous, i.e. a subset V of $G\backslash M$ is declared to be open in $G\backslash M$ if and only if the G-invariant subset $\pi^{-1}(V)$ is open in M.

In general the orbit structure can be very complicated, already for flows of complete vector fields. For instance, orbits need not be closed, not even embedded submanifolds, and the topology in $G\backslash M$ need not be Hausdorff. For this reason we will make additional assumptions about the action, when we need these in order to get further conclusions.

For instance, the orbit structure is already much better in the following case.

Definition 2.6 (Palais [25, Def. 1.2.2] A mapping f from a topological space X to a topological space Y is called *proper* if for every compact subset K of Y the pre-image $f^{-1}(K)$ is a compact subset of X.

The *action* of G on M is called *proper* if the mapping

$$(g,\, m) \mapsto (m,\, g \cdot m) : G \times M \to M \times M$$

is proper. This is equivalent to the condition that if m_j and g_j is an infinite sequence in M and G, respectively, such that for $j \to \infty$ the sequences m_j and $g_j \cdot m_j$ converge in M, then there is a subsequence $j = j(k) \to \infty$ such that the $g_{j(k)}$ converge in G as $k \to \infty$. $\oslash$

If the action is proper, then for every $m \in M$ the stabilizer subgroup G_m is a compact, hence closed, hence Lie subgroup of G, cf. [10, Cor. 1.10.7]. Moreover, the orbit $G \cdot m$ is a properly embedded submanifold of M, and therefore a closed subset of M. Finally, the topology of the orbit space $G\backslash M$ is Hausdorff, cf. [10, Lemma 1.11.3].

Although proper Lie group actions are quite special among the general Lie group actions, they do occur quite often in applications. For instance, an effective Lie group action on a paracompact smooth manifold M is proper if and only if there exists a Riemannian structure on M which is invariant under the Lie group action, cf. [10, Prop. 2.5.2 and p.106].

Definition 2.7 The action A is called *free at the point* $m \in M$ if $G_m = \{1\}$, and it is called *free* if it is free at every point of M. $\oslash$

The situation is particularly nice if the action is proper and free. Then $G \backslash M$ has a unique smooth manifold structure such that the projection $\pi : M \to G \backslash M$ is a smooth fibration, and $\dim(G \backslash M) = \dim M - \dim G$. In this case $\pi : M \to G \backslash M$ is called a *principal fiber bundle with structure group equal to* G, cf. [10, Th. 1.11.4].

Remark 2.8 Let us explain this terminology in more detail.

The fact that $\pi : M \to G \backslash M$ is a smooth fibration implies that there exist local smooth sections. This means that for every $b \in G \backslash M$ there is an open neighborhood V of b in $G \backslash M$ and a smooth mapping $\sigma : V \to M$ of $\pi : \pi^{-1}(V) \to V$, such that $\pi \circ \sigma$ is equal to the identity in V. The mapping $\tau = \tau_\sigma : (v,\, g) \mapsto g \cdot \sigma(v)$ is a diffeomorphism from $V \times G$ onto $\pi^{-1}(V)$, such that $\pi \circ \tau$ is equal to the projection $(v,\, g) \mapsto v : V \times G \to V$ on the first components. In other words, $\tau^{-1} : \pi^{-1}(V) \to V \times G$ is a local trivialization of the fibration over the open subset V of the base space.

If $\sigma' : V' \to M$ is another smooth local section for π defined on another open subset V' of $G \backslash M$, then we have for each $v \in V \cap V'$ a unique $\rho(v) \in G$, which depends smoothly on v, such that $\sigma'(v) = \rho(v) \cdot \sigma(v)$. Then $\tau_{\sigma'}(v,\, g) = g \cdot \rho(v) \cdot \sigma(v) = \tau_\sigma(v,\, g \cdot \rho(v))$, or

$$\tau_\sigma^{-1} \circ \tau_{\sigma'}(v,\, g) = (v,\, g \cdot \rho(v)), \quad v \in V \cap V',\ g \in G.$$

Now a principal fiber bundle with structure group equal to G is defined as a smooth fibration $\pi : M \to B$ with an open covering V_i, $i \in I$, of B and local trivializations $\tau_i^{-1} : \pi^{-1}(V_i) \to V_i \times G$, such that

$$\tau_i^{-1} \circ \tau_j(v,\, g) = (v,\, g \cdot \rho_{ij}(g)), \quad i,\, j \in I,\ v \in V_i \cap V_j,\ g \in G$$

for smooth mappings $\rho_{ij} : V_{ij} \to G$. It is easily verified that the actions $(g',\, (v,\, g)) \mapsto (v,\, g'\, g) : G \times (V_i \times G)$ of G on the spaces $V_i \times G$ are intertwined by the mappings $\tau_i : V_i \times G \to M$ with a unique smooth action of G on M which is free and proper. In this way "free and proper G-action" and "principal fibration with structure group G" are equivalent concepts. $\oslash$

Remark 2.9 The condition that the action is proper and free is equivalent to Axiom (FP) on [4, p. 6-05] who, in the more general framework of a continuous action of a locally compact topological group G on a locally compact topolological space M, took this as the definition of a principal fiber bundle. He then observed that (FP) implies that, for each $m \in M$, $G \cdot m$ is a closed subset of M and $g \mapsto G \cdot m$ is a homeomorphism from G onto $G \cdot m$, and that $G \backslash M$ is a locally compact Hausdorff space.

On [4, p. 6-08] he mentioned the theorem of Gleason that, for a free and smooth action of a compact Lie group on a smooth manifold, the fibration is smooth and locally trivial. He also remarked that the fibration is locally trivial if G is locally connected and metrisable, and $G\backslash M$ is a manifold, but that "the proof is difficult and would take take us out of the framework of these expositions". We think that it is fair to say that the equivalence between "proper free actions" and "principal fiber bundles" originates in H. Cartan [4].

$\oslash$

3 Isotropy Types and Orbit Types

This section can be ignored if the action is free.

3.1 Reduction to Free Actions

Definition 3.1 For each $m \in M$ the isotropy group G_m of m is a closed, hence Lie subgroup of G, cf. [10, Cor. 10.7]. For each closed Lie subgroup H of G we introduce the subset

$$M_H := \{m \in M \mid G_m = H\} \tag{3.1}$$

of M, which is called the *isotropy type* in M for the subgroup H of G. The sets M_H, where H ranges over the closed Lie subgroups of G for which M_H is non-empty, form a partition of M, and therefore are the equivalence classes of an equivalence relation in M. $\oslash$

Definition 3.2 The *normalizer* of H in G is defined as

$$\mathrm{N}(H) = \mathrm{N}_G(H) := \{g \in G \mid g\, H\, g^{-1} = H\}. \tag{3.2}$$

$\oslash$

$\mathrm{N}(H)$ is a closed, hence Lie subgroup of G. It is the largest subgroup N of G which contains H as a normal subgroup, in the sense that $H \subset N \subset \mathrm{N}(H)$ for every such subgroup N of G. As a consequence $\mathrm{N}(H)/H$ is a Lie group, cf. [10, Cor. 1.11.5].

The following lemma can be viewed as a reduction of a general action to a free one.

Lemma 3.3 *Let H be a closed Lie subgroup of G such that $M_H \neq \emptyset$, and let $g \in G$. Then*

$$g \in \mathrm{N}(H) \quad \Longleftrightarrow \quad g \cdot M_H = M_H \quad \Longleftrightarrow \quad (g \cdot M_H) \cap M_H \neq \emptyset.$$

The action of $\mathrm{N}(H)$ on M_H induces a free action of $\mathrm{N}(H)/H$ on M_H.

Proof If $m \in M_H$, then $G_m = H$, and it follows from (2.4) that $g \cdot m \in M_H$, i.e. $G_{g \cdot m} = H = G_m$, if and only if $g \in \mathrm{N}(H)$. $\square$

It follows from Lemma 3.3 that the subset M_H of M is not G-invariant, unless $\mathrm{N}(H) = G$, i.e. unless H is a normal subgroup of G.

Definition 3.4 We say that $H' \subset G$ is *conjugate* to H by means of an element of G if there exists an element $g \in G$ such that $H' = g\,H\,g^{-1}$. Clearly H' is a closed Lie subgroup of G if H' is conjugate to the closed Lie subgroup H of G, and we see from (2.4) that $M_{H'} \neq \emptyset$ if and only if $M_H \neq \emptyset$. The *conjugacy class* of H, the set of all subgroups H' of G which are conjugate to H by means of an element of G, will be denoted by $[H]$. $\oslash$

Definition 3.5 The set

$$M_{[H]} := \{m \in M \mid G_m = g\,H\,g^{-1} \quad \text{for some} \quad g \in G\} \tag{3.3}$$

is called the *orbit type* of $[H]$ in M. $\oslash$

This name comes from the fact that m and m' belong to the same orbit type if and only if there exists a G-equivariant bijection from $G \cdot m$ onto $G \cdot m'$, cf. [10, Lemma 2.6.2,(i)]. This defines an equivalence relation in M, which is coarser than the relation of belonging to the same isotropy type. It follows that M is partitioned into orbit types, and that each orbit type is partitioned into isotropy types for conjugate subgroups of G.

Definition 3.6 The subset $G\backslash M_{[H]} := \pi(M_{[H]})$ is called the *orbit type in the orbit space* of the conjugacy class $[H]$. $\oslash$

Also the orbit space $G\backslash M$ is partitioned into orbit types. Note that if G is commutative, then the orbit types types are equal to the isotropy types. We summarize the situation in the following lemma.

Lemma 3.7 *Let H be a closed Lie subgroup of G such that $M_H \neq \emptyset$. Then*

a) *$M_{[H]}$ is the smallest G-invariant subset of M which contains M_H. It is partitioned into the isotropy types $M_{H'}$, $H' \in [H]$, and we have*

$$g \cdot M_H = M_{g\,H\,g^{-1}} \quad \text{for every} \quad g \in G. \tag{3.4}$$

The mappings $g \mapsto g\,H\,g^{-1} \mapsto M_{g\,H\,g^{-1}}$ induce a G-equivariant bijection from $G/\,\mathrm{N}(H)$ onto the collection of the isotropy types in the orbit type $M_{[H]}$, where in $G/\,\mathrm{N}(H)$ we use the action of G by left multiplications.

b) *If π denotes the canonical projection from M onto the orbit space $G\backslash M$, then $\pi(M_H) = G\backslash M_{[H]}$. The fibers of $\pi|_{M_H}$ are the $\mathrm{N}(H)/H$-orbits in M_H.*

3.2 When the Action is Proper

Until now all the considerations in this section have been purely set-theoretical. In this subsection, we will assume that the action is proper.

Let $m \in M$. Recall that $\alpha_m(\mathfrak{g})$ is equal to the tangent space at m of the G-orbit through the point m, cf. (2.6). For every $g \in G_m$, the linear transformation $\mathrm{T}_m\, g_M$ in $\mathrm{T}_m\, M$ leaves

$\alpha_m(\mathfrak{g})$ invariant, and we obtain an induced linear action of the compact group $H := G_m$ on the vector space $E := \mathrm{T}_m\, M/\alpha_m(\mathfrak{g})$. For any open H-invariant subset B of E, we have the action of $h \in H$ on $G \times B$ which sends $(g,\, b) \in G \times B$ to $(g\, h^{-1},\, h \cdot b)$. This action is proper and free, and therefore its orbit space $G \times_H B$ is a smooth manifold. Because the action of G on $G \times B$ by multiplication from the left on the first factor commutes with the H-action on $G \times B$, it passes to an action of G on $G \times_H B$.

Theorem 3.8 (Tube theorem) *Let $m \in M$ and write $H := G_m$. There exists a G-invariant open neighborhood U of m in M, an open H-invariant neighborhood of the origin in E, and a diffeomorphism Φ from $G \times_H B$ onto U which intertwines the G-action on $G \times_H B$ with the G-action on U.*

In the proof of the tube theorem, see for instance [10, 2.4.1], B is identified with a suitable submanifold of M through m, called a *slice*, and the diffeomorphism Φ is induced by the mapping $(g,\, b) \mapsto g \cdot b$ from $G \times B$ to M.

The tube theorem is the basic tool in the investigation of the action of G in G-invariant neighborhoods of orbits, and in the local study of the orbit space $G\backslash M$.

Theorem 3.9 *Suppose that $H = G_m$ for some $m \in M$, which implies that H is a compact Lie subgroup of G. Then*

a) *M_H is a locally closed smooth submanifold of M, where it is allowed that different connected components have different dimensions.*

b) *The Lie group $\mathrm{N}(H)/H$ acts smoothly on M_H, and this action is free and proper. In view of Lemma* 3.7, b)*, there is a unique smooth manifold structure on the orbit type $G\backslash M_{[H]}$ such that $\pi|_{M_H} : M_H \to G\backslash M_{[H]}$ is a principal fiber bundle with structure group $\mathrm{N}(H)/H$.*

c) *$M_{[H]}$ is a G-invariant locally closed smooth submanifold of M, where it is allowed that different connected components have different dimensions. The G-action in Lemma* 3.7, a) *induces a G-equivariant diffeomorphism from the associated fiber bundle $(G/H) \times_{\mathrm{N}(H)/H} M_H$ onto $M_{[H]}$. Furthermore, the projection $\pi : M_{[H]} \to G\backslash M_{[H]}$ is a smooth fibration, of which the fibers are G-equivariantly diffeomorphic to the homogeneous space G/H.*

Proof Let

$$M^H := \{m \in M \mid h \cdot m = m \quad \text{for every} \quad h \in H\} = \{m \in M \mid H \subset G_m\} \tag{3.5}$$

denote the set of all the common fixed points of all the elements of H.

Obviously $M_H \subset M_{[H]} \cap M^H$. Now assume that $m \in M_{[H]} \cap M^H$. This means that G_m is conjugate to H by means of an element of G and that $H \subset G_m$. The properness of the action implies that G_m is compact. Because H is conjugate to G_m, H is compact as well and has the same dimension and the same finite number of connected components as G_m. In combination with $H \subset G_m$ this leads to the conclusion that $H = G_m$, and therefore $m \in M_H$. We therefore have proved that

$$M_H = M_{[H]} \cap M^H, \tag{3.6}$$

and the theorem now follows from [10, Th. 2.6.7], in the proof of which the tube theorem is the essential ingredient. □

Remark 3.10 Lemma 3.7 and Theorem 3.9 are due to Borel [3, Ch. XII, §1], in the framework of compact Lie group actions, and with $M_{[H]} \cap M^H$ instead of the isotropy types M_H, which are not mentioned explicitly. Still for compact Lie group actions, Theorem 3.9 is formulated in terms of the M_H in Jänich [17, §1.5], with a reference to Borel [3, Ch. XII, §1], but without an argument for the inclusion $M_{[H]} \cap M^H \subset M_H$.

Jänich [17, p. 6] warned that different connected components of the orbit type $M_{[H]}$ can have different dimensions, and it follows from Theorem 3.9, c) that the same holds for the isotropy type M_H. This is one of the reasons why in [10, Th. 2.6.7] the orbit type $M_{[H]}$ has been replaced by the *local action type* $M_x^{\approx}$, the equivalence class of x for the equivalence relation $x \approx y \iff$ there exists a G-equivariant diffeomorphism from a G-invariant open neighborhood of x onto a G-invariant open neighborhood of y in M. The local action types are open and closed subsets of the orbit types, and have a constant dimension. They are determined by the conjugacy class of the isotropy subgroup $H = G_x$ together with the equivalence class of the linear representation of H defined by the induced action on $T_xM/\mathrm{T}_x(G \cdot x)$.

The proofs in the aforementioned references for compact G carry over without change to proofs for proper Lie group actions. ⊘

Definition 3.11 A *stratification* of a smooth manifold M is a collection $\mathcal{S}$ of locally closed connected smooth submanifolds of M, called the *strata* of $\mathcal{S}$, such that the following is satisfied.

i) $\mathcal{S}$ is a locally finite partition of M, i.e.

   a) $S \cap S' = \emptyset$ if S, $S' \in \mathcal{S}$, $S \neq S'$,

   b) M is equal to the union of all $S \in \mathcal{S}$, and

   c) For each $m \in M$ there is a neighborhood U of m in M such that the set of all $S \in \mathcal{S}$ such that $S \cap U \neq \emptyset$ is finite.

ii) For each $S \in \mathcal{S}$ the closure $\overline{S}$ of S in M is equal to the union of S and a collection of $S' \in \mathcal{S}$ such that $\dim S' < \dim S$.

The stratification $\mathcal{S}$ is called a *Whitney stratification* if in addition the following conditions are met.

A) If $S,\, S' \in \mathcal{S}$, $S' \subset \overline{S}$, $S' \neq S$, and s_j is a sequence in S which converges to $s' \in S'$ and for which $\mathrm{T}_{s_j} S$ converges to the linear subspace L of $\mathrm{T}_{s'} M$, then $\mathrm{T}_{s'} S' \subset L$.

B) If s_j is a sequence as in A) and $s'_j \in S'$ which also also converges to s', then each limit of the one-dimensional subspaces $\mathbf{R}\, \lambda(s_j,\, s'_j)$ is contained in L. Here λ is a diffeomorphism from an open neighborhood of the diagonal in $M \times M$ to an open neighborhood of the zero section in $\mathrm{T}\, M$ such that $\lambda(m,\, m) = 0 \in \mathrm{T}_m\, M$ for every $m \in M$.

⊘

The connected components of the orbit types in M form a Whitney stratification in M, cf. [10, Th. 2.7.4] or Pflaum [26, Th. 4.3.7]. Note that the orbit type M_α itself is fibered by the isotropy types M_H, $H \in \alpha$, where for each $H \in \alpha$ the codimension of M_H in M_α is equal to $\dim G - \dim \mathrm{N}(H)$, cf. c) in Theorem 3.9. This implies that, as soon as $\dim \mathrm{N}(H) < \dim G$, the isotropy types do not form a stratification.

In order to be able to state that the orbit types in $G\backslash M$ form a Whitney stratification, we need that, at least locally, $G\backslash M$ is embedded in some natural way in a smooth manifold. This will be discussed, among other things, in Section 4

Example 3.12 An example of a highly non-free action is given by $M = G$ and the action of G on itself by means of *conjugation*, i.e.

$$A : (g,\, x) \mapsto g\, x\, g^{-1} : G \times G \to G.$$

Indeed, if G is nontrivial, i.e. $G \neq \{1\}$, then there are no points at which the action is free, because $G_1 = G \neq \{1\}$ and, if $x \neq 1$, the fact that $x \in G_x$ implies that $G_x \neq \{1\}$.

If G is compact and connected, and x is a so-called principal element of G, then G_x is equal to a maximal torus T in G, cf. [10, Cor. 3.3.2]. The isotropy type of a maximal torus T is equal to the set of principal elements in T, and $\mathrm{N}(T)/T$ is a finite reflection group, called the *Weyl group* of T, cf. [10, 3.7.2]. The corresponding orbit type is equal to the set of all principal elements in G, which is a dense open subset of G, and the corresponding orbit type in in the orbit space is a connected dense open subset of the orbit space, cf. [10, 2.8.5].

It was the strategy in [10, Ch. 3] that the structure theory of compact Lie groups is an application of the theory of proper group actions to the action of conjugation. ⊘

4 Smooth Structure on the Orbit Space

We keep the assumption that the action of G on M is proper, but not neccessarily free.

4.1 Differential Spaces

Definition 4.1 For every open subset V of $G\backslash M$, the function $f : V \to \mathbf{R}$ is called *smooth* if the function $\pi^*(f) := f \circ \pi : \pi^{-1}(V) \to \mathbf{R}$ is smooth. The space of smooth functions on V will be denoted by $\mathrm{C}^\infty(V)$. ⊘

It follows from the definition that π^* is an isomorphism of algebras from $\mathrm{C}^\infty(V)$ onto the algebra $\mathrm{C}^\infty(U)^G$ of all G-invariant smooth functions on the G-invariant open subset $U = \pi^{-1}(V)$ of M. The mapping $V \mapsto \mathrm{C}^\infty(V)$, where V ranges over all the open subsets of $G\backslash M$, defines a sheaf of functions on $G\backslash M$, which is called the *smooth structure* of $G\backslash M$. See for instance Gunning [13, §2] for basic facts about sheaves.

Definition 4.2 (Sikorski [31], [32]) A *differential space* is a pair $(Q, \mathrm{C}^\infty(Q))$, in which Q is a topological space, and $\mathrm{C}^\infty(Q)$ is a set of continuous real-valued functions on Q, with the following properties.

i) The sets $f^{-1}(I)$, with $f \in \mathrm{C}^\infty(Q)$ and I an open interval in $\mathbf{R}$, form a subbasis for the topology of Q.

ii) For every positive integer n, $F \in \mathrm{C}^\infty(\mathbf{R}^n)$, and $f_1, \ldots, f_n \in \mathrm{C}^\infty(Q)$, we have $F \circ f \in \mathrm{C}^\infty(Q)$, where we have written $f(q) = (f_1(q), \ldots, f_n(q)) \in \mathbf{R}^n$ for every $q \in Q$.

iii) If $f : Q \to \mathbf{R}$ and for every $q \in Q$ there is an open neighborhood V_q of q in Q and $f_q \in \mathrm{C}^\infty(V_q)$ such that $f|_{V_q} = f_q|_{V_q}$, then $f \in \mathrm{C}^\infty(Q)$. Here $g|_A$ denotes the restriction of a function g to a subset A of the domain of definition of g.

$\mathrm{C}^\infty(Q)$ is called the *differential structure* of $(Q, \mathrm{C}^\infty(Q))$. When there is no danger of confusion, one also refers to Q as the differential space. $\oslash$

Note that ii) implies that $\mathrm{C}^\infty(Q)$ is an algebra of functions.

Definition 4.3 If $(P, \mathrm{C}^\infty(P))$ and $(Q, \mathrm{C}^\infty(Q))$ are differential spaces, then a *smooth mapping* f from $(P, \mathrm{C}^\infty(P))$ to $(Q, \mathrm{C}^\infty(Q))$ is defined as a continuous mapping $\varphi : P \to Q$ such that $\varphi^*(\mathrm{C}^\infty(Q)) \subset \mathrm{C}^\infty(P)$. φ is called a *diffeomorphism* from $(P, \mathrm{C}^\infty(P))$ to $(Q, \mathrm{C}^\infty(Q))$ if φ is a homeomorphism from P onto Q and both φ and φ^{-1} are smooth. Note that this is equivalent to the condition that φ is a homeomorphism from P onto Q such that $\varphi^*(\mathrm{C}^\infty(Q)) = \mathrm{C}^\infty(Q)$, because this implies that

$$(\varphi^{-1})^*(\mathrm{C}^\infty(P)) = (\varphi^{-1})^*(\varphi^*(\mathrm{C}^\infty(Q))) = (\varphi \circ \varphi^{-1})^*(\mathrm{C}^\infty(Q)) = \mathrm{C}^\infty(Q).$$

$\oslash$

With the smooth mappings as morphisms, the differential spaces form a category.

Definition 4.4 Let $(Q, \mathrm{C}^\infty(Q))$ be a differential space and let S be any subset of Q. Then $\mathrm{C}^\infty(S)$ is defined as the set of all $f : S \to \mathbf{R}$ such that for every $s \in S$ there exists an open neighborhood V_s of s in S and an $f_s \in \mathrm{C}^\infty(Q)$, such that $f|_{V_s} = f_s|_{V_s}$. $\oslash$

Property iii) in Definition 4.2 just says that this $C^\infty(S)$ is equal to the $\mathrm{C}^\infty(Q)$ we started out with, if $S = Q$.

For any subset S of Q, provided with the induced topology, $(S, \mathrm{C}^\infty(S))$ is a differential space and the inclusion $S \to Q$ is a smooth mapping from $(S, C^\infty(S))$ to $(Q, \mathrm{C}^\infty(Q))$. Actually, $\mathrm{C}^\infty(S)$ is the smallest differential structure on S such that the inclusion $S \to Q$ is smooth.

Remark 4.5 The mapping $V \mapsto \mathrm{C}^\infty(V)$, where V ranges over all open subsets of Q, defines a sheaf of functions over Q. In the literature a *smooth structure* on a given speace is usually defined as a sheaf of functions of locally defined "smooth functions" on the space.

If $q \in Q$ and V is a neighborhood of q in Q, then it follows from i) in Definition 4.2 that there exists a positive integer n, open subset W of $\mathbf{R}^n$, and $f_1, \ldots, f_n \in \mathrm{C}^\infty(Q)$, such that $f^{-1}(W) \subset V$. There exists a cutoff function $F \in \mathrm{C}^\infty(\mathbf{R}^n)$ such that $F = 1$ on a neighborhood of $f(q)$ in $\mathbf{R}^n$, and the support of F is equal to a compact subset K of W. It follows from ii) in Definition 4.2 that $F \circ f \in \mathrm{C}^\infty(Q)$. Furthermore, the support of $F \circ f$ is equal to the closed subset $f^{-1}(K)$ of V, which is compact if V is compact. This shows that if the topological space Q is locally compact, then we have cutoff functions in $\mathrm{C}^\infty(Q)$.

If X is a topological space, $\mathcal{F}(X)$ is a space of functions on X and $\mathcal{V}$ is an open covering of X, then a *partition of unity in $\mathcal{F}(X)$ which is subordinate to $\mathcal{V}$* is a family $\chi_j \in \mathcal{F}(X)$, $j \in J$, with the following properties. First, for each $j \in J$, the support of χ_j is a compact subset of some $V_j \in \mathcal{V}$. Second, the supports of the χ_j form a locally finite family of compact subsets of X, and $\sum \chi_j = 1$ on X.

A Hausdorff topological space X is called *paracompact* if every open covering of X has a locally finite refinement. If X is Hausdorff and locally compact, then X is paracompact, if and only if every connected component of X is equal to the union of a countable collection of compact subsets, cf. Dieudonné [7]. Cutoff functions in $\mathcal{F}(X)$ now can be used to obtain partitions of unity as in Dieudonné [8, 6.1.4.(ii) and 16.4.1].

This means that if the topological space Q of the differential space $(Q, \mathrm{C}^\infty(Q))$ is Hausdorff, locally compact and paracompact, then for every open covering $\mathcal{V}$ of Q there exists a partition of unity in $\mathrm{C}^\infty(Q)$ which is subordinate to $\mathcal{V}$. In other words, the sheaf $V \mapsto \mathrm{C}^\infty(V)$ over Q is *fine*, cf. Gunning [13, p. 36].

The sheaf of analytic functions on an analytic manifold of positive dimension is not fine. In this respect differential spaces are quite different from analytic spaces. $\oslash$

If M is a smooth manifold, in particular if $M = \mathbf{R}^n$, and $C^\infty(M)$ denotes the space of all smooth functions on M, then $(M, \mathrm{C}^\infty(M))$ is a differential space. The property ii) then expresses that $f : Q \to \mathbf{R}^n$ is a smooth mapping from the differential space $(Q, \mathrm{C}^\infty(Q))$ to the differential space $(\mathbf{R}^n, \mathrm{C}^\infty(\mathbf{R}^n))$.

If S is a subset of a smooth manifold M, then S is an embedded smooth submanifold of M if and only if the differential space $(S, \mathrm{C}^\infty(S))$, defined as in Definition 4.4 with $Q = M$, is diffeomorphic as a differential space to a smooth manifold. For the "if" part, let S have the structure of a smooth manifold such that $\mathrm{C}^\infty(S)$ is equal to the space of smooth functions on S with respect to the manifold structure of S. This implies that the identity $S \to M$ is a smooth mapping from the manifold S into the manifold M, and also the manifold topology of S agrees with the restriction topology. If $s \in S$ and $\dim S = r$, then there exist smooth functions $f_1, \ldots, f_r$ on M, such that the $f_i|_S$ form a local system of coordinates for S in an open neighborhood of s in S. But this implies that the identity $S \to M$ has injective tangent mappings, and therefore it is a smooth embedding.

If S is a smooth submanifold of M, then $\mathrm{C}^\infty(S)$ is equal to the space of all smooth functions on S. For the proof one uses smooth cutoff functions in order to show that a smooth function on a neighborhood in S of a point in S agrees on some smaller neighborhood with a smooth function on M.

These considerations allow to refer, for any differential space $(Q,\, \mathrm{C}^\infty(Q))$, to $\mathrm{C}^\infty(Q)$ as the *space of smooth functions* on Q.

4.2 The Orbit Space as a Differential Space

We now return to the orbit space $G\backslash M$ of our proper G-acton on the smooth manifold M, with $\mathrm{C}^\infty(G\backslash M)$ as in Definition 4.1.

Lemma 4.6 *For every $q \in G\backslash M$ and every open neighborhood V of q in $G\backslash M$, there exists a cutoff function $\chi \in \mathrm{C}^\infty(G\backslash M)$ such that $\chi = 1$ on a neighborhood of q and $\chi = 0$ on the complement of a compact neighborhood of q in V.*

Let M be paracompact. Then $G\backslash M$ is paracompact, and there exists for every open covering $\mathcal{V}$ of $G\backslash M$ a partition of unity in $\mathrm{C}^\infty(G\backslash M)$, which is subordinate to $\mathcal{V}$.

Proof It follows from the tube theorem 3.8 with $U = \pi^{-1}(V)$ and $H = G_m$ that the mapping $H \cdot b \mapsto G \cdot b$ defines a homeomorphism from $H\backslash B$ onto $G\backslash U = V$.

Because H is a compact group acting linearly on E, averaging of an arbitrary inner product on E over H leads to an H-invariant inner product on E. There is an $\epsilon > 0$ such that $x \in E$, $\|x\| \leq \epsilon$ implies that $x \in B$. There exists a $\psi \in \mathrm{C}^\infty(\mathbf{R})$ such that $\psi = 1$ on a neighborhood of 0 in $\mathbf{R}$ and $\psi(r) = 0$ when $r \geq \epsilon$. Then $f(g,\, b) = \psi(\|b\|)$ defines a smooth and G-invariant function f on $G \times B$, which corresponds to a $g \in \mathrm{C}^\infty(V)$ of which the support is a compact subset of V. Extending g by zero in the complement of V in $G\backslash M$, we obtain the desired function χ.

The paracompactness of M implies that the Hausdorff space $G\backslash M$ is paracompact. For the proof, let C be a connected component of $G\backslash M$. Because G need not be connected, $\pi^{-1}(C)$ need not be connected, but $C = \pi(D)$ for every connected component D of M. Because M is paracompact, D is equal to the union of a countable family K_i of compact subsets. Because π is continuous, the $\pi(K_i)$ form a countable family of compact subsets of C, with union equal to $\pi(D) = C$.

The cutoff functions now can be used to obtain partitions of unity as in Remark 4.5. $\square$

Proposition 4.7 *With Definition 4.1, $(G\backslash M,\, \mathrm{C}^\infty(G\backslash M))$ is a differential space, and the orbit map $\pi : M \to G\backslash M$ is a smooth mapping from $(M,\, \mathrm{C}^\infty(M))$ to $(G\backslash M,\, \mathrm{C}^\infty(G\backslash M))$.*

For any open subset V of $G\backslash M$, the space $\mathrm{C}^\infty(V)$ in Definition 4.1 is equal to the space $\mathrm{C}^\infty(V)$ in Definition 4.4 with $Q = G\backslash M$ and $S = V$.

Proof If χ is a cutoff function as in Lemma 4.6, then $\chi^{-1}(]1/2,\, 3/2[\subset V$. This proves i) in Definition 4.2.

Let $f_1, \dots f_n \in \mathrm{C}^\infty(G\backslash M)$ and $F \in \mathrm{C}^\infty(\mathbf{R}^n)$. Then $f_j \circ \pi \in \mathrm{C}^\infty(M)^G$ hence $F \circ f \circ \pi \in \mathrm{C}^\infty(M)^G$, and therefore $F \circ f \in \mathrm{C}^\infty(G\backslash M)$. This proves ii) in Definition 4.2.

Now let $f : G\backslash M \to \mathbf{R}$ and assume that for every $q \in G\backslash M$ there exists an open neighborhood V_q of q in $G \setminus M$ and an $f_q \in \mathrm{C}^\infty(G\backslash M)$, such that $f|_{V_q} = f_q|_{V_q}$. Then $F \circ \pi$ is G-invariant, and equal to the smooth function $f_q \circ \pi$ on $\pi^{-1}(V_q)$. Because the $\pi^{-1}(V_q)$ form a covering of M, it follows that $f \circ \pi$ is smooth, $f \circ \pi \in \mathrm{C}^\infty(M)^G$, and therefore $f \in \mathrm{C}^\infty(G\backslash M)$. This proves iii) in Definition 4.2.

π is smooth because $\pi^*(\mathrm{C}^\infty(G\backslash M)) = \mathrm{C}^\infty(M)^G \subset \mathrm{C}^\infty(M)$.

Finally let V be an open subset of $G\backslash M$ and $f : V \to \mathbf{R}$. Write $U = \pi^{-1}(V)$. If f locally agrees with elements of $\mathrm{C}^\infty(G\backslash M)$ then we obtain as in the proof of iii) that $f \circ \pi \in \mathrm{C}^\infty(U)^G$, hence $f \in \mathrm{C}^\infty(V)$ according to Definition 4.1. Now assume conversely that $f : V \to \mathbf{R}$ and $f \circ \pi \in \mathrm{C}^\infty(U)^G$. For every $q \in V$ there exists, according to Lemma 4.6, a function $\chi \in \mathrm{C}^\infty(G\backslash M)$, such that the support of χ is a compact subset of V and $\chi = 1$ on a neighborhood of q in V. Define $f_q : G\backslash M \to \mathbf{R}$ by $f_q = \chi\, f$ in V and $f_q = 0$ in the complement of V in $G\backslash M$. Then $f_q = 0$ in the complement W of K in $G\backslash M$. V and W are open subsets of $G\backslash M$ and $V \cup W = G\backslash M$. It follows that $\pi^{-1}(V)$ and $\pi^{-1}(W)$ are open subsets of M and $\pi^{-1}(V) \cup \pi^{-1}(W) = M$. On $\pi^{-1}(V)$ we have $f_q \circ \pi = (\chi \circ \pi)\,(f \circ \pi)$, which is smooth and G-invariant. On $\pi^{-1}(W)$ we have $f_q \circ \pi = 0$, which is also smooth and G-invariant. The conclusion is that $f_q \in \mathrm{C}^\infty(G\backslash M)$. Because $f = f_q$ in a neighborhood of q, we have proved that $f \in \mathrm{C}^\infty(V)$ according to Definition 4.4 with $Q = G\backslash M$ and $S = V$. $\square$

Lemma 4.8 *In the notation of the tube theorem 3.8, the mapping $\varphi : b \mapsto \Phi(1,\, b) : B \to U$ induces a homeomorphism ϕ from $H\backslash B$ onto $G\backslash U$ and the restriction of $\varphi^* : \mathrm{C}^\infty(U) \to \mathrm{C}^\infty(B)$ to $\mathrm{C}^\infty(U)^G$ is an isomorphism from $\mathrm{C}^\infty(U)^G$ onto $\mathrm{C}^\infty(B)^H$.*

In other words, ϕ is a diffeomorphism from the differential space $H\backslash B$ onto the differential space $G\backslash U$, where $\mathrm{C}^\infty(H\backslash B) \simeq \mathrm{C}^\infty(B)^H$ and $\mathrm{C}^\infty(G\backslash U) \simeq \mathrm{C}^\infty(U)^G$.

Proof Write $i : b \mapsto (1,\, b) : B \to G \times B$ and ψ for the canonical projection from $G \times B$ onto $G \times_H B$. Then i induces a homeomorphism from $H\backslash B$ onto $(G \times H)\backslash(G \times B)$ and ψ induces a homeomorphism from $(G \times H)\backslash(G \times B)$ onto $G\backslash(G \times_H B)$. Because the G-equivariant diffeomorphism $\Phi : G \times_H B \to U$ induces a homeomorphism from $G\backslash(G \times_H B)$ onto $G\backslash U$, it follows that $\varphi = \Phi \circ \psi \circ i$ induces a homeomorphism from $H\backslash B$ onto $G\backslash U$.

If $f : U \to \mathbf{R}$, then $f \in \mathrm{C}^\infty(U)^G$ if and only if $g := \Phi^*(f) := f \circ \Phi \in \mathrm{C}^\infty(G \times_H B)$. The latter condition is equivalent to the condition that $h := \psi^*(g) \in \mathrm{C}^\infty(G \times B)^{G \times H}$. If π_2 denotes the projection from $G \times B$ onto the second factor B, then π_2^* is an isomorphism from $\mathrm{C}^\infty(B)$ onto $\mathrm{C}^\infty(G \times B)^G$, which restricts to an isomorphism $\pi_2^* : \mathrm{C}^\infty(B)^H \to \mathrm{C}^\infty(G \times B)^{G \times H}$. The restriction of $i^* : \mathrm{C}^\infty(G \times B) \to \mathrm{C}^\infty(B)$ to $\mathrm{C}^\infty(G \times B)^{G \times H}$ is equal to the inverse of π_2^*. We conclude that $\varphi^* = (\Phi \circ \psi \circ i)^* = i^* \circ \psi^* \circ \Phi^*$ is an isomorphism from $\mathrm{C}^\infty(U)^G$ onto $\mathrm{C}^\infty(B)^H$. $\square$

The isotropy group $H := G_m$ is compact and acts linearly on the finite-dimensional vector space E. It is a classical result of invariant theory that the algebra $\mathcal{P}(E)^H$ of all

H-invariant polynomial functions $f : E \to \mathbf{R}$ is finitely generated, i.e. there exist $n \in \mathbf{Z}_{\geq 0}$ and $p_1, \ldots, p_n \in \mathcal{P}(E)^H$ such that every $f \in \mathcal{P}(E)^H$ can be written in the form $f = F(p_1, \ldots, p_n)$ for some polynomial function $F : \mathbf{R}^n \to \mathbf{R}$. See Pflaum [26, Th. 4.4.2], also for further references to the literature. One can choose the p_i homogeneous of degrees $d_i > 0$. Even with a minimal set of such generators, called a *Hilbert basis*, in general the polynomial $F : \mathbf{R}^n \to \mathbf{R}$ is not unique, as there can exist nontrivial polyomial relations between the p_i. Because $\mathcal{P}(E)^H$ separates the H-orbits in E, the mapping $p : x \mapsto (p_1(x), \ldots, p_n(x))$ induces a continuous and injective mapping $\widetilde{p}$ from $H\backslash E$ onto the subset $p(E)$ of $\mathbf{R}^n$.

The Tarski-Seidenberg theorem, cf. Hörmander [15, Appendix A.2] for a nice proof, states that the image of any semi-algebraic set under a polynomial mapping is semi-algebraic, and therefore $p(E)$ is a semi-algebraic subset of $\mathbf{R}^n$. Because $p(t\, x) = (t^{d_1}\, p_1(x), \ldots, t^{d_n}\, p_n(x))$, the set $p(E)$ is quasi-homogeneous in the sense that $(t^{d_1}\, y_1, \ldots, t^{d_n}\, y_n) \in p(E)$ if $y \in p(E)$ and $t \in \mathbf{R}$. In particular $p(E)$ is contractible to the origin in $\mathbf{R}^n$. Averaging an arbitrary inner product on E over H, we obtain an H-invariant inner product β on E. Because $x \mapsto \beta(x, x) \in \mathcal{P}(E)^H$, there exists a polynomial $F : \mathbf{R}^n \to \mathbf{R}$ such that $\beta(x, x) = F(p(x))$ for every $x \in E$. This implies that the mapping $p : E \to \mathbf{R}^n$ is proper, which in turn implies that $p(E)$ is a closed subset of $\mathbf{R}^n$. Because the mapping $\widetilde{p} : H\backslash E \to p(E)$ is continuous, bijective and proper, it is a homeomorphism from the locally compact Hausdorff space $H\backslash E$ onto $p(E)$. By shrinking B (and therefore U) if necessary, we can arrange that $B = \{x \in E \mid \beta(x, x) < c\}$ for a suitable $c > 0$. Then $\widetilde{p}$ defines a homeomorphism from $G\backslash U \simeq H\backslash B$ onto the set $p(B) = \{y \in p(E) \mid F(y) < c\}$, which is an open semi-algebraic subset of the closed semi-algebraic subset $p(E)$ of $\mathbf{R}^n$.

The theorem of Schwarz [29], see also Mather [23], says that for any set $p_1, \ldots, p_n$ of generators of $\mathcal{P}(E)^H$ and every $f \in \mathrm{C}^\infty(E)^H$ there exists a $\varphi \in \mathrm{C}^\infty(\mathbf{R}^n)$ such that $f = \varphi \circ p$.

Lemma 4.9 *Let ϕ be as in Lemma 4.8. Then the mapping $\widetilde{p} \circ \phi^{-1}$ is a diffeomorphism from the differential space $G \setminus U$ onto the differential space $p(B)$, viewed as a subset of $\mathbf{R}^n$.*

Proof Let $f \in \mathrm{C}^\infty(p(B))$ and $b \in B$. Then there exists an open neighborhood Y of $p(b)$ in $\mathbf{R}^n$ and an $g \in \mathrm{C}^\infty(\mathbf{R}^n)$, such that $f = g$ on $p(B) \cap Y$. It follows that $f \circ p = g \circ p$ is smooth on the open neignborhood $p^{-1}(Y)$ of b in B. Because this holds for every $b \in B$, the conclusion is that $f \circ p \in \mathrm{C}^\infty(B)$. Because $f \circ p$ is invariant under $H = G_m$, it follows that $p^*(\mathrm{C}^\infty(p(B)) \subset \mathrm{C}^\infty(B)^H$.

On the other hand, the theorem of Schwarz states that $\mathrm{C}^\infty(B)^H \subset p^*(\mathrm{C}^\infty(E))$. If ι denotes the embedding $p(E) \to E$, $\iota^*(\mathrm{C}^\infty(E)) \subset \mathrm{C}^\infty(p(B))$, whereas $p = \iota \circ p$, hence

$$\mathrm{C}^\infty(B)^H \subset p^*(\mathrm{C}^\infty(E)) = p^*(\iota^*(\mathrm{C}^\infty(E))) \subset p^*(\mathrm{C}^\infty(p(B))).$$

We conclude that $p^*(\mathrm{C}^\infty(p(B))) = \mathrm{C}^\infty(B)^H$, or equivalently that $\widetilde{p}^*(\mathrm{C}^\infty(p(B))) = \mathrm{C}^\infty(H\backslash B)$. Because $\widetilde{p}$ is a homeomorphism from $H\backslash B$ onto $p(B)$, this completes the proof that $\widetilde{p}$ is a diffeomorphism from the differential space $H\backslash B$ onto the the differential space $p(B)$. $\square$

Definition 4.10 A differential space is Q called *subcartesian*, if Q is Hausdorff and locally diffeomorphic to a subset of a Cartesian space $\mathbf{R}^n$.

That is, if for every point $q \in Q$ there is a neighborhood V of q in Q, an $n \in \mathbf{Z}_{\geq 0}$, a subset S of $\mathbf{R}^n$, and a diffomorphism φ from $(V, \mathrm{C}^\infty(V))$ to $(S, \mathrm{C}^\infty(S))$. $\oslash$

Because the $G\backslash U$ in Lemma 4.9 form an open covering of $G\backslash M$, we conclude:

Corollary 4.11 *The differential space $G\backslash M$, with $\mathrm{C}^\infty(G\backslash M) \simeq \mathrm{C}^\infty(M)^G$, is subcartesian. More precisely, $G\backslash M$ has a covering by open subsets which are diffeomorphic as differential spaces with open subsets $p(B)$ of the closed semi-algebraic sets $p(E)$ in front of Lemma 4.9.*

Remark 4.12 The fact that $p(E)$ is a closed semi-algebraic subset of $\mathbf{R}^n$, means that $p(E)$ is determined by polynomial equalities and inequalities (of the type ≥ 0) between the coordinates of $\mathbf{R}^n$. The polynomial equalities represent the polynomial relations between the generators $p_1, \ldots, p_n$ of $\mathcal{P}(E)^H$. More information about the polynomial inequaties is given by Procesi and Schwarz [27]. $\oslash$

4.3 The Orbit Type Stratification of the Orbit Space

We now investigate the orbit types $G\backslash M_{[H]}$ in the differential space. $G\backslash M$. For a characterization of the connected components of the orbit types in $G\backslash M$ in terms of the smooth structure on $G\backslash M$ only, see Corollary 6.11.

If M_H denotes the isotropy type of $H = G_m$ in M, then it follows from b) in Theorem 3.9 that $\pi(M_H) = G\backslash M_{[H]}$, and there is a unique smooth manifold structure on $G\backslash M_{[H]}$ such that $\pi|_{M_H} : M_H \to G\backslash M_{[H]}$ is a principal $\mathrm{N}(H)/H$-bundle.

In the situation of the tube theorem 3.8, Let $g \in G$, $e \in E$, $h \in H$. Then $g \cdot (1, e) = (g, e) = (1\, h^{-1}, h \cdot e)$ if and only if $g = h^{-1}$ and $h \cdot e = e$. That is, the isotropy subgroup in G for the G-action on $G \times_H E$ of the H-orbit $[(1, e)]$ through $(1, e)$ is equal to the isotropy subgroup in H of e for the H-action on E. This isotropy group is equal to H, i.e. $[(1, e)]$ belongs to the isotropy type of H in $G \times_H E$, if and only if $h \cdot e = e$ for every $h \in H$. Let $E^H = \{e \in E \mid h \cdot e = e\}$ denote the set of common fixed points for the action of H on E. Then $\pi(M_H) = G\backslash M_{[H]}$, in combination with Lemma 4.8, yields that $G\backslash U_{[H]} = \phi(H\backslash(B \cap E^H))$. Here $H\backslash(B \cap E^H)$ denotes the image of $B \cap E^H$ under the canonical projection from B onto the H-orbit space $H\backslash B$.

Because H acts on E by means of linear transformations, E^H is a linear subspace of E. If F denotes the orthogonal complement of E^H in E with respect to the H-invariant inner product β, then F is H-invariant and $F^H = F \cap E^H = \{0\}$. The mapping $(x, y) \mapsto x + y$ is a linear isomorphism from $E^H \times F$ onto E, which is H-equivariant if we let $h \in H$ act on $E^H \times F$ by sending (x, y) to $(x, h \cdot y)$.

We note in passing that the common β-orthogonal complement in E of all vectors of the form $h \cdot e - e$, $h \in H$, $e \in E$, is equal to E^H. Therefore F is equal to the linear span of all the vectors $h \cdot e - e$, $h \in H$, $e \in E$, which implies that F is independent of the choice of the H-invariant inner product β in E.

If $x_1, \dots, x_l$ is a coordinate system in E^H, then every $p \in \mathcal{P}(E^H \times F)$ can be written in a unique fashion as a finite sum

$$p(x,\, y) = \sum_{\alpha} x^{\alpha}\, q_{\alpha}(y),$$

in which $\alpha = (\alpha_1,\, \alpha_n)$, $\alpha_i \in \mathbf{Z}_{\geq 0}$, x^{α} denotes the monomial

$$x^{\alpha} = \prod_{i=1}^{l} {x_i}^{\alpha_i},$$

and q_{α} is a polynomial on F. We have $p \in \mathcal{P}(E^H \times F)^H$ if and only if $q_{\alpha} \in \mathcal{P}(F)^H$ for every multi-index α. This shows that a Hilbert basis of $\mathcal{P}(E)^H \simeq \mathcal{P}(E^H \times F)^H$ is given by $x_1, \dots, x_l, q_1, \dots q_m$, in which the q_j form a Hilbert basis of $\mathcal{P}(F)^H$.

It follows that under the map $p : E \to \mathbf{R}^n = \mathbf{R}^l \times \mathbf{R}^m$ the set $B \cap E^H$ is sent to an open subset of $\mathbf{R}^l \times \{0\}$, which is a smooth l-dimensional submanifold of $\mathbf{R}^n$. It follows that the differential subspace $p(B \cap E^H)$ of the differential space $p(E)$ is a smooth l-dimensional manifold. In view of the identification of $H\backslash(B \cap E^H) \simeq p(B \cap E^H)$ with the orbit type $G\backslash U_{[H]}$ in $G\backslash U$, and the identification in Lemma 4.9 of the differential space $G \setminus U$ with the differential subspace $p(B)$ of $\mathbf{R}^n$, we conclude that each connected component C of each orbit type $G\backslash M_{[H]}$ in $G\backslash M$ is a smooth manifold, when regarded as a differential subspace of the differential space $G\backslash M$. Moreover, this manifold structure is equal to the one for which $\pi|_{M_H} : M_H \to G\backslash M_{[H]}$ is a smooth principal fibration, cf. b) in Theorem 3.9.

The orbit types for the action of H in $E \simeq E^H \times F$ are of the form $E^H \times R$, in which R is an orbit type for the action of H in F. Furthermore, if $t \in \mathbf{R}$, $t \neq 0$, then the multiplication by t in F is a linear isomorphism of F which commutes with the action of H in F, and it follows that $t\, R = R$. We conclude that if S denotes the unit sphere in F with respect to β, then $R \mapsto R \cap S$ is a bijective mapping from the set of all orbit types in F which are not equal to the origin onto the set of all orbit types for the action of H on S. It follows by induction on the dimension of S, that there are only finitely many orbit types for the action of H on a compact manifold S, and we conclude that there are only finitely many orbit types for the action of H on E.

It follows from the quasihomogeneity of the mapping $p : E \to \mathbf{R}^n = \mathbf{R}^l \times \mathbf{R}^m$ that $p(E)$ is invariant under the transformations

$$(x_1,\, \dots,\, x_l,\, q_1,\, \dots,\, q_m) \mapsto \left(x_1,\, \dots,\, x_l,\, t^{d_1}\, q_1,\, \dots,\, t^{d_m}\, q_m\right), \quad t \in \mathbf{R}_{>0},$$

in which $d_j = \deg q_j$. Note that the right hand sides are different for different t's in $\mathbf{R}_{>0}$ when $q \neq 0$. Therefore each orbit type in $p(E)$ which is different from $\mathbf{R}^l \times \{0\}$ is equal to the Cartesian product of $\mathbf{R}^l$ with a submanifold of $\mathbf{R}^m$ of dimension at least equal to one. It follows that each orbit type near a given orbit type in $G\backslash M$ has a strictly larger dimension, and we have proved:

Proposition 4.13 *The connected components of the orbit types in $G\backslash M$, viewed as differential subspaces of the differential space $G\backslash M$, are smooth manifolds. These manifolds define a stratification $\mathcal{S}$ of $G\backslash M$, called the* orbit type stratification *of the orbit space.*

The following simple but important observations are due to Bierstone [2, Lemma 2.12].

Let $q \in \mathcal{P}(F)^H$ be homogeneous of degree one, which implies that q is an element of the dual space F^* of all linear forms on F. Note that $\beta : f \mapsto (g \mapsto \beta(f,\, g))$ is a bijective linear mapping from F onto F^*. Then the H-invariance of q implies that $\beta^{-1}(q) \in F^H = \{0\}$, which in turn implies that $q = 0$. This shows that $d_j := \deg q_j \geq 2$ for every $1 \leq j \leq m$. Because $y \mapsto \beta(y,\, y)$ is H-invariant and homogenous of degree 2, it is a linear combination of the q_j with $d_j = 2$, which means that we can arrange that $q_1(y) = \beta(y,\, y)$, $y \in F$.

Let $S = \{y \in F \mid q_1(y) = 1\}$ denote the unit sphere in F with respect to β and let, for each $2 \leq j \leq m$, C_j be the maximum of the $|q_j(y)|$ such that $y \in S$. For any $y \in F \setminus \{0\}$ and $1 \leq j \leq m$, write $t = d_1(y)^{1/2}$, which implies that $t^{-1}\, y \in S$. Because $q_j(y) = q_j(t\,(t^{-1}\, y)) = t^{d_j}\, q_j(t^{-1}\, y)$ and $|q_j(t^{-1}\, y)| \leq C_j$, we conclude that $|q_j(y)| \leq d_1(y)^{d_j/2}\, C_j$, which inequality also holds for $y = 0$. This leads to the estimate

$$p(E) \subset \left\{ (x,\, q) \in \mathbf{R}^l \times \mathbf{R}^m \mid q_1 \geq 0 \text{ and } |q_j| \leq C_j\, {q_1}^{d_j/2} \text{ for every } 2 \leq j \leq m \right\} \tag{4.1}$$

for the subset $p(E)$ of $\mathbf{R}^n$.

Lemma 4.14 *Let I be an open interval in $\mathbf{R}$, $\gamma : I \to p(E)$, and write $\gamma(t) = (x(t),\, q(t))$, $t \in I$. Assume that $0 \in I$, $q_1(0) = 0$, and that $t \mapsto q(t)$ is diffferentiable at $t = 0$ as a function from I to $\mathbf{R}^m$. Then $q'(0) = 0$.*

Proof It follows from $\gamma(I) \subset p(E)$ and (4.1) that $q_1(t) \geq 0$ for every $t \in I$. Because $q_1(0) = 0$, we obtain in view of the variational principle that $q_1'(0) = 0$, i.e. $q_1(t)/t \to 0$ as $t \to 0$. On the other hand, if $2 \leq j \leq m$, then $\gamma(I) \subset p(E)$ and (4.1) imply that

$$|q_j(t)|/|t| \leq C_j\, q_1(t)^{d_j/2}/|t| = C_j\, |q_1(t)/t|^{d_j/2}\, |t|^{(d_j/2)-1},$$

where the right hand side converges to zero as $t \to 0$ becuase $q_1(t)/t \to 0$ and $(d_j/2) - 1 \geq 0$. This proves that $q_j'(0) = 0$. $\square$

It follows from Lemma 4.14 that if S is a C^1 submanifold of $\mathbf{R}^n = \mathbf{R}^l \times \mathbf{R}^m$, $S \subset p(E)$, and $0 \in S$, then $\mathrm{T}_0\, S \subset \mathbf{R}^l \times \{0\}$. In particular $\dim S \leq l$. Because all strata in $p(E)$ different from $\mathbf{R}^l \times \{0\}$ have dimension $> l$, this in turn implies that no union of $\mathbf{R}^l \times \{0\}$ with different strata in $p(E)$ can be a C^1 manifold through the origin. This leads to the following

Corollary 4.15 *The orbit type stratification of $G\backslash M$ is minimal in the sense that no union of different strata can be a connected smooth manifold in the differential space $G\backslash M$.*

If $G\backslash M$ is connected, then the differential space $G\backslash M$ is a smooth manifold if and only if there is only one orbit type.

Note that actually no union of different strata can be a connected C^1 manifold in the differential space $G\backslash M$, where we replace the C^∞ differential structure by a C^1 differential structure in an obvious way. Also, if $G\backslash M$ is a connected C^1 manifold, then there is only one orbit type and $G\backslash M$ is a C^∞ manifold.

Remark 4.16 A theorem of Bierstone [1, Th. A] states that in the local model $p(B) \subset \mathbf{R}^n$, cf. Lemma 4.9, the orbit type stratification coincides with the primary semi-analytic stratification of the semi-algebraic subset $p(B)$ of $\mathbf{R}^n$. This implies that in the local model $p(B) \subset \mathbf{R}^n$ the orbit type stratification is a Whitney stratification. $\oslash$

Remark 4.17 If C is a connected component of $G\backslash M$, then $\pi^{-1}(C)$ is an open and closed G-invariant subset of M (connected if G is connected), and in all the discussions we may replace M and $G\backslash M$ by $\pi^{-1}(C)$ and C, respectively. Therefore we may assume without loss of generality that $G\backslash M$ is connected.

If $G\backslash M$ is connected, then $G\backslash M$ contains a unique open orbit type, which moreover is a dense subset of $G\backslash M$, and it is connected, cf. [10, Thm. 2.8.5]. It is called the *prinicpal orbit type* $G\backslash M^{\text{princ}}$ in $G\backslash M$. Because $G\backslash M^{\text{princ}}$ is a connected orbit type, it is a stratum of the orbit type stratification of $G\backslash M$, and it is the unique one which is an open subset of $G\backslash M$. Also, because $G\backslash M^{\text{princ}}$ is dense in $G\backslash M$, it approaches every point of every other stratum.

Because every other stratum has a strictly smaller dimension than $G\backslash M^{\text{princ}}$, the dimension of $G\backslash M^{\text{princ}}$ is called the *dimension* $\dim(G\backslash M)$ *of* $G\backslash M$. $\oslash$

4.4 Cohomology of the Orbit Space

A smooth differential form ω on M is called *basic* if

i) ω is G-invariant in the sense that $g_M^*\omega = \omega$ for every $g \in G$, and

ii) ω is horizontal in the sense that $\mathrm{i}_{X_M}\,\omega = 0$ for every $X \in \mathfrak{g}$.

We write $\Omega^p_{\text{basic}}(M)$ for the space of all basic smooth differential forms of degree p on M. For smooth functions = differential forms of degree zero, we only have condition i), and therefore $\Omega^0_{\text{basic}}(M) = \mathrm{C}^\infty(M)^G \simeq \mathrm{C}^\infty(M\backslash G)$.

It follows from i) that, for every $X \in \mathfrak{g}$,

$$0 = \mathcal{L}_{X_M}\omega = \mathrm{d}\ \mathrm{i}_{X_M}\,\omega + \mathrm{i}_{X_M}\ \mathrm{d}\omega,$$

in which $\mathcal{L}_u$ and i_u denotes the Lie derivative with respect to and the inner product with the vector field u, respectively. The second identity is the homotopy identity for the Lie derivative. Therefore $\mathrm{i}_{X_M}\ \mathrm{d}\omega = 0$ if $\mathrm{i}_{X_M}\,\omega = 0$. Because the exterior derivative commutes with pull-backs by means of any smooth mapping we have that $\mathrm{d}\omega$ is G-invariant if ω is G-invariant, and therefore $\mathrm{d}\omega$ is basic if ω is basic. The cohomology of the complex of the basic differential forms on M with the exterior derivative as the boundary operator is called the *basic cohomology* of M. That is,

$$\mathrm{H}^p_{\text{basic}}(M) := \ker \mathrm{d}_p \,/\, \mathrm{d}_{p-1}(\Omega^{p-1}_{\text{basic}}(M)), \quad \mathrm{d}_r = \mathrm{d} : \Omega^r_{\text{basic}}(\Omega) \to \Omega^{r+1}_{\text{basic}}(\Omega).$$

The $V \mapsto \Omega^p_{\text{basic}}(\pi^{-1}(V))$, where V ranges over the open subsets of $G\backslash M$, defines a sheaf of vector spaces over $G\backslash M$. Because $\Omega^p_{\text{basic}}(\pi^{-1}(V))$ is a module over $\Omega^0(\pi^{-1}(V)) \simeq \mathrm{C}^\infty(V)$, the partitions of unity of Lemma 4.6 make that the sheaf $V \mapsto \Omega^p_{\text{basic}}(\pi^{-1}(V))$ is fine.

In the local model of Thm. 3.8, the radial contractions in B lead to a Poincaré lemma for basic differential forms. That is, if $p > 0$, $\omega \in \Omega^p_{\text{basic}}(U)$ and $\mathrm{d}\omega = 0$, then there exists an $\alpha \in \Omega^{p-1}_{\text{basic}}(U)$, such that $\omega = \mathrm{d}\alpha$. In other words, the sequence of sheaves

$$0 \to \mathbf{R} \to \Omega^0_{\text{basic}} \xrightarrow{\mathrm{d}_0} \Omega^1_{\text{basic}} \xrightarrow{\mathrm{d}_1} \Omega^2_{\text{basic}} \xrightarrow{\mathrm{d}_2} \ldots$$

is exact. Because the sheaves Ω^p_{basic} are fine, this exact sequence is a fine resolution of the sheaf of locally constant functions on $G\backslash M$. Because a fine resolution of a sheaf $\mathcal{S}$ induces a canonical isomorphism of the cohomology of the sheaf $\mathcal{S}$ with the cohomology of the operator d, cf. [13, §3, Thm. 3], we arrive at the following conclusion.

Theorem 4.18 (Koszul [18]) *For a proper action of a Lie group G on a smooth manifold M, the Čech chomology $\mathrm{H}^p(G\backslash M, \mathbf{R})$ of the orbit space $G\backslash M$ is canonically isomorphic to the basic cohomomolgy $\mathrm{H}^p_{\text{basic}}(M)$ of the G-space M.*

When $G = \{1\}$, hence $G\backslash M = M$, this is the De Rham theorem for smooth manifolds. In this sense Theorem 4.18 is a generalization of the De Rham theorem.

It is a general fact in algebraic topology that the Čech cohomology is canonically isomorphic to any other cohomology of a complex which satisfies the Steenrod axioms, such as the singular cohomology with values in $\mathbf{R}$. If the orbit space does not possess strata S of codimension one, i.e. such that $\dim S = \dim(G\backslash M) - 1$, cf. Remark 4.17, then the basic cohomology of the G-space M is also canonically isomorphic to what is called the *de Rham cohomology* of $G\backslash M$, cf. Pflaum [26, Thm. 5.3.5].

5 Dynamical System with Symmetry

We assume that we have a vector field v and an action of a Lie group G on the manifold M as in Section 1 and 2, respectively. Recall that I_m is the interval of definition of the maximal solution of (1.1) which starts at m.

Lemma 5.1 *The following conditions are equivalent.*

i) *The vector field v is invariant under the action of G, in the sense that*

$$(\mathrm{T}_m\, g_M)\; v(m) = v(g_M(m)) \tag{5.1}$$

for every $g \in G$ and $m \in M$.

ii) *The flow of v commutes with the action of G in the sense that $I_m = I_{g\cdot m}$ and*

$$g_M\left(\mathrm{e}^{t\,v}(m)\right) = \mathrm{e}^{t\,v}(g_M(m)) \tag{5.2}$$

for every $g \in G$, $m \in M$, and $t \in I_m$.

Proof Suppose that i) holds. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\, g_M\left(\mathrm{e}^{t\,v}(m)\right) = (\mathrm{T}_m\, g_M)\; v\left(\mathrm{e}^{t\,v}(m)\right) = v\left(g_M\left(\mathrm{e}^{t\,v}(m)\right)\right)$$

and $g_M\left(\mathrm{e}^{0\,v}(m)\right) = g_M(m)$. This shows that $\gamma(t) := g_M\left(\mathrm{e}^{t\,v}(m)\right)$ is equal to the solution of (1.1) with $\gamma(0) = g_M(m)$. This implies ii).

Conversely i) follows by differentiating ii) with respect to t at $t = 0$. □

Definition 5.2 We will say that the action of G is a *symmetry* of the dynamical system defined by the vector field v, or that v is *invariant* under the action of G, if one of the equivalent conditions in Lemma 5.1 holds. ⊘

In the remainder of these notes we assume that the group action is a symmetry for the dynamical system.

The first basic observation is that (5.2) implies that, for each $m \in M$ and $t \in I_m$, the time t flow $\mathrm{e}^{t\,v}$ maps the G-orbit through m, in a G-equivariant way, onto the G-orbit through $\mathrm{e}^{t\,v}(m)$. It follows that the time t flow induces a transformation Φ^t in the orbit space, which is the unique mapping $\Phi^t : \pi(D_t) \to G\backslash M$ such that

$$\Phi^t \circ \pi = \pi \circ \mathrm{e}^{t\,v} \quad \text{on } D_t. \tag{5.3}$$

Let $t \in \mathbf{R}$. The set D_t is an open G-invariant subset of of M and therefore the domain of definition $\pi(D_t)$ of Φ^t is an open subset of $G\backslash M$, equal to $G\backslash M$ if the vector field v in M is complete. The mapping Φ^t is a homeomorphism from $\pi(D_t)$ onto $\pi(D_t)$, with Φ^{-t} as its inverse. Furthermore (1.2) implies the group property that if $s,\, t \in \mathbf{R}$, $x \in \pi(D_s)$, $\Phi^s(x) \in \pi(D_t)$, then $x \in \pi(D_{s+t})$ and $\Phi^t(\Phi^s(x)) = \Phi^{t+s}(x)$. In other words, the transformations Φ^t define a continuous flow in the orbit space $G\backslash M$, which is called the *reduced flow*, or the *reduced dynamical system*.

In the case that the reduced system in the orbit space $G\backslash M$ is simpler than the system in M, the strategy will be to first analyse the reduced system and then try to obtain conclusions about the flow in M from the properties of the flow in $G\backslash M$. This last step is called *reconstruction*.

Lemma 5.3 *If the action is proper and free, then there is a unique vector field $w = \pi_*(v)$ on $G\backslash M$, such that $w(\pi(m)) = (\mathrm{T}_m\,\pi)\; v(m)$ for every $m \in M$. The vector field w is smooth and the flow Φ^t in $G\backslash M$ is equal to the flow $\mathrm{e}^{t\,w}$ of the vector field w.*

Proof If $g \in G$, then $\pi \circ g_M = \pi$ and therefore we have for every $m \in M$ that $\mathrm{T}_{g\cdot m}\,\pi \circ \mathrm{T}_m\, g_M = \mathrm{T}_m\,\pi$. Applying $\mathrm{T}_{g\cdot m}\,\pi$ to (5.1), we therefore obtain that $(\mathrm{T}_m\,\pi)\, v(m) = (\mathrm{T}_{g\cdot m}\,\pi)\; v(g\cdot m)$. Defining $w(\pi(m))$ as the common value of all $(\mathrm{T}_{g\cdot m}\,\pi)\; v(g\cdot m)$, $g \in G$, we obtain the first statement in the lemma.

It follows from (5.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Phi^t(m)(\pi(m)) = \frac{\mathrm{d}}{\mathrm{d}t}\,\pi \circ \mathrm{e}^{t\,v}(m) = \left(\mathrm{T}_{\mathrm{e}^{t\,v}(m)}\,\pi\right)\; v(\mathrm{e}^{t\,v}(m)) = w(\pi \circ \mathrm{e}^{t\,v}(m)) = w(\Phi^t(m)),$$

which in combination with $\Phi^0(m)(\pi(m)) = \pi(m)$ shows that $\Phi^t(\pi(m)) = \mathrm{e}^{t\,w}(\pi(m))$. $\square$

Because for a proper and free action the dimension of $G\backslash M$ is smaller than the dimension of M if the Lie group G has a positive dimension, there is a good chance that the dynamical system in $G\backslash M$ is simpler than the one in M.

5.1 Reconstruction in a Principal Fiber Bundle

For the reconstruction we follow the description of Marsden [22, Sec. 6.2]. Let $\delta : I \to G\backslash M$, in which I is an open interval around 0, be the solution curve of

$$\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} = w(\delta(t)), \quad t \in I \tag{5.4}$$

such that $\delta(0) = \pi(m)$. In order to find the solution curve $\gamma : I \to M$ of (1.1) such that $\gamma(0) = m$, we start with an arbitrary smooth curve $\beta : I \to M$ which is a lift of δ in the sense that $\pi(\beta(t)) = \delta(t)$ for every $t \in I$, and satisfies $\beta(0) = m$.

Because $\pi \circ \gamma$ is a solution curve of (5.4) which starts at $\pi(m)$, we have $\pi \circ \gamma = \delta = \pi \circ \beta$, which means that for every $t \in I$ there is an element $g(t) \in G$ such that $\gamma(t) = g(t)_M(\beta(t)$. Because $\pi : M \to G\backslash M$ is a principal fibration, the element $g(t)$ is uniquely determined and $t \mapsto g(t) : I \to G$ is a smooth curve in G, with $g(0) = 1$, the identity element of G. Differentiating $\gamma(t) = g(t) \cdot \beta(t) = g(t)_M(\beta(t))$ with repect to t and using the sum rule for differentiation, we obtain that

$$\left(\mathrm{T}_{\beta(t)}\, g(t)_M\right)\, v(\beta(t)) = v(\gamma(t)) = \gamma'(t) = \mathrm{T}_{\beta(t)}\, g(t)_M\, \left(\beta'(t) + X(t)_M(\beta(t))\right), \tag{5.5}$$

where in the first identity we have used (5.1). The last term comes from differentiating

$$g(t+h)_M(\beta(t)) = g(t)_M \circ \left(g(t)^{-1}\, g(t+h)\right)_M (\beta(t))$$

with respect to h at $h = 0$, and we have written

$$X(t) := \left.\frac{\mathrm{d}}{\mathrm{d}h}\, g(t)^{-1}\, g(t+h)\right|_{h=0} \in \mathfrak{g}. \tag{5.6}$$

The equation (5.5) is equivalent to

$$X(t)_M(\beta(t)) = v(\beta(t)) - \beta'(t), \quad t \in I. \tag{5.7}$$

The curve $t \mapsto X(t) : I \to \mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ of G is given by the formula (5.10) below, from which we read off that this curve is smooth.

The reconstruction is completed by solving the ordinary differential equation (5.6) for $g(t)$, in which the smooth curve $t \mapsto X(t) : I \to \mathfrak{g}$ is given by (5.10), with the initial condition $g(0) = 1$. For any Lie group G with Lie algebra $\mathfrak{g}$ and any continuous curve $t \mapsto X(t) : I \to \mathfrak{g}$, in which I is an open interval around 0 in $\mathbf{R}$, there is a unique C^1 solution $t \mapsto g(t) : I \to G$

of (5.6) such that $g(0) = 1$. See for instance the first statement in [10, Prop. 1.13.4], which is also valid if we replace right multiplications by left multiplications. If $\mathfrak{g}$ is abelian, then

$$g(t) = \mathrm{e}^{\int_0^t X(s)\ \mathrm{d}s}.$$

For noncommutative groups the solution $g(t)$ is the *product integral* over the interval $[0, t]$ of $s \mapsto X(s)$ introduced by Volterra [35].

In order to obtain an explicit formula for $X(t)$, we can use a *connection form* for the principal fiber bundle $M \to G\backslash M$, which is a $\mathfrak{g}$-valued smooth one-form θ on M which has the following properties.

a) For every $m \in M$ and $X \in \mathfrak{g}$ we have

$$\theta_m(X_M(m)) = X. \tag{5.8}$$

b) θ is equivariant in the sense that for every $m \in M$, $v \in \mathrm{T}_m M$ and $g \in G$ we have

$$\theta_{g\cdot m}(\mathrm{T}_m\, g_M(v)) = (\mathrm{Ad}\, g)(\theta_m(v)). \tag{5.9}$$

The linear subspace $H_m = \ker \theta_m$ of $\mathrm{T}_m M$ is complementary to the fiber $T_m(G \cdot m)$ of the fiber through m, and is called the *horizontal subspace* defined by the connection form θ. Together the horizontal subspaces H_m, $m \in M$, define a smooth vector subbundle H. (A smooth vector subbundle of $\mathrm{T}\, M$ is also called a *distribution* in M.) It follows from a) and b) that H is invariant under the induced action of G on $\mathrm{T}\, M$. Conversely, if H is a G-invariant smooth vector subbundle which is complementary to the tangent spaces of the fibers, then there is a unique connection form θ on M such that $H = \ker \theta$. In general a smooth vector subbundle which is complementary to the fibers of a fibration is called an *infinitesimal connection* for the fibration. Therefore, giving a connection form is equivalent to giving a G-invariant infinitesimal connection.

Piecing together connection forms in local trivializations of the bundle by means of a partition of unity in the base space $G\backslash M$, cf. Lemma 4.6, one can construct a connection form θ in any paracompact principal fiber bundle M. If we now apply θ to (5.7), then we obtain the explicit formula

$$X(t) = \theta_{\beta(t)}\,(v(\beta(t)) - \beta'(t)) \tag{5.10}$$

for $X(t)$ in terms of the lift $\beta(t)$ in M of the curve $\delta(t)$ in $G\backslash M$.

The formula (5.10) simplifies to $X(t) = \theta_{\beta(t)}(v(\beta(t)))$ if β is a so-called *horizontal lift* of δ, i.e. $\pi \circ \beta = \delta$ and, for every $t \in I$, $\beta'(t) \in H_{\beta(t)} = \ker \theta_{\beta(t)}$. With a proof similar to the proof of the first statement in [10, Prop. 1.13.4], one can show that, for any $m \in M$, every smooth curve $\delta : I \to G\backslash M$ with $\delta(0) = \pi(m)$ has a unique horizontal lift $\beta : I \to M$ such that $\beta(0) = m$. This completes the proof that the above reconstruction procedure always can be carried out. This also shows that

Corollary 5.4 *The domain of definition of the maximal solution of* (5.4) *which starts at* $\pi(m)$ *is equal to the domain of definition of the maximal solution of* (1.1) *which starts at* m.

In the G-invariant open subset M' of M where $v(m) \notin \mathrm{T}_m\, G{\cdot}m$, corresponding to the domain in $G\backslash M$ where w has no zeros, there exists a connection form θ such that $\theta_m(v(m)) = 0$ for every $m \in M'$. Such connection forms can be constructed by piecing together such connection forms in local trivializations by means of a partition of unity in the base space. In this case the equation (5.10) simplifies further to $X(t) = 0$, and the solutions γ of (1.1) such that $\pi \circ \gamma = \delta$ are just the horizontal lifts of δ.

In general the equation $\beta'(t) \in H_{\beta(t)}$ for horizontal lifts is a system of ordinary diffferential equations with no obvious explicit formulas for the solutions, which means that in general the above reconstruction procedure is not so explicit as one might wish. However, in a local trivialization $\pi^{-1}(U) = U \times G$ over some open subset U of $G\backslash M$, one will take the $H_{(u,\, g)} := \mathrm{T}_u\, U \times \{0\}$ as the horizontal spaces. In this case the horizontal lifts of δ are just the curves $t \mapsto (\delta(t),\, g)$ in which g is a constant element of G. The disadvantage of this procedure is that it will only work for the whole curve δ if $\delta(t) \in U$ for all $t \in I$.

5.2 Non-free Actions

We now apply the reduction in Lemma 3.3, of non-free actions to free actions, to our dynamical system with symmetry. We begin with the following observation, which holds without any assumption on the action. Recall that I_m is the interval of definition of the maximal solution of (1.1) which starts at m.

Lemma 5.5 *Each path component C of each isotropy type is invariant under the flow of v, in the sense that if $m \in C$, then $\mathrm{e}^{t\, v}(m) \in C$ for every $t \in I_m$.*

Proof We have

$$g \in G_m \Longleftrightarrow g_M(m) = m \Longleftrightarrow g_M(\mathrm{e}^{t\, v}(m)) = \mathrm{e}^{t\, v}(g_M(m)) = \mathrm{e}^{t\, v}(m) \Longleftrightarrow g \in G_{\mathrm{e}^{t\, v}(m)},$$

where we have used (5.2) in the middle statement. In other words, $G_m = G_{\mathrm{e}^{t\, v}(m)}$, which means that the v-flow leaves each isotropy type invariant. Because $t \mapsto \mathrm{e}^{t\, v}(m)$ is a continuous curve in M, it follows that the v-flow actually leaves every path component of every isotropy type invariant. □

Remark 5.6 For compact G, Lemma 5.5 follows from Field [11, Prop. A2]. ⊘

There is only one isotropy type if and only if there is a closed normal subgroup H of G such that $G_m = H$ for every $m \in M$, in which case one replaces the action of G on M by the free action of the Lie group G/H on M. For this reason Lemma 5.5 can be viewed as a *law of conservation as a consequence of non-freeness of the action.* Such a law of conservation is stronger if the path component C of the isotropy type is smaller.

Remark 5.7 If m is a path-isolated point of M_{G_m} in the sense that $C = \{m\}$, then Lemma (5.5) implies that m is an equilibrium point of the v-flow, i.e. $v(m) = 0$.

This may be compared with the theorem of Michel [24, Th. 1], which states that for a smooth action of a compact Lie group G, every smooth G-invariant function is critical at the point m, if and only if $G \cdot m$ is an isolated element of the orbit type $G\backslash M_{[G_m]}$ of $G \cdot m$ in the orbit space $G\backslash M$. (Michel's theorem actually holds for proper Lie group actions.) $\oslash$

5.3 Reconstruction for Proper Non-free Actions

It follows from a) in Theorem 3.9 that each isotropy type M_H is a smooth embedded submanifold of M, where it is allowed that different connected components C (= the path components of M_H) have different dimensions. Because C is invariant under the v-flow, v is tangent to C and the restriction to C of the v-flow is equal to the flow of the vector field $v|_C$ in C. Because $\mathrm{N}(H)$ is a subgroup of G which preserves v (commutes with the v-flow), the free and proper action of $\mathrm{N}(H)/H$ on M_H is a symmetry of the restriction of the dynamical system to M_H. It follows from b) in Theorem 3.9 and Lemma 5.3, that there is a unique smooth vector field $w = \pi_*(v)$ in the orbit type $G\backslash M_{[H]}$ in the orbit space $G\backslash M$, such that $w(\pi(m)) = (\mathrm{T}_m\,\pi)\,v(m)$ for every $m \in M_H$, where $\pi = \pi|_{M_H} : M_H \to G\backslash M_{[H]}$ denotes the canonical projection in Theorem 3.9, b).

Because $\mathrm{e}^{t\,w_H} = \Phi^t = \mathrm{e}^{t\,w_{H'}}$ if H' is conjugate to H', i.e. if $[H'] = [H]$, the vector field w_H in $G\backslash M_{[H]}$ does not depend on the choice of $H \in [H]$, which is why in the sequel we will delete the subscript H from the vector field w. In other words, we have a smooth vector field $w = \pi_*(v)$ in each orbit type in the orbit space, such that, for each $m \in M$ and each $t \in I_m$, we have that $\mathrm{e}^{t\,v}(m)$ belongs to the same orbit type as m and $\pi(\mathrm{e}^{t\,v}(m)) = \Phi^t(\pi(m)) = \mathrm{e}^{t\,w}(\pi(m))$. We will call $w = \pi_*(v)$ the vector field on $G\backslash M_{[H]}$ *induced by* v.

The reconstruction of Subsection 5.1 can now be applied to each orbit type, in the following way. For each $p \in G\backslash M$ there exists a unique conjugacy class α of compact Lie subgroups of G such that p belongs to the orbit type $G \setminus M_\alpha$ in the orbit space $G\backslash M$. Let $\delta : I \to G \setminus M_\alpha$ be the maximal solution curve of (5.4) in $G \setminus M_\alpha$ such that $\delta(0) = p$. Choose $H \in \alpha$, i.e. $\alpha = [H]$. In view of b) in Theorem 3.9, we can apply the reconstruction procedure of Subsection 5.1 with M and G replaced by M_H and $\mathrm{N}(H)/H$, respectively. This yields for any $m \in M_H$ the maximal solution $\gamma : I \to M_H$ in M_H of (1.1) such that $\gamma(0) = m$. Note that $\delta(I)$ has a paracompact open neighborhood in $G \setminus M_\alpha$, over which there exists a connection form.

Because of Lemma 5.5 we have that $I = I_m$, the interval of definition of the maximal solution γ of (1.1) in M which starts at m. Because of Corollary 5.4, $\pi \circ \gamma$ is the maximal solution curve of (1.1) with M and v replaced by $G\backslash M_{[H]}$ and w, respectively. Therefore we have proved:

Corollary 5.8 *Let $\gamma : I \to M$ be a maximal solution of* (1.1)*, $0 \in I$, and $\gamma(0) = m$. Write $H = G_m$. Then $\gamma(I)$ is contained in the isotropy type M_H in M and $\pi \circ \gamma(I)$ is contained in the orbit type $G\backslash M_{[H]}$ in $G\backslash M$. If γ runs out of every compact subset of M in a finite time, then $\pi \circ \gamma$ runs out of every compact subset of $G\backslash M_{[H]}$ in the same time.*

Remark 5.9 Corollary 6.9 below implies that actually $\pi \circ \gamma$ runs out of every compact subset of $G\backslash M$ in the same time. Note that this conclusion is trivial if G is compact, because then the canonical projection $\pi : M \to G\backslash M$ is a proper mapping. $\oslash$

For any $g \in G$ the maximal solution of (1.1) in M which starts at $g \cdot m$ is given by $t \mapsto g \cdot \gamma(t) : I_m \to M$, cf. (5.2). In view of a) in Lemma 3.7 we obtain in this way all the solutions of (1.1) in $M_{[H]} = \pi^{-1}(G \backslash M_{[H]})$. Because M is equal to the union of the disjoint orbit types $M_{[H]}$, this leads to the reconstruction of all solutions of (1.1) in M. The reconstructions can be made as explicit as the reconstruction in Subsection 5.1, with M and G replaced by M_H and $\mathrm{N}(H)/H$, respectively, can be made explicit.

5.4 A Local Model for the Invariant Vector Fields

We keep the assumption that the action is proper, but not necessarily free.

Let $m \in M$. Identifying U with $G \times_H B$ as in the tube theorem 3.8, we view v as a G-invariant vector field in $G \times_H B$. Because $H = G_m$ is compact and the adjoint representation is continuous, $\mathrm{Ad}\, H$ is a compact group of linear transformations in $\mathfrak{g}$, and averaging an arbitrary inner product in $\mathfrak{g}$ over $\mathrm{Ad}\, H$ we obtain an $\mathrm{Ad}\, H$-invariant inner product β on $\mathfrak{g}$, i.e. $\mathrm{Ad}\, h$ is a β-orthogonal linear transformation in $\mathfrak{g}$ for every $h \in H$. The Lie algebra $\mathfrak{h} = \mathfrak{g}_m$ of H is an $\mathrm{Ad}\, H$-invariant linear subspace of $\mathfrak{g}$, and it follows that the β-orthogonal linear complement $\mathfrak{q}$ of $\mathfrak{h}$ is H-invariant.

If $X \in \mathfrak{g}$, we denote by X^{L} the vector field on G which is invariant under the action of G on itself by means of multiplications from the left, and which is equal to X at the identity element. That is, for any $g \in G$, $X^{\mathrm{L}}(g)$ is equal to the derivative with respect to t at $t = 0$ of $g\, \mathrm{e}^{t\, X}$. If $\mathfrak{q}$ is any $\mathrm{Ad}\, H$-invariant linear complement of $\mathfrak{h}$ in $\mathfrak{g}$, then we write

$$\mathfrak{q}^{\mathrm{L}}(g) := \left\{ X^{\mathrm{L}}(g) \mid X \in \mathfrak{q} \right\}, \quad g \in G.$$

Then

$$C_{(g,\, b)} := \mathfrak{q}^{\mathrm{L}}(g) \times E \subset \mathrm{T}_g\, G \times E \simeq \mathrm{T}_{(g,\, b)}(G \times B), \quad (g,\, b) \in G \times B,$$

is a linear subspace of $\mathrm{T}_{(g,\, b)}(G \times B)$ which is complementary to the tangent space at $(g,\, b)$ of the H-orbit through $(g,\, b)$. In other words, the $C_{(g,\, b)}$, $(g,\, b) \in G \times B$, form an infinitesimal connection $\mathcal{C}$ for the principal H-bundle $G \times B$. For any $h \in H$, $X \in \mathfrak{g}$, $t \in \mathbf{R}$ we have

$$g\, \mathrm{e}^{t\, X}\, h^{-1} = g\, h^{-1}\, h\, \mathrm{e}^{t\, X}\, h^{-1} = g\, h^{-1}\, \mathrm{e}^{t\, (\mathrm{Ad}\, h)\, X}, \tag{5.11}$$

where $(\mathrm{Ad}\, h)\, X \in \mathfrak{q}$ if $X \in \mathfrak{q}$. This shows that the infinitesimal connection $\mathcal{C}$ is H-invariant. It is obviously also G-invariant.

Let $\mathcal{C}$ be any H-invariant and G-invariant connection in the G-invariant principal H-bundle $G \times B$. Write $\pi : G \times B \to G \times_H B$ for the canonical projection. Every G-invariant vector field v on $G \times_H B$ has a unique horizontal lift with respect to $\mathcal{C}$, a vector field v^{hor} on $G \times B$ such that $v^{\mathrm{hor}}(g,\, b) \in C_{(g,\, b)}$ for every $(g,\, b) \in G \times B$ and such that π intertwines v^{hor} with v in the sense that

$$\left(\mathrm{T}_{(g,\, b)}\, \pi\right)\, v^{\mathrm{hor}}(g,\, b) = v(\pi(g,\, b))$$

for every $(g,\, b) \in G \times B$. v^{hor} is smooth and G-invariant if v is smooth and G-invariant, respectively. Actually, the mapping $v \mapsto v^{\mathrm{hor}}$ defines an isomorphism from the space of all smooth G-invariant vector fields on $G \times_H B$ onto the space of all smooth horizontal vector fields in $G \times B$ which are both H-invariant and G-invariant.

The fact that v^{hor} is G-invariant means that there are smooth mappings $X : B \to \mathfrak{g}$ and $u : B \to E$, such that $v^{\mathrm{hor}}(g,\, b) = (X(b)^{\mathrm{L}},\, u(b))$ for every $(g,\, b) \in G \times B$. The fact that v^{hor} is horizontal then is equivalent to the condition that $X(b) \in \mathfrak{q}$ for every $b \in B$. Finally the condition that v^{hor} is H-invariant is equivalent to the conditions that the mapping $X : B \to \mathfrak{q}$ is H-equivariant and the vector field u in B is H-invariant. The H-equivariance of the mapping $X : B \to \mathfrak{q}$ means that $X(h \cdot b) = (\mathrm{Ad}\, h)(X(b))$ for every $h \in H$, $b \in B$, which makes sense because the linear subspace $\mathfrak{q}$ of $\mathfrak{g}$ is $\mathrm{Ad}\, H$-invariant. The equivariance of $X : B \to \mathfrak{q}$ is based on (5.11). This leads to the following conclusion.

Proposition 5.10 *The mapping which assigns to each H-equivariant smooth mapping $X : B \to \mathfrak{g}$ and each smooth H-invariant vector field u on B the vector field $(g,\, b) \mapsto (X(b)^{\mathrm{L}},\, u(b))$ leads to an isomorphism from*

$$\mathrm{C}^\infty(B,\, \mathfrak{q})^H \times \mathcal{X}^\infty(B)^H \tag{5.12}$$

onto the space of all smooth G-invariant vector fields on $G \times_H B$, and therefore onto the space $\mathcal{X}^\infty(U)^G$ of all smooth G-invariant vector fields on the G-invariant open neighborhood U of m in M. Here $\mathrm{C}^\infty(B,\, \mathfrak{q})^H$ and $\mathcal{X}^\infty(B)^H$ denotes the space of all H-equivariant smooth mappings from B to $\mathfrak{q}$ and H-invariant smooth vector fields on B, respectively.

The dynamical system in $G \times B$ defined by v^{hor} is determined by the system of ordinary differential equations

$$\frac{\mathrm{d}g}{\mathrm{d}t} = X(b)^{\mathrm{L}}, \quad \frac{\mathrm{d}b}{\mathrm{d}t} = u(b). \tag{5.13}$$

The second equation in (5.13) is the general H-invariant dynamical system in B defined by a vector field w which does not depend on the first component g.

Substituting a maximal solution $b : I \to B$ of $\mathrm{d}b/\,\mathrm{d}t = u(b)$ in the first equation in (5.13), we arrive at $\mathrm{d}g/\,\mathrm{d}t = \xi(t)^{\mathrm{L}}$, in which $\xi(t) := X(b(t))$ is a smooth curve in $\mathfrak{g}$, which actually runs inside $\mathfrak{q}$. This equation has a unique solution $g_1 : I \to G$ such that $g_1(0) = 1$, and then the general solution is given by $g(t) = g(0)\, g_1(t)$. Note that $g_1(t)$ is obtained as a Volterra product integral over the interval from 0 to t of $s \mapsto \xi(s)$. Under the G-equivariant diffeomorphism Φ from $G \times_H B$ onto U, the solution curve $(g_1(t),\, b(t))$ is mapped to the solution curve $t \mapsto g_1(t) \cdot \beta(t)$ of (1.1), in which the curve $\beta(t) := \Phi(\pi(1,\, b(t)))$ runs in the slice through the point m which has been used in the proof of the tube theorem 3.8.

Remark 5.11 For compact Lie groups G acting linearly on $\mathbf{R}^n$, the decompositions (5.13) and $\mathrm{e}^{t\,v}(\beta(0)) = g_1(t) \cdot \beta(t)$ have been found by Krupa [20, Th. 2.1 and Th. 2.2]. To these statements we have added the G_m-equivariance of the mapping $X : B \to \mathfrak{q}$, and the isomorphism of $\mathcal{X}^\infty(U)^G$ with (5.12). ⊘

The projection from $G \times B$ onto the second factor leads to an identification of the G-orbits in $U \simeq G \times_H B$ with the H-orbits in B, and therefore to an identification of the open neighborhood $G \setminus U$ of $G \cdot m$ in $G \backslash M$ with the space $H \backslash B$ of the H-orbits in B. This leads to an identification of the flow in $G\backslash U$ induced by the v-flow in U with the flow in $H\backslash B$ induced by the u-flow in B. In this way the study of the flow in the orbit space can locally be reduced to the case when the symmetry group is a compact group acting by means of linear transformations in an invariant open neighborhood of the origin in a vector space.

Lemma 5.12 *Let M be paracompact. For every stratum S of the orbit type stratification in $G\backslash M$ as in Proposition 4.13, every subset K of S which is closed in M, and every smooth vector field w in S, there exists a G-invariant smooth vector field v on M such that $w = \pi_*(v)$ on K. Here $\pi_*(v)$ is the smooth vector field in S as defined in Subsection 5.3.*

Proof In the notation of Subsection 4.3, the isotropy type $M_{[H]} \cap V$ corresponds to $G \times_H (E^H \cap B)$, in which E^H denotes the set of all fixed points in E of all $h \in H$. E^H is a linear subspace of E. Let π denote the linear projection from E onto E^H along the H-invariant linear complement F of E^H in E. Then for any smooth vector field $w : E^H \to E^H$ the vector field $u : w \circ \pi : E \to E^H \subset E$ is an H-invariant vector field u in E, which in turn, with the choice $X(b) \equiv 0$, leads to a G-invariant smooth vector field v in U, such that $\pi_*(v) = w$ on E^H.

It follows that for every $s \in S$ there is an open neighborhood V_s of s in $G\backslash M$ and a smooth G-invariant vector field v_s on $\pi^{-1}(V_s)$, such that $\pi_*(v_s) = w$ on $S \cap V_s$. Let $\mathcal{V}$ be the open covering of $G\backslash M$ which consists of the V_s, $s \in S$, together with the complement K^{c} of K in $G\backslash M$. Let χ_j be a partition of unity in $\mathrm{C}^\infty(G\backslash M)$ which is subordinate to $\mathcal{V}$, cf. Lemma 4.6. For each j we have that either the support $\operatorname{supp}\chi_j$ of χ_j is a compact subset of some $V_{s(j)}$, or of K^{c}. In the first case we define $v_j = (\chi_j \circ \pi)\, v_{s(j)}$ on $\pi^{-1}(V_{s(j)})$ and $v_j = 0$ in the complement of $\pi^{-1}(\operatorname{supp}\chi_j)$ in M. In the second case we take $v_j = 0$. Then the v_j are smooth G-invariant vector fields on M, their supports form a locally finite collection of subsets of M, and hence their sum v is a smooth G-invariant vector field on M.

If $s \in K$, then $s \notin K^{\mathrm{c}}$, hence $v_j = (\chi_j \circ \pi)\, v_{s(j)}$ on $\pi^{-1}(s)$ and therefore $(\pi_*(v_j))(s) = \chi_j(s)\, w(s)$ for every j. Summing over j yields that $(\pi_*(v))(s) = w(s)$. □

For the subsets K in Lemma 5.12 one can take any compact subset of S. Or $K = S$, if the stratum S is a closed subset of $G\backslash M$, a stratum of locally minimal dimension.

6 Smooth Vector Fields in the Orbit Space

We keep the assumption that the action is proper, but not necessarily free.

6.1 Derivations and Vector Fields in a Differential Space

Definition 6.1 If $\mathcal{A}$ is an algebra over $\mathbf{R}$, then a *derivation* of $\mathcal{A}$ is a linear mapping $D : \mathcal{A} \to \mathcal{A}$ such that $D(f\,g) = (Df)\,g + f\,(Dg)$ for all $f,\, g \in \mathcal{A}$. The set of all derivations of $\mathcal{A}$ will be denoted by $\mathrm{Der}(\mathcal{A})$, which is a Lie algebra with the brackets $[D,\, D'] := D \circ D' - D' \circ D$. ⊘

Let $(Q,\, \mathrm{C}^\infty(Q))$ be a differential space as in Definition 4.2. If Q is a smooth manifold and w is a smooth vector field on Q, i.e. a smooth section of the tangent bundle $\mathrm{T}Q$ of Q, then for each $f \in \mathrm{C}^\infty(Q)$ the partial derivative $\partial_v f(x) := \langle v(x),\, \mathrm{d}f(x)\rangle$ of f in the direction of the vector field v defines an element $\partial_v f \in \mathrm{C}^\infty(Q)$. The mapping $D := \partial_v : \mathrm{C}^\infty(Q) \to \mathrm{C}^\infty(Q)$ is a derivation of $\mathrm{C}^\infty(Q)$. Moreover, the mapping $\partial : v \mapsto \partial_v$ is an isomorphism of Lie algebras from $\mathcal{X}^\infty(Q)$ onto $\mathrm{Der}(\mathrm{C}^\infty(Q))$. Actually, in many expositions smooth vector fields are *defined* as derivations of $\mathrm{C}^\infty(Q)$, after which tangent spaces are introduced and it is shown that the derivations correspond to smooth sections of the tangent bundle.

Definition 6.2 Let $(Q,\, \mathrm{C}^\infty(Q))$ be a differential space, not necessarily equal to a smooth manifold, and $D \in \mathrm{Der}(\mathrm{C}^\infty(Q))$ a derivation. Then an *integral curve* of D is a smooth mapping $\gamma : I \to Q$, in which I is an interval in $\mathbf{R}$, such that

$$\frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t} = (D(f))(\gamma(t))$$

for every $f \in \mathrm{C}^\infty(Q)$ and every $t \in I$. In other words, if $(f \circ \gamma)' = (Df) \circ \gamma$ for every $f \in \mathrm{C}^\infty(Q)$. Or, still more abstractly, if $\,' \circ \gamma^* = \gamma^* \circ D$, where $'$ denotes the derivation $f \mapsto f'$ of $\mathrm{C}^\infty(I)$. $\oslash$

Proposition 6.3 *Let $(Q,\, \mathrm{C}^\infty(Q))$ be a locally compact and subcartesian differential space, and let $D \in \mathrm{Der}(\mathrm{C}^\infty(Q))$. Assume that for every $q \in Q$ there is an* open *interval I in $\mathbf{R}$ and an integral curve $\gamma : I \to Q$ of D such that $\gamma(0) = q$. Then we have the following conclusions.*

For each $q \in Q$ there is a unique integral curve $\gamma = \gamma_q : I_q \to Q$ of D, defined on a maximal open interval I_q arond 0 in $\mathbf{R}$ such that $\gamma(0) = q$. The set Ω of all $(t,\, q) \in \mathbf{R} \times Q$ such that $t \in I_q$ is open in $\mathbf{R} \times Q$, and the mapping $\Phi : (t,\, q) \mapsto \gamma_q(t)$ is a smooth mapping from Ω to Q. For each $t \in \mathbf{R}$, the set Q_t of all $q \in Q$ such that $t \in I_q$ is an open subset of Q, and the mapping $\Phi^t : q \mapsto \gamma_q(t)$, called the flow of D after time t*, is a smooth mapping from Q_t to Q. If $s,\, t \in \mathbf{R}$, $q \in Q_s$ and $\Phi^s(q) \in Q_t$, then $q \in Q_{s+t}$, and $\Phi^t(\Phi^s(q)) = \Phi^{s+t}(q)$. It follows that Φ^t is a diffeomorphism from Q_t onto Q_{-t}, with inverse equal to Φ^{-t}.*

One has $s := \sup I_q < \infty$, if and only if for each compact subset K of Q there exists an $\epsilon > 0$ such that $\Phi^t(q) \notin K$ for every $t \in\,]s - \epsilon,\, s[\, \cap I_q$. Similarly $i := \inf I_q > -\infty$, if and only if for each compact subset K of Q there exists an $\epsilon > 0$ such that $\Phi^t(q) \notin K$ for every $t \in\,]i,\, i + \epsilon[\, \cap I_q$.

Proof For a subcartesian differential space $(Q,\, \mathrm{C}^\infty(Q))$, the condition that the topological space Q is locally compact is equivalent to the following condition. For every $q \in Q$ we have, for every (some) diffeomorphism ψ from an open neighborhood V of q in Q onto a subset $\psi(V)$ in $\mathbf{R}^n$ as in Definition 4.10, that $\psi(V)$ is a locally closed subset of $\mathbf{R}^n$.

In order to simplify the notation, we identitfy V with a locally closed subset of $\mathbf{R}^n$. On $\mathbf{R}^n$ we use the coordinate functions x_i, $1 \leq i \leq n$. Then, for each i, $D(x_i) \in \mathrm{C}^\infty(V)$, which implies that there exists an open neighborhood U_i of q in $\mathbf{R}^n$ and $\delta_i \in \mathrm{C}^\infty(U_i)$, such that $D(x_i) = \delta_i|_{U_i}$. If U denotes the intersection of the U_i, $1 \leq i \leq n$, then U is an open neighborhood of q in $\mathbf{R}^n$, and $\delta : U \to \mathbf{R}^n$ is a smooth vector field in U such that $D(x_i) = \delta_i|_U$ for every $1 \leq i \leq n$. By shrinking U further if necessary, we can arrange that $V \cap U$ is a closed subset of U.

If $\gamma : I \to V \cap U$ is an integral curve of D, then

$$\frac{\mathrm{d}\gamma_i(t)}{\mathrm{d}t} = \frac{\mathrm{d}x_i(\gamma(t))}{\mathrm{d}t} = (D(x_i))(\gamma(t)) = \delta_i(\gamma(t)), \quad 1 \leq i \leq n,$$

shows that γ is a solution of (1.1) with $v = \delta$. Therefore the local uniqueness theorem for solutions of (1.1) with smooth v implies a local uniqueness theorem for the integral curves of D with prescribed initial values. This implies that for each $q \in V \cap U$ there is a unique integral curve $\gamma : I \to V \cap U$ of D, defined on a maximal open interval I arond 0 in $\mathbf{R}$ such that $\gamma(0) = q$. Moreoever, if $\widetilde{I}$ denotes the maximal domain of definition of $\widetilde{\gamma} : t \mapsto \mathrm{e}^{t\,\delta}(q)$, then $I \subset \widetilde{I}$ and $\gamma = \widetilde{\gamma}|_I$.

We will now prove that $I = \widetilde{I}$. Suppose that $s := \sup I \in \widetilde{I}$. Write $r = \widetilde{\gamma}(s) \in U$. Then

$$r = \lim_{t \uparrow s} \widetilde{\gamma}(t) = \lim_{t \uparrow s} \gamma(t) \in V \cap U,$$

because $\gamma(t) \in V \cap U$ for every $t \in I$ and $V \cap U$ is closed in U. The assumption in the proposition implies that we have an open interval J around 0 in $\mathbf{R}$ and an integral curve $\beta : J \to V \cap U$ of D such that $\beta(0) = r$. According to the previous paragraph,

$$\beta(t - s) = \mathrm{e}^{(t-s)\,\delta}(r) = \mathrm{e}^{(t-s)\,\delta}(\mathrm{e}^{s\,\delta}(q)) = \mathrm{e}^{t\,\delta}(q) = \gamma(t)$$

for all $t \in I \cap (s + J)$. It follows that γ and $t \mapsto \beta(t - s)$ piece together to an integral curve of D on $I \cup (s + J)$, in contradiction with the maximility of the interval I. A similar argument shows that $i := \inf I \notin \widetilde{I}$, and the conclusion is that $I = \widetilde{I}$.

If $\widetilde{\Omega} \subset \mathbf{R} \times U$ denotes the domain of definition of the flow $\widetilde{\Phi} : \widetilde{\Omega} \to U$ of $\delta \in \mathcal{X}^\infty(U)$, and $\Phi : \Omega \to V \cap U$ is defined as in the proposition with Q replaced by $V \cap U$, then we have just proved that $\Omega = \widetilde{\Omega} \cap (\mathbf{R} \times (V \cap U))$ and $\Phi = \widetilde{\Phi}|_\Omega$. Because $\widetilde{\Omega}$ is an open subset of $\mathbf{R} \times U$ and the mapping $\widetilde{\Phi} : \widetilde{\Omega} \to U$ is smooth, it follows that Ω is an open subset of $\mathbf{R} \times (V \cap U)$ and the mapping $\Phi : \Omega \to V \cap U$ is smooth as a mapping between differential spaces. Because these are local properties, we have proved that the set Ω in the proposition is an open subset of $\mathbf{R} \times Q$ and the mapping $\Phi : \Omega \to Q$ is smooth. The other statements in the proposition now follow in the same way as for the solutions of (1.1) for a smooth vector field v on a smooth manifold M. $\square$

The integral curves of the derivation D in Proposition 6.3 define a smooth flow Φ in Q in the same way as the solutions of (1.1) for a smooth vector field v on a smooth manifold define a smooth flow in M. Note that the definition of the Φ^t, $t \in \mathbf{R}$, implies that

$$\left.\frac{\mathrm{d}f(\Phi^t(q))}{\mathrm{d}t}\right|_{t=0} = (D(f))(q), \quad f \in \mathrm{C}^\infty(Q),\ q \in Q,$$

or more abstractly, D is equal to the derivative of $(\Phi^t)^*$ with respect to t at $t = 0$. This is the motivation for the following

Definition 6.4 Let $(Q, \mathrm{C}^\infty(Q))$ be a locally compact and subcartesian differential space, and let $D \in \mathrm{Der}(\mathrm{C}^\infty(Q))$. Then D is called a *flow derivation* of $(Q, \mathrm{C}^\infty(Q))$, if for every $q \in Q$ there is an *open* interval I in $\mathbf{R}$ and an integral curve $\gamma : I \to Q$ of D such that $\gamma(0) = q$. The set of all flow derivations of $(Q, \mathrm{C}^\infty(Q))$ will be denoted by $\mathcal{X}(Q, \mathrm{C}^\infty(Q))$. ⊘

Definition 6.5 (Pflaum [26, 2.1.5 and Prop. 2.2.6]) Let $(Q, \mathrm{C}^\infty(Q))$ be a differential space and $\mathcal{S}$ a stratification of Q as in Definition 3.11, with the smooth manifold M replaced by the differential space Q. A *stratified vector field* on $(Q, \mathcal{S})$ is a mapping w which assigns to each $S \in \mathcal{S}$ a smooth vector field $w_S \in \mathcal{X}^\infty(S)$ on S.

If $f \in \mathrm{C}^\infty(Q)$ and $S \in \mathcal{S}$, then $f|_S \in \mathrm{C}^\infty(S)$, hence $\partial_{w_S}(f|_S) \in \mathrm{C}^\infty(S)$. The functions $\partial_{w_S}(f|_S)$, $S \in \mathcal{S}$, piece together to a function $\partial_w f : Q \to \mathbf{R}$. The stratified vector field w is called *smooth* if $\partial_w f \in \mathrm{C}^\infty(Q)$ for every $f \in \mathrm{C}^\infty(Q)$. The set of all smooth stratified vector fields on $(Q, \mathcal{S})$ will be denoted by $\mathcal{X}^\infty(Q, \mathcal{S})$. ⊘

Lemma 6.6 *Let $(Q, \mathrm{C}^\infty(Q), \mathcal{S})$ be a stratified differential space. Then the mapping $\partial : w \mapsto \partial_w$ is an injective homomorphism of Lie algebras from $\mathcal{X}^\infty(Q, \mathcal{S})$ to $\mathrm{Der}(\mathrm{C}^\infty(Q))$. If the differential space $(Q, \mathrm{C}^\infty(Q))$ is locally compact and subcartesian, then*

$$\partial(\mathcal{X}^\infty(Q, \mathcal{S})) \subset \mathcal{X}(Q, \mathrm{C}^\infty(Q)). \tag{6.1}$$

Proof We prove the injectivity of ∂. Let $w \in \mathcal{X}^\infty(Q, \mathcal{S})$, $\partial_w = 0$, $S \in \mathcal{S}$, $f \in \mathrm{C}^\infty(S)$ and $q \in S$. Then there exists an open neighborhood U of q in Q and a function $g \in \mathrm{C}^\infty(Q)$ such that $f|_{S \cap U} = g|_{S \cap U}$. Because $\partial_w g = 0$, we have $\partial_{w_S}(f|_{S \cap U}) = 0$, and because this holds for every $q \in S$, we have $\partial_{w_S} f = 0$. Because this holds for every $f \in \mathrm{C}^\infty(S)$ and because we have for the smooth manifold S that $\partial : \mathcal{X}^\infty(S) \to \mathrm{Der}(\mathrm{C}^\infty(S))$ is an isomorphism, it follows that $w_S = 0$. Because this holds for every $S \in \mathcal{S}$, the conclusion is that $w = 0$.

Let $w \in \mathcal{X}^\infty(Q, \mathcal{S})$ and $q \in Q$. Then there exists an $S \in \mathcal{S}$ such that $q \in S$. For the smooth vector field w_S on S, we have a smooth solution $\gamma : I \to S$ of (1.1) with v replaced by w_S, where I is an open interval around 0 in $\mathbf{R}$ and $\gamma(0) = q$. For any $f \in \mathrm{C}^\infty(Q)$, we have that $f|_S$ is a smooth function on the smooth manifold S, hence

$$\begin{aligned}\frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t} &= \langle \gamma'(t), \mathrm{d}(f|_S)(\gamma(t)) \rangle = \langle w_S(\gamma(t)), \mathrm{d}(f|_S)(\gamma(t)) \rangle \\ &= (\partial_{w_S}(f|_S))(\gamma(t)) = (\partial_w f)(\gamma(t)).\end{aligned} \tag{6.2}$$

Therefore γ is an integral curve of ∂_w such that $\gamma(0) = q$. Because this holds for every $q \in Q$, this shows that $\partial_w \in \mathcal{X}(Q, \mathrm{C}^\infty(Q))$ according to Definition 6.4. □

Corollary 6.7 *Let $(Q,\, \mathrm{C}^\infty(Q),\, \mathcal{S})$ be a stratified differential space, with Q locally compact and subcartesian. Let $w \in \mathcal{X}^\infty(Q,\, \mathcal{S})$, $S \in \mathcal{S}$, and let $\gamma : I \to S$ be a maximal solution curve in S of* (1.1) *with $v = w_S$. If γ runs out of every compact subset of S in a finite time, then γ runs out of every compact subset of Q in the same time.*

Proof γ is an integral curve of $D := \partial_w$. Let $\widetilde{\gamma} : \widetilde{I} \to Q$ be the maximal integral curve of D which agrees with γ on I, cf. Proposition 6.3. Suppose that $s := \sup I \in \widetilde{I}$. Then $\gamma(t) = \widetilde{\gamma}(t)$ converges to $\widetilde{q} := \widetilde{\gamma}(s) \in Q$ as $t \uparrow s$, and the maximality of γ implies that $\widetilde{q} \notin S$. Let $\widetilde{S} \in \mathcal{S}$ be the stratum which contains $\widetilde{q}$, we have $\widetilde{S} \cap S = \emptyset$ because $\widetilde{S} \neq S$. Let $\beta : J \to \widetilde{S}$ be the maximal solution of (1.1) with $v = w_{\widetilde{S}}$ such that $\beta(s) = \widetilde{q}$. Because β is an integral curve of D and $\beta(s) = \widetilde{q} = \widetilde{\gamma}(s)$, we have that $J \subset \widetilde{I}$ and $\beta = \widetilde{\gamma}$ on J. In combination with $\widetilde{\gamma} = \gamma$ on I this yields that $\beta = \gamma$ on the non-empty interval $I \cap J$. Because for every $t \in I \cap J$ we have $\gamma(t) \in S$ and $\beta(t) \in \widetilde{S}$, this leads to a contradiction with $S \cap \widetilde{S} = \emptyset$. It therefore follows that $s = \sup \widetilde{I}$ and the last statement in Proposition 6.3 now implies that $\gamma(t)$ runs out of every compact subset of Q as $t \uparrow s$. $\square$

6.2 In the Orbit Space

We now turn to the case that Q is equal to the orbit space $G\backslash M$ for a proper action of a Lie group G on a smooth manifold M, with the differential structure $\mathrm{C}^\infty(G\backslash M)$ as defined in Definition 4.1 and Proposition 4.7. In this subsection, $\mathcal{S}$ denotes the orbit type stratification of $G\backslash M$, as introduced in Proposition 4.13.

Let $v \in \mathcal{X}^\infty(M)^G$. In Subsection 5.3 we found, for each orbit type $G\backslash M_{[H]}$ in $G\backslash M$, a smooth vector field $w_{[H]}$ on $G\backslash M_{[H]}$ such that $\pi_*(v_{[H]}) = w_{[H]}$. Here $v_{[H]}$ denotes the restriction of v to the orbit type $M_{[H]}$ in M. The $w_{[H]}$ together define a stratified vector field $w = \pi_*(v)$ on $(G\backslash M,\, \mathcal{S})$.

Lemma 6.8 *We have*

$$\pi_*(\mathcal{X}^\infty(M)^G) \subset \mathcal{X}^\infty(G\backslash M,\, \mathcal{S}), \tag{6.3}$$

and the mapping π_* is a homomorphism of Lie algebras from $\mathcal{X}^\infty(M)^G$ to $\mathcal{X}^\infty(G\backslash M,\, \mathcal{S})$.

Proof Let $v \in \mathcal{X}^\infty(M)^G$ and $f \in \mathrm{C}^\infty(G\backslash M)$. Then $\pi^*(f) \in \mathrm{C}^\infty(M)^G$, and it follows that

$$\pi^*(\partial_{\pi_*(v)} f) = \partial_v(\pi^*(f)) \in \mathrm{C}^\infty(M)^G.$$

But this implies that $\partial_{\pi_*(v)} f \in \mathrm{C}^\infty(G\backslash M)$. Because this holds for every $f \in \mathrm{C}^\infty(G\backslash M)$, this proves that the stratified vector field $\pi_*(v)$ on $(G\backslash M,\, \mathcal{S})$ is smooth. $\square$

Corollary 6.9 *Let v be a smooth G-invariant vector field on M and let $\gamma : I \to M$ be a maximal solution of* (1.1). *If γ runs out of every compact subset of M in a finite time, then $\pi \circ \gamma$ runs out of every compact subset of $G\backslash M$ in the same time.*

Proof It follows from Corollary 5.8 that $\pi \circ \gamma$ runs out of every compact subset of its orbit type in the same time, where $\pi \circ \gamma$ is a maximal solution of (1.1) with v replaced by $\pi_* v$. It now follows from Corollary 6.7 that $\pi \circ \gamma$ runs out of every compact subset of $G\backslash M$ in the same time. $\square$

Theorem 6.10 *Let $G\backslash M$ be the orbit space for a proper action of a Lie group G on a smooth manifold M, provided with the differential structure $\mathrm{C}^\infty(G\backslash M)$ of Definition 4.1 and Proposition 4.7, and with the orbit type stratification $\mathcal{S}$ of Proposition 4.13. Then the inclusion in* (6.3) *is an equality:*

$$\pi_*(\mathcal{X}^\infty(M)^G) = \mathcal{X}^\infty(G\backslash M,\, \mathcal{S}), \tag{6.4}$$

and the inclusion in (6.1) *is an equality:*

$$\partial(\mathcal{X}^\infty(G\backslash M,\, \mathcal{S})) = \mathcal{X}(G\backslash M,\, \mathrm{C}^\infty(G\backslash M)). \tag{6.5}$$

The flow derivations of $\mathrm{C}^\infty(G\backslash M)$ *form a Lie subalgebra of* $\mathrm{Der}(\mathrm{C}^\infty(G\backslash M))$ *and* $\partial : \mathcal{X}^\infty(G\backslash M,\, \mathcal{S}) \to \mathcal{X}(G\backslash M,\, \mathrm{C}^\infty(G\backslash M))$ *is an isomorphism of Lie algebras.*

Proof The identity (6.4) is due to Schwarz [30], with a deep proof. There the result is formulated for compact Lie groups G. The result for general proper actions follows from the result for the action in any slice of the isotropy subgroup, which is a compact group. When all orbits have the same dimension, the result had been obtained before by Bierstone [1].

For the proof of (6.5), we begin with the observation that $G\backslash M$ is locally compact, because M is locally compact and the canonical projection $\pi : M \to G\backslash M$ is a continuous, open and surjective mapping. Furthermore Corollary 4.11 implies that the differential space $G\backslash M$ is subcartesian.

Let $D \in \mathcal{X}(G\backslash M,\, \mathrm{C}^\infty(G\backslash M))$ be a flow derivation in $G\backslash M$. For any given point in $G\backslash M$, we use an identification $\varphi = \widetilde{p} \circ \phi^{-1}$ of an open neighborhood of of the point in $G\backslash M$ with the subset $p(B)$ in $\mathbf{R}^n$ as in Lemma 4.9. We also use the Cartesian product structure $\mathbf{R}^n = \mathbf{R}^l \times \mathbf{R}^m$ introduced in Subsection 4.3, such that the orbit types in $p(E)$ are of the form $\mathbf{R}^l \times R$, where R is an orbit type for the action of H in F. We may assume that the given point in $G\backslash M$ corresponds to the origin in $\mathbf{R}^l \times \mathbf{R}^l$ and that its orbit type corresponds locally to $\mathbf{R}^l \times \{0\}$.

Let $\delta = (\dot{x},\, \dot{q})$ be the smooth vector field in an open neighborhood of the origin in $\mathbf{R}^n$ which is defined by D as in the proof of Proposition 6.3, and let $\gamma(t) = (x(t),\, q(t))$ be an integral curve of D, defined on an open interval around 0 in $\mathbf{R}$, such that $q(0) = 0$. Because $\gamma(t) \in p(E)$ for all t, it follows from Lemma 4.14 that $q'(0) = 0$. Because the integral curves of D are solutions of (1.1) with $v = \delta$, it follows that $\dot{q} = 0$ when $q = 0$, which means that the smooth vector field δ is tangent to $S := \mathbf{R}^l \times \{0\}$, which in turn implies that the restriction of δ to S near the origin is equal to a smooth vector field w in S. Therefore the integral curves of D, which are solutions of (1.1) with $v = \delta$, remain in S when they start in S.

Let f be any smooth function on an open neighborhood of the origin in $\mathbf{R}^n$. For any integral curve γ of D in S we have $(D(f))(\gamma(t)) = \mathrm{d}f(\gamma(t))/\,\mathrm{d}t = (\partial_w f)(\gamma(t))$, cf. (6.3), which implies that $D(f)|_S = \partial_w(f|_S)$. Going back to $G\backslash M$, we have proved that for every $S \in \mathcal{S}$ and every $s \in S$, there is an open neighborhood U of s in $G\backslash M$ and a smooth vector field w_T in $T := S \cap U$, such that $D(f)|_T = \partial_{w_T}(f|_T)$ for every $f \in \mathrm{C}^\infty(U)$. Because these equations determine the $w_T = w_{S\cap U}$ uniquely in terms of D, the $w_{S\cap U}$ patch together to a smooth vector field w_S on S, and we have $D(f)|_S = \partial_{w_S}(f|_S)$ for every $f \in \mathrm{C}^\infty(G\backslash M)$. Because this holds for every $S \in \mathcal{S}$ we obtain $w : S \mapsto w_S \in \mathcal{X}^\infty(G\backslash M,\, \mathcal{S})$ such that $D = \partial_w$. $\square$

Theorem 6.10 motivates to call the elements of

i) $\pi_*(\mathcal{X}^\infty(M)^G)$, the stratified vector fields in $G\backslash M$ induced by the G-invariant smooth vector fields in M, or

ii) $\mathcal{X}^\infty(G\backslash M,\, \mathcal{S})$, the smooth stratified vector fields in $G\backslash M$, or

iii) $\mathcal{X}(G\backslash M,\, \mathrm{C}^\infty(G\backslash M))$, the flow derivations of $\mathrm{C}^\infty(G\backslash M)$,

the *smooth vector fields on* $G\backslash M$. Here we used in i) and ii) the orbit type stratification of Proposition 4.13.

Note that iii) is defined in terms of the differential structure $G\backslash M$ only, whereas for ii) we also need the orbit type stratification $\mathcal{S}$ of $G\backslash M$, and for i) we need the smooth manifold M on which the Lie group G acts properly. The Lie algebra of all smooth vector fields on $G\backslash M$ could be denoted shortly by $\mathcal{X}^\infty(G\backslash M)$.

The following corollary of Lemma 5.12, (6.3) and (6.5) yields a characterization of the orbit type stratification of $G\backslash M$ in terms of the differential structure $\mathrm{C}^\infty(G\backslash M)$ only.

Corollary 6.11 *For each $q \in G\backslash M$, the smallest subset of $G\backslash M$ which contains q and which is invariant under the flows of all $D \in \mathcal{X}(G\backslash M,\, \mathrm{C}^\infty(G\backslash M))$, is equal to the connected component of the orbit type in $G\backslash M$ to which q belongs.*

Example 6.12 Let $M = \mathbf{R}$ and $G = \{\pm 1\}$. Then $p : x \mapsto y := x^2$ generates $\mathrm{C}^\infty(M)^G$ freely, hence p^* is an isomorphism from $\mathrm{C}^\infty([0,\, \infty[)$ onto $\mathrm{C}^\infty(M)^G$, which means that p is a diffeomorphism from $G\backslash M$ onto $Q := [0,\, \infty[\, \subset \mathbf{R}$. The derivations of $\mathrm{C}^\infty(Q)$ are the ∂_w, acting on $\mathrm{C}^\infty(Q)$, for arbitrary smooth vector fields w on $\mathbf{R}$. Clearly ∂_w is a flow deriviation on $(Q,\, \mathrm{C}^\infty(Q))$ if and only if $\mathrm{e}^{t\,w}(0) \in Q$ for all t in an open neighborhood of 0 in $\mathbf{R}$, which is the case if and only if $w(0) = 0$. This shows that not every derivation of $\mathrm{C}^\infty(G\backslash M)$ is a smooth vector field on $G\backslash M$, actually $\mathcal{X}(G\backslash M,\, \mathrm{C}^\infty(G\backslash M))$ is a codimension one linear subspace of $\mathrm{Der}(\mathrm{C}^\infty(G\backslash M))$ in this example.

Similarly, if $M = \mathbf{C}$ and G is equal to the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$ acting on $\mathbf{C}$ by multiplications, then the real polynomial $p : z \mapsto z\,\overline{z}$ generates $\mathrm{C}^\infty(M)^G$ freely and defines a diffeomorphism from $G\backslash M$ onto the same differential space $Q = [0,\, \infty[\, \subset \mathbf{R}$ as above. So again not every derivation of $\mathrm{C}^\infty(G\backslash M)$ is a smooth vector field on $G\backslash M$. In contrast with the previous example, in this example the Lie group G is connected. The embedding of $\mathbf{R}$ into $\mathbf{C}$ induces the diffeomorphism between the two orbit spaces. $\oslash$

Remark 6.13 Let $(Q,\, \mathrm{C}^\infty(Q))$ be a differential space and let S be a locally closed subset of Q be such that $(S,\, \mathrm{C}^\infty(S))$ is a smooth manifold, cf. the definitions in the beginning of Subsection 4.1. We say that the derivation $D \in \mathrm{Der}(\mathrm{C}^\infty(Q))$ is *tangent to* S, if for every $s \in S$ there exists a $v \in \mathrm{T}_s S$ such that $(D(f))(s) = \langle v,\, \mathrm{d}(f|_S)(s)$ for every $f \in \mathrm{C}^\infty(Q)$.

Suppose now that we have a proper action of a Lie group G on a smooth manifold M and $G \setminus M$ is connected. Let $\mathcal{S}$ be the orbit type stratification of $G\backslash M$ of Proposition 4.13. Recall from Remark 4.17 that if $S \in \mathcal{S}$, then $\dim S \leq \dim(G\backslash M)$, with equality if and only if S is equal to the principal stratum $G\backslash M^{\mathrm{princ}}$ of $G\backslash M$.

It has been proved by Schwarz [30, Prop. 3.5] that a derivation $D \in \mathrm{Der}(\mathrm{C}^\infty(G\backslash M))$ belongs to $\partial(\mathcal{X}^\infty(G\backslash M,\, \mathcal{S}))$, already if it is tangent to the codimension one strata in the orbit type stratification $\mathcal{S}$ of $G\backslash M$, i.e. the $S \in \mathcal{S}$ such that $\dim S = \dim(G\backslash M) - 1$. This implies that all derivations of $\mathrm{C}^\infty(G\backslash M)$ are smooth vector fields on $G\backslash M$ if the orbit type stratification of $G\backslash M$ has no codimension one strata.

In the notation of the proof of Proposition 6.3, this means that a smooth vector field δ in U defines s smooth vector field on the local model $Q := \psi(V) \subset U$ of $G\backslash M$, if and only if δ is tangent to the principal stratum P in Q and to the codimension one strata in Q (if they exist). The condition that δ is tangent to P implies that ∂_δ leaves the ideal I in $\mathrm{C}^\infty(U)$ of all functions which vanish on Q invariant, and therefore defines a derivation of $\mathrm{C}^\infty(Q) \simeq \mathrm{C}^\infty(U)/I$. $\oslash$

Remark 6.14 In this subsection we have introduced smooth vector fields on $G\backslash M$ without defining a tangent bundle of $G\backslash M$ of which these smooth vector fields are supposed to be the smooth sections.

Let $\mathcal{S}$ be the orbit type stratification of $G\backslash M$, which in the local models for $G\backslash M$ is a Whitney stratification, cf. Remark 4.16. Let $\mathrm{T}^{\mathcal{S}}(G\backslash M)$ denote the disjoint union of the tangent bundles $\mathrm{T}S$, of the strata $S \in \mathcal{S}$, this is called the *stratified tangent bundle* of $G\backslash M$. It can be provided with the structure of a differential space, such the $\mathrm{T}S$, $S \in \mathcal{S}$, form a stratification of $\mathrm{T}^{\mathcal{S}}(G\backslash M)$ and the projection $\pi : \mathrm{T}^{\mathcal{S}}(G\backslash M) \to G\backslash M$ is a smooth mapping of stratified differential spaces, cf. Pflaum [26, Th. 2.1.2]. Furthermore, the space $\mathcal{X}^\infty(G\backslash M,\, \mathcal{S})$ of smooth stratified vector fields is equal to the space of smooth sections of $\pi : \mathrm{T}^{\mathcal{S}}(G\backslash M) \to G\backslash M$, that is, smooth mappings $w : G\backslash M \to \mathrm{T}^{\mathcal{S}}(G\backslash M)$, such that $\pi \circ w$ is equal to the identity in $G\backslash M$. This follows from Pflaum [26, Prop. 2.2.6 and Prop. 2.2.8]. $\oslash$

Remark 6.15 Let $(Q,\, \mathrm{C}^\infty(Q))$ be a differential space. For any $q \in Q$,

$$\mathcal{M}_q = \{f \in \mathrm{C}^\infty(Q) \mid f(q) = 0\}$$

is a maximal ideal in the algebra $\mathrm{C}^\infty(Q)$. Mimicking definitions from algebraic geometry, $\mathcal{M}_q/\mathcal{M}_q^2$ is called the *Zariski cotangent space* of Q at q, and its topological dual

$$\mathrm{T}_q^{\mathrm{Z}}\, Q := \left(\mathcal{M}_q/\mathcal{M}_q^2\right)^*$$

is called the *Zariski tangent space* of Q at the point q.

If S is a smooth submanifold of Q as in Remark 6.13 then for each $w \in \mathrm{T}_s\, S$ the mappnig $f \mapsto \langle w,\, \mathrm{d}(f|_S)(q)\rangle$ defines an injective linear mapping $\partial_w : \mathrm{T}_q\, S \to \mathrm{T}^{\mathrm{Z}}\, Q$, which is used to identify the "ordinary" tangent space $\mathrm{T}_q\, S$ of S at q with a linear subspace of the Zariski tangent space of Q at q. In this way the stratified tangent bundle $\mathrm{T}^{\mathcal{S}}\, Q$ is contained in the Zariski tangent bundle $\mathrm{T}^{\mathrm{Z}}\, Q$ if $\mathcal{S}$ is a stratification of Q.

In a similar way the Zariski tangent space is identified with a linear subspace of $\mathrm{T}_q\, R$, if Q is contained in a smooth manifold R.

When $Q = G\backslash M$, and we use the local model of Lemma 4.9, then a lemma of Mather [23, §3] states that

$$\mathrm{T}_0^{\mathrm{Z}}(p(E)) \simeq \mathrm{T}_0^{\mathrm{Z}}(\mathbf{R}^n) \simeq \mathbf{R}^n.$$

This implies that, at each $G \cdot m \in G\backslash M$, the dimension of the Zariski tangent space is equal to the number of elements in a Hilbert basis of the algebra of G_m-invariant polynomials on $E := \mathrm{T}_m\, M / \mathrm{T}_m(G \cdot m)$.

Mimicking the definition of Whitney [36] for complex analytic varietes, the *tangent cone* of $p(E)$ at the origin in $\mathbf{R}^n \simeq \mathrm{T}_0^{\mathrm{Z}}\, p(E)$ can be defined as the set of all limits of sequences $\tau_j\, p(e_j)$, where the τ_j are positive real numbers and the $p(e_j)$ converge to zero. The conic structure of the tangent cone consists of multiplication of elements of $\mathbf{R}^n$ by positive real numbers. Because in general the quasi-homogeneous mapping $p : E \to \mathbf{R}^n$ is not homogeneous, this conic structure is different from the conic structure on $p(E) \simeq H\backslash E$ which is induced by the H-invariant conic structure on E defined by multiplication with positive real numbers in E.

$\oslash$

Remark 6.16 Over each G-orbit O in M, the vector spaces $\mathrm{T}_m\, M / \mathrm{T}_m(G \cdot m)$, $m \in O \Leftrightarrow G \cdot m = O$, form a smooth vector bundle $\mathrm{N}O$ over O, called the *normal bundle* of O in M. On the normal bundle of O we have the induced tangent action of G. The corresponding orbit space $G\backslash \mathrm{N}O$ is called the *tangent wedge* at $O \in G\backslash M$ in Cushman and Śniatycki [6, Sec. 5].

The injection of the fiber $E = \mathrm{T}_m\, M / \mathrm{T}_m(G \cdot m)$ into $\mathrm{N}O$ leads to an identification of the tangent wedge with $H\backslash E$, in which $H = G_m$. The discussion in the beginning of Subsection 4.3 showed that $H\backslash E \simeq E^H \times (H\backslash F)$, in which the linear subspace E^H of E corresponds to the tangent space at O of the orbit type in $G\backslash M$, and $F \simeq E/E^H$. The tangent wedge is called a *wedge over the cone* $H\backslash F$ in [6, Sec. 5], where presumably the conic structure in $H\backslash F$ is induced by the H-invariant conic structure in F.

It follows from Lemma 4.8 that an open neighborhood of the origin in the tangent wedge at O is diffeomorphic, as a differential space, to an open neighborhood of O in the orbit space $G\backslash M$. In general the tangent wedge does not coincide with the tangent cone in Remark 6.15.

$\oslash$

Remark 6.17 Several of the statements in this subsection have been proved in Śniatycki [33], [34] for more general differential spaces than our orbit space of a proper action of a Lie group on a smooth manifold. $\oslash$

7 Relative Equilibria

Recall that I_m is the interval of definition of the maximal solution of (1.1) which starts at m.

Lemma 7.1 *Assume that G has countably many connected components and let $m \in M$. Then the following conditions are equivalent.*

i) *The vector $v(m)$ is tangent at m to the G-orbit through m, i.e. there exists an $X \in \mathfrak{g}$ such that*
$$v(m) = X_M(m). \tag{7.1}$$

ii) *The solution of* (1.1) *which starts at m is equal to the action on m of a one-parameter subgroup of G, i.e. $I_m = \mathbf{R}$ and there exists an $X \in \mathfrak{g}$ such that*
$$\mathrm{e}^{t\,v}(m) = \left(\mathrm{e}^{t\,X}\right)_M(m) \quad \text{for every} \quad t \in \mathbf{R}. \tag{7.2}$$

iii) *The solution of* (1.1) *which starts at m is contained in the G-orbit through m, i.e. $\mathrm{e}^{t\,v}(m) \in G \cdot m$ for every $t \in I_m$.*

iv) *$G{\cdot}m$ is an equilibrium point of the reduced system in $G \backslash M$, in the sense that $\Phi^t(G{\cdot}m) = G \cdot m$ for every $t \in I_m$.*

The element $X \in \mathfrak{g}$ satisfies (7.2) *if and only if it satisfies* (7.1).

Proof Assume that (7.1) holds. If we substitute $g = \mathrm{e}^{s\,X}$ in (5.2) and differentiate the resulting equation with respect to s at $s = 0$, then we obtain that

$$X_M\left(\mathrm{e}^{t\,v}(m)\right) = \left(\mathrm{T}_m\,\mathrm{e}^{t\,v}\right)\,X_M(m) = \left(\mathrm{T}_m\,\mathrm{e}^{t\,v}\right)\,v(m) = v\left(\mathrm{e}^{t\,v}(m)\right),$$

in which the third identity follows by differentiating (1.2) with respect to s at $s = 0$. It follows that
$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{t\,v}(m) = v\left(\mathrm{e}^{t\,v}(m)\right) = X_M\left(\mathrm{e}^{t\,v}(m)\right),$$
where $\mathrm{e}^{0\,v}(m) = m$, which implies that

$$\mathrm{e}^{t\,v}(m) = \mathrm{e}^{t\,X_M}(m) = \left(\mathrm{e}^{t\,X}\right)_M(m)$$

for every $t \in I_m$. Because the right hand side does not run out every compact subset of M in a finite time, we conclude also that $I_m = \mathbf{R}$. This proves ii) with X as in (7.1). Conversely (7.1) follows from differentiating (7.2) with respect to t at $t = 0$.

The implication ii) $\Longrightarrow$ iii) is obvious, as well as the equivalence between iii) and iv).

The implication iii) $\Longrightarrow$ i) is obvious if the orbit $G \cdot x$ is an embedded submanifold of M, i.e. the mapping $A_m : g\,G_m \mapsto A(g,\, m)$ is an embedding from G/G_m into M. In order to prove iii) $\Longrightarrow$ i), we will use the assumption that G has countably many connected components, which is equivalent to the assumption that G is equal to the union of a countable family of compact subsets K_n of G, cf. [10, Thm. 1.9.1].

Choose a linear complement $\mathfrak{q}$ of $\mathfrak{g}_m = \mathrm{T}_1 G_m$ in $\mathfrak{g}$. Let Q be a smooth submanifold of G such that $1 \in Q$ and $\mathrm{T}_1 Q = \mathfrak{q}$. Similarly, choose a linear complement E in $\mathrm{T}_m M$ of $\mathrm{T}_m(G \cdot m) = \{X_M(m) \mid X \in \mathfrak{g}\}$. Let $\mathcal{E}$ be a smooth submanifold of M such that $m \in \mathcal{E}$ and $\mathrm{T}_m \mathcal{E} = E$. Then the tangent mapping at $(1, m)$ of the smooth mapping $(q, e) \mapsto q \cdot e : Q \times \mathcal{E} \to M$ is a bijective linear mapping from $\mathfrak{q} \times E$ onto $\mathrm{T}_m M$, and it follows from the inverse function theorem that, after shrinking Q and $\mathcal{E}$ if necessary, $(q, e) \mapsto q \cdot e$ defines a diffeomorphism from $Q \times \mathcal{E}$ onto an open neighborhood R of m in M.

Suppose that e_j is an infinite sequence in $\mathcal{E} \cap (G \cdot m)$, which converges to an element $e \in \mathcal{E} \cap (G \cdot m)$, both in the topology of $\mathcal{E}$ and in the orbit topogy of $G \cdot m$. The latter means that $e_j = g_j \cdot m$ for a sequence g_j in G which converges in G to an element $g \in G$, which implies that $e = g \cdot m$.

Because the tangent mapping at 1 of the mapping $q \mapsto q \cdot e : Q \to M$ is injective, we have that $\mathfrak{q} \cap \mathfrak{g}_e = \{0\}$. Because $e = g \cdot m$ implies that $\mathfrak{g}_e = (\mathrm{Ad}\, g)(\mathfrak{g}_m)$, cf. (2.4), and therefore $\dim \mathfrak{g}_e = \dim \mathfrak{g}_m = \dim \mathfrak{g} - \dim \mathfrak{q}$, we obtain that the mapping $(q, h) \mapsto p\, h$ is a diffeomorphism from an open neighborhood of $(1, 1)$ in $Q \times G_e$ onto an open neighborhood of 1 in G. Because $g_j\, g^{-1} \to 1$ in G as $j \to \infty$, we can write, for sufficiently large j, $g_j\, g^{-1} = q_j\, h_j$, in which $q_j \in Q$ and $h_j \in G_e$ both converge to 1. It follows that

$$1 \cdot e_j = e_j = g_j \cdot m = (g_j\, g^{-1}) \cdot e = (q_j\, h_j) \cdot e = q_j \cdot e.$$

In view of the injectivity of the mapping $(q, e) \mapsto q \cdot e : Q \times \mathcal{E} \to M$, this implies that $e_j = e$ if j is sufficiently large.

Now suppose that $\mathcal{E}_0$ is a compact neighborhood of m in $\mathcal{E}$. Let K be a compact subset of G. If $\mathcal{E} \cap (K \cdot m)$ is infinite, then there is an infinite sequence $g_j \in K$ such that the $e_j := g_j \cdot m$ are distinct elements of $\mathcal{E}$. Passing to a subsequence if necessary, we may assume that the g_j converge in G to some element $g \in G$. Then the e_j converge in M to an element e, and $e \in \mathcal{E}_0 \subset \mathcal{E}$ because $\mathcal{E}_0$ is compact. This leads to a contradiction with the previous conclusion that $e_j = e$ for sufficiently large j.

It follows that for each n the set $\mathcal{E}_0 \cap (K_n \cdot m)$ is finite, which in turn implies that $\mathcal{E}_0 \cap (G \cdot m)$ is countable. On the other hand there is an open interval $I \subset I_m$ around 0 in $\mathbf{R}$ and there are smooth curves $t \mapsto q(t) : I \to Q$ and $t \mapsto e(t) : I \to \mathcal{E}_0$, such that $\mathrm{e}^{t\, v}(m) = q(t) \cdot e(t)$ for every $t \in I$. Under the assumption iii) this implies that, for every $t \in \mathbf{R}$, $e(t) \in \mathcal{E}_0 \cap G \cdot m$. Because $\mathcal{E}_0 \cap G \cdot m$ is countable, this implies that the continuous function $t \mapsto e(t)$ is constant and therefore $e(t) = e(0) = m$, hence $\mathrm{e}^{t\, v}(m) = q(t) \cdot m$ for every $t \in I$. Differentiating this identity with respect to t at $t = 0$, we obtain (7.1) with $X = q'(0) \in \mathfrak{q} \subset \mathfrak{g}$. □

Remark 7.2 The element $X \in \mathfrak{g}$ in (7.1) and in (7.2) is uniquely determined if and only if the linear mapping $\alpha_m : X \mapsto X_M(m) : \mathfrak{g} \to \mathrm{T}_m M$ is injective, i.e. if and only if the action is *locally free at the point* $m \in M$. Because $\ker \alpha_m$ is equal to the Lie algebra $\mathfrak{g}_m$ of G_m, the action is locally free at m if and only if G_m is a discrete subgroup of G, which is certainly the case if $G_m = \{1\}$, i.e. the action is free at the point m. ⊘

Definition 7.3 The point $m \in M$, and also the solution of (1.1) which starts at m, is called a *relative equilibrium* if one (each) of the conditions i) — iv) in Lemma 7.1 is satisfied. $\oslash$

Many discussions of dynamical systems with symmetry start with descriptions of special solutions which turn out to be relative equilibria. An early example is Huygens [16, Th. VIII], in which the relative equilibria are described for the motion of a particle under the influence of gravity in a bowl which is symmetric with respect to the rotations about a vertical axis.

Remark 7.4 Every equilibrium point of v is a relative equilibrium, where one can take $X = 0$ in (7.1). Conversely, if m is a relative equilibrium, then m is an equilibrium point of v if and only if the element X in (7.1) belongs to the Lie algebra $\mathfrak{g}_m = \ker \alpha_m$ of the isotropy subgroup G_m of the point m. $\oslash$

Remark 7.5 If the one-parameter subgroup of G generated by the element X in (7.2) is periodic, i.e. there exists a $\tau \neq 0$ such that $\mathrm{e}^{\tau X} = 1$, then it follows from (7.2) that

$$\mathrm{e}^{\tau\, v}(m) = \left(\mathrm{e}^{\tau\, X}\right)_M (m) = 1_M(m) = m,$$

which implies that the relative equilibrium is a periodic solution of (1.1), with τ as a period.

In general the set P of periods of the relative equilibrium, the set of all $\tau \in \mathbf{R}$ such that $\mathrm{e}^{\tau\, v}(m) = m$, is a closed subgroup of $\mathbf{R}$. We have $P = \mathbf{R}$, or $P = \mathbf{Z}\,\tau$ for a unique positive real number τ, or $P = \{0\}$, cf. [10, Lemma 1.12.2]. In the first case the solution is constant and m is an equilibrium point of v as in Remark 7.4. In the second case the solution of (1.1) which starts at m is a non-constant periodic function and τ is called *the period* of this periodic solution. In the third case the solution of (1.1) which starts at m is not periodic. $\oslash$

7.1 Quasi-periodic Relative Equilibria

Definition 7.6 A curve $\gamma : \mathbf{R} \to M$ is called *quasi-periodic with at most k frequencies* in the continuous (smooth, analytic) category, if there is a continuous (smooth, analytic) mapping $\Gamma : (\mathbf{R}/\mathbf{Z})^k \to M$ from the standard torus $(\mathbf{R}/\mathbf{Z})^k \simeq \mathbf{R}^k/\mathbf{Z}^k$ to M, and there are constants $\nu_1, \nu_2, \ldots, \nu_k \in \mathbf{R}$, the frequencies, such that $\gamma(t) = \Gamma\left(\nu_1\, t + \mathbf{Z}, \ldots, \nu_k\, t + \mathbf{Z}\right)$ for every $t \in \mathbf{R}$. $\oslash$

The phrase "there is a continuous (smooth, analytic) mapping $\Gamma : (\mathbf{R}/\mathbf{Z})^k \to M$ and there are constants $\nu_1, \nu_2, \ldots, \nu_k \in \mathbf{R}$" in Definition 7.6 makes the definition of quasi--periodicity not very specific. In our applications, both the mapping Γ and the frequencies will be described explicitly in terms of the solution of (1.1) which starts at m, and the action of G on the point m. This explicit description will lead to some more specific conclusions about the solution.

Definition 7.7 A *torus group* is a compact, connected and commutative Lie group. $\oslash$

Let $\mathfrak{t}$ be the Lie algebra of the abstract torus group T. Then the exponential mapping $X \mapsto \mathrm{e}^X$ is a surjective homomorphism of Lie groups, from the additive group $(\mathfrak{t}, +)$ onto T. Its kernel $\Lambda := \ker \exp$ is a discrete additive subgroup of the vector space $\mathfrak{t}$. According to [10, Th.1.12.3], there exist elements $\lambda_j \in \Lambda$, $1 \leq j \leq k$, which, as elements of $\mathfrak{t}$ are linearly independent over $\mathbf{R}$, such that each $\lambda \in \Lambda$ is of the form

$$\lambda = \sum_{j=1}^{n} n_j \, \lambda_j \tag{7.3}$$

for suitable integers n_j. Because of the linear independence of the λ_j over $\mathbf{R}$, the integers n_j are uniquely determined, i.e. the mapping which assigns to $(n_1, \ldots, n_k)$ the right hand side of (7.3) is an isomorphism from $\mathbf{Z}^k$ onto Λ. In this case one says that the λ_j, $1 \leq j \leq k$, form a $\mathbf{Z}$-*basis* of the additive group Λ. Because $\mathbf{Z}^k$ is called the integral lattice in $\mathbf{R}^n$, the subgroup $\Lambda = \ker \exp$, which is isomorphic to $\mathbf{Z}^k$, is called the *integral lattice* in $\mathfrak{t}$.

If l denotes the dimension of $\mathfrak{t}$ over $\mathbf{R}$, then $k \leq l$, and we can extend the λ_j to an $\mathbf{R}$-basis λ_j, $1 \leq j \leq l$ of $\mathfrak{t}$. Now the mapping

$$(\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_l) \mapsto \exp\left(\sum_{j=1}^{l} \theta_j \, \lambda_j\right) : \mathbf{R}^l \to T$$

induces a bijective homomorphism of Lie groups ψ from $\left(\mathbf{R}^k/\mathbf{Z}^k\right) \times \mathbf{R}^{l-k}$ onto T. It follows from [10, Cor. 1.10.10] that ψ^{-1} is continuous, hence $\left(\mathbf{R}^k/\mathbf{Z}^k\right) \times \mathbf{R}^{l-k}$ is compact, which in turn implies that $k = l$ and ψ is an isomorphism from the standard torus $\mathbf{R}^k/\mathbf{Z}^k \simeq (\mathbf{R}/\mathbf{Z})^k$ onto T.

Note that we have also proved that every $\mathbf{Z}$-basis of Λ is an $\mathbf{R}$-basis of $\mathfrak{t}$. It follows that a sequence of vectors λ_i', $1 \leq i \leq k'$, is a $\mathbf{Z}$-basis of Λ if and only if $k' = \dim \mathfrak{t} = k$, and

$$\lambda_i' = \sum_{j=1} A_{ji} \, \lambda_j, \quad 1 \leq i \leq k,$$

in which the coefficients of the $k \times k$-matrix A_{ji} are integers, the matrix A is invertible and A^{-1} also has integral coefficients. In other words, $A \in \mathrm{GL}(k, \mathbf{Z})$, the group of invertible matrices A with integral coefficients and $\det A = \pm 1$.

Therefore the isomorphism ψ from the standard torus $\mathbf{R}^k/\mathbf{Z}^k \simeq (\mathbf{R}/\mathbf{Z})^k$ onto the abstract torus T is by no means unique: the freedom is precisely the choice of the $\mathbf{Z}$-bases of Λ (or of $\mathbf{Z}^k$), which is parametrized by the discrete group $\mathrm{GL}(k, \mathbf{Z})$. If $k > 1$, then this group is infinite and noncommutative.

Lemma 7.8 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $X \in \mathfrak{g}$. Then either $t \mapsto \mathrm{e}^{t\,X}$ is a dense one-parameter subgroup of a torus subgroup T of G, or the mapping $t \mapsto \mathrm{e}^{t\,X} : \mathbf{R} \to G$ is proper.*

Proof Let A denote the closure in G of the set E of all $\mathrm{e}^{t\,X}$ such that $t \in \mathbf{R}$. Because E is a commutative and connected subsgroup of G, A is a commutative and connected subgroup of G. Because A is a closed subgroup of G, it is a Lie subgroup of G, cf. [10, Cor. 1.10.7]. Because A is a connected and commutative Lie group, there are $k,\, l \in \mathbf{Z}_{\geq 0}$ such that A is isomorphic, as a Lie group, to $(\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$, cf. [10, Cor. 1.12.4].

Because $\mathrm{e}^{t\,X} \in A$ for every $t \in \mathbf{R}$, a differentiation with respect to t at $t = 0$ shows that X belongs to the Lie algebra of A. If $(Y,\, Z) \in \mathbf{R}^k \times \mathbf{R}^l$ denotes the corresponding element of the Lie algebra of $(\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$, then $\mathrm{e}^{t\,X}$ corresponds to the element $(t\,Y + \mathbf{Z}^k,\, t\,Z)$ of $(\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$. The density of E in A implies that the set L of all $(t\,Y + \mathbf{Z}^k,\, t\,Z)$ such that $t \in \mathbf{R}$ is dense in $(\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$.

If $Z \neq 0$, then the mapping $t \mapsto (t\,Y + \mathbf{Z}^k,\, t\,Z) : \mathbf{R} \to (\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$ is proper, because already its projection $t \mapsto t\,Z : \mathbf{R} \to \mathbf{R}^l$ to the second component is proper. Hence the mapping $t \mapsto \mathrm{e}^{t\,X} : \mathbf{R} \to A$ is proper, and because A is a closed subset of G, it follows that the mapping $t \mapsto \mathrm{e}^{t\,X} : \mathbf{R} \to G$ is proper.

If $Z = 0$, then L is contained in the torus subgroup $(\mathbf{R}/\mathbf{Z})^k \times \{0\}$ of $(\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$. The density of L implies that $l = 0$ and therefore A is a torus subgroup of G. □

Definition 7.9 Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $X \in \mathfrak{g}$. Then X is called an *elliptic element* of $\mathfrak{g}$ if $t \mapsto \mathrm{e}^{t\,X}$ is a dense one-parameter subgroup of a torus subgroup of G. According to Lemma 7.8, this condition is equivalent to the condition that there exists a compact subset K of G and an unbounded sequence of real numbers t_i, such that $\mathrm{e}^{t_i\,X} \in K$ for every i. ⊘

Lemma 7.10 *Let X be an elliptic element in the Lie algebra $\mathfrak{g}$ of a Lie group G. Then we have the following conclusions.*

i) *The closure T in G of the set S of all $\mathrm{e}^{t\,X}$ such that $t \in \mathbf{R}$ is a torus subgroup of G. We have $X \in \mathfrak{t}$:= the Lie algebra of T.*

ii) *For every $\mathbf{Z}$-basis λ_j, $1 \leq j \leq k$, of the integral lattice of $\mathfrak{t}$, the mapping*

$$(\theta_1,\, \ldots,\, \theta_k) \mapsto \exp\left(\sum_{j=1}^{k} \theta_j\, \lambda_j\right) : \mathbf{R}^k \to T \tag{7.4}$$

induces an isomorphism of Lie groups ψ from the standard torus $\mathbf{R}^k/\mathbf{Z}^k \simeq (\mathbf{R}/\mathbf{Z})^k$ onto T.

iii) (Kronecker [19]) *If $\nu \in \mathbf{R}^k$, then the set L of all $t\,\nu + \mathbf{Z}^k$ such that $t \in \mathbf{R}$ is dense in the standard torus $\mathbf{R}^k/\mathbf{Z}^k \simeq (\mathbf{R}/\mathbf{Z})^k$, if and only if the coordinates ν_j, $1 \leq j \leq k$ are linearly independent over $\mathbf{Q}$.*

iv) *If the $X_j \in \mathbf{R}$ are the coefficients of X with respect to the $\mathbf{R}$-basis λ_j, $1 \leq j \leq k$, of $\mathfrak{t}$, then the real numbers X_j are linearly independent over $\mathbf{Q}$, and the set L of all $(X_1\, t,\, \ldots,\, X_k\, t) + \mathbf{Z}^k$, where t ranges over all real numbers, is dense in the standard torus $\mathbf{R}^k/\mathbf{Z}^k$.*

Proof i) is just the definition of the ellipticity of X.

ii) follows from the discussion preceding Definition 7.9.

Suppose that there is a linear relation

$$\sum_{j=1}^{k} q_j\, \nu_j = 0 \tag{7.5}$$

with $q_j \in \mathbf{Q}$ and not all q_j equal to zero. If we multiply the relation with the smallest common multiple of the denominators of the q_j, we arrive at (7.5) with $q_j \in \mathbf{Z}$, and not all $q_j = 0$. The non-zero linear mapping

$$Q : \theta \mapsto \sum_{j=1}^{k} q_j\, \theta_j : \mathbf{R}^k \to \mathbf{R}$$

maps $\mathbf{Z}^k$ to $\mathbf{Z}$ and therefore induces a surjective homomorphism of Lie groups from $\mathbf{R}^k/\mathbf{Z}^k$ onto $\mathbf{R}/\mathbf{Z}$, which we also denote by Q. We have that $L \subset \ker Q$, where $\ker Q$ is a closed Lie subgroup of dimension $k-1$ in $\mathbf{R}^k/\mathbf{Z}^k$, and therefore L is not dense in $\mathbf{R}^k/\mathbf{Z}^k$.

If conversely L is not dense in $\mathbf{R}^k/\mathbf{Z}^k$, then i) implies that the closure of L in $\mathbf{R}^k/\mathbf{Z}^k$ is a subtorus S of $\mathbf{R}^k/\mathbf{Z}^k$ of dimension $l < k$. It follows that $(\mathbf{R}^k/\mathbf{Z}^k)/S$ is a torus of dimension $k - l > 0$. Using an isomorphism of Lie groups from $(\mathbf{R}^k/\mathbf{Z}^k)/S$ onto $(\mathbf{R}/\mathbf{Z})^{k-l}$ followed by the projection onto the first factor, we obtain a surjective homomorphism of Lie groups from $(\mathbf{R}^k/\mathbf{Z}^k)/S$ onto $\mathbf{R}/\mathbf{Z}$, which after precomposing it with the projection $\mathbf{R}^k/\mathbf{Z}^k \to (\mathbf{R}^k/\mathbf{Z}^k)/S$ yields a surjective homomorphism of Lie groups $Q : \mathbf{R}^k/\mathbf{Z}^k \to \mathbf{R}/\mathbf{Z}$ such that $S \subset \ker Q$. The tangent mapping of Q, which we also denote by Q, is a linear mapping from $\mathbf{R}^k$ onto $\mathbf{R}$, which maps $\mathbf{Z}^k$ into $\mathbf{Z}$, which means that the coefficients q_j of Q are integers, not all equal to zero. Because $\nu \in \mathfrak{s} \subset \ker Q$, we have a linear relation of the form (7.5), with $q_j \in \mathbf{Z}$ and not all q_j equal to zero. This completes the proof of iii).

iv) follows from iii) and the fact that $L = \psi^{-1}(S)$, S is dense in T and ψ^{-1} is continuous. $\square$

After these preparations, we are ready for the following proposition about quasi-periodic relative equilibria.

Proposition 7.11 *Assume that m is a relative equilibrium and suppose that the element X in* (7.1), (7.2) *is an elliptic element of $\mathfrak{g}$. Then the solution of* (1.1) *which starts at m is quasi-periodic.*

More precisely, the closure of the one-parameter subgroup of G generated by X is a torus subgroup T of G. If λ_j, $1 \leq j \leq k = \dim T$, is a $\mathbf{Z}$-basis of the integral lattice $\Lambda = \ker \exp$ of the Lie algebra $\mathfrak{t}$ of T, then the mapping Γ in Definition 7.6 can be defined by

$$\Gamma : (\theta_1, \ldots, \theta_k) \to \exp\left(\sum_{j=1}^{k} \theta_j\, \lambda_j\right) \cdot m,$$

and the frequencies ν_j, $1 \leq j \leq k$, as the coordinates of X with respect to the $\mathbf{R}$-basis λ_j, $1 \leq j \leq k$ of $\mathfrak{t}$. These frequencies are linearly independent over $\mathbf{Q}$.

The closure of v-orbit through m is equal to $\Gamma(\mathbf{R}^k/\mathbf{Z}^k)$. The pre-image $\Gamma^{-1}(\{m\})$ of m is a closed sugroup of $\mathbf{R}^k/\mathbf{Z}^k$ and Γ induces a smooth embedding Γ_0 of the torus $T_0 := (\mathbf{R}^k/\mathbf{Z}^k)/\Gamma^{-1}(\{m\})$ into M, with image equal to the closure of v-orbit through m, making the latter diffeomorphic to T_0. The minimal number of frquencies is equal to the dimension of T_0 = the dimension of the closure of the v-orbit through m.

If the action of G is free at m, then $T_0 = \mathbf{R}^k/\mathbf{Z}^k$, $\Gamma_0 = \Gamma$, and the minimal number of frequencies of the quasi-periodic solution is equal to k.

Proof It only remains to prove the statements about the closure of the v-orbit through m and the minimal number of frequencies, as the other statements follow from Lemma 7.10 or are obvious.

It follows from iv) in Lemma 7.10 that the set of all $t\,\nu + \mathbf{Z}^k$, $t \in \mathbf{R}$ is dense in $\mathbf{R}^k/\mathbf{Z}^k$, and because

$$\mathrm{e}^{t\,v}(m) = \mathrm{e}^{t\,X} \cdot m = \Gamma(t\,\nu + \mathbf{Z}^k),$$

cf. (7.2), it follows that the v-orbit through m is dense in $\Gamma(\mathbf{R}^k/\mathbf{Z}^k)$. Because $\Gamma(\mathbf{R}^k/\mathbf{Z}^k)$ is a compact and therefore closed subset of M, it follows that $\Gamma(\mathbf{R}^k/\mathbf{Z}^k)$ is equal to the closure of the v-orbit through m.

$\Gamma^{-1}(\{m\})$ is equal to the pre-image of the closed subgroup $T \cap G_m$ of T under the isomorphism of Lie groups (7.4) from $\mathbf{R}^k/\mathbf{Z}^k$ onto T. This isomorphism induces an isomorphism of Lie groups from T_0 onto $T/(T \cap G_m)$. Because $T/(T \cap G_m)$ is a compact, connected and commutative Lie group, it is a torus group. Γ_0 is equal to the isomorphism from T_0 to $T/(T \cap G_m)$, followed by the injective immersion from $T/(T \cap G_m)$ into M defined by the action $g \mapsto g \cdot m$. Because T_0 is compact, it follows that the injective immersion from T_0 into M is a smooth embedding.

Let l be the minimal number of frequencies, attained with a smooth mapping Δ from $\mathbf{R}^l/\mathbf{Z}^l$ to M, $\omega \in \mathbf{R}^l$ and $\mathrm{e}^{t\,v}(m) = \Delta(t\,\omega + \mathbf{Z}^l)$, $t \in \mathbf{R}$. Applying Lemma 7.10 to the closure of the set of all $t\,\omega + \mathbf{Z}^l$, $t \in \mathbf{R}$, in $\mathbf{R}^l/\mathbf{Z}^l$, we see that the minimality of l implies that the set of all $t\,\omega + \mathbf{Z}^l$, $t \in \mathbf{R}$, is dense in $\mathbf{R}^l/\mathbf{Z}^l$. With the same argument as for Γ, we obtain that $\Delta(\mathbf{R}^l/\mathbf{Z}^l)$ is equal to the closure of the v-orbit through m, hence $\Delta(\mathbf{R}^l/\mathbf{Z}^l) = \Gamma_0(T_0)$. According to Sard's theorem, cf. [28], the set of regular values of the smooth mapping $\Delta : \mathbf{R}^l/\mathbf{Z}^l \to \Gamma(T_0)$ has full measure, which implies that there exists a point where the tangent mapping is surjective, which in turn implies that $l \geq \dim T_0$. This completes the proof that $\dim T_0$ is equal to the minimal number of frequencies. $\square$

Remark 7.12 If the identity component G° of G is compact, then every relative equilibrium is quasi-periodic. If M is compact and the mapping $g \mapsto g \cdot m : G \to M$ is proper, then G is compact. If the G-action is proper, then the mapping $g \mapsto g \cdot m : G \to M$ is proper. Therefore every relative equilibrium is quasi-periodic if M is compact and the G-action is proper. $\oslash$

Remark 7.13 When G is compact, then Proposition 7.11 follows from applying Proposition B1 of Field [11] to the G-orbit $G \cdot m \simeq G/G_m$ through the point m. $\oslash$

7.2 Runaway Relative Equilibria

Let I be an open interval in $\mathbf{R}$ and $\gamma : I \to M$ a continuous curve in M. The mapping $\gamma : I \to M$ is proper, if and only if for every compact subset K of M the pre-image $\gamma^{-1}(K)$ is a compact subset of I. Because the continuity of γ implies that the pre-image of every closed subset is closed, and because the compact subsets of I are the closed subsets of I which are contained in a subset of the form $[a,\, b]$ for some $a,\, b \in I$, we see that γ is proper if and only if for every compact subset K there exist $a,\, b \in I$ such that $\gamma(t) \in K \Longrightarrow t \in [a,\, b]$, or equivalently $t \in I \setminus [a,\, b] \Longrightarrow \gamma(t) \notin K$. That is, $\gamma(t)$ runs out of every compact subset of M if t runs runs to either end of I.

Definition 7.14 A *runaway curve* in M is a continuous curve $\gamma : I \to M$ such that I is an open interval in $\mathbf{R}$ and the mapping $\gamma : I \to M$ is proper. $\oslash$

Proposition 7.15 *Let m be a relative equilibrium and X as in* (7.1), (7.2). *Assume that the mapping $g \mapsto g \cdot m : G \to M$ is proper, which is certainly the case if the G-action on M is proper. Then the following conditions are equivalent.*

i) *The element X of $\mathfrak{g}$ is not elliptic.*

ii) *The solution of* (1.1) *starting at m is a runaway curve in M.*

iii) *The solution of* (1.1) *starting at m is not quasi-periodic.*

Proof Suppose that i) holds. Then it follows from Lemma 7.8 that the mapping $t \mapsto \mathrm{e}^{t\,X} : \mathbf{R} \to G$ is proper. Because the mapping $g \mapsto g \cdot m : G \to M$ is proper by assumption, and the composition of two proper mappings is proper, the mapping

$$t \mapsto \mathrm{e}^{t\,X} \cdot m = \mathrm{e}^{t\,v}(m) : \mathbf{R} \to M$$

is proper. Here we have used the identity (7.2). This concludes the proof of i) $\Longrightarrow$ ii).

If the solution of (1.1) starting at m is quasi-periodic, then its image is contained in $\Gamma(\mathbf{R}^k/\mathbf{Z}^k)$, which is compact, because the standard torus $\mathbf{R}^k/\mathbf{Z}^k$ is compact and the mapping Γ is continuous. This is in contradiction with ii) and we have proved ii) $\Longrightarrow$ iii).

The implication iii) $\Longrightarrow$ i) follows from Proposition 7.11. $\square$

Remark 7.16 Without any additional assumption on the action, like the assumption of properness of the mapping $g \mapsto g \cdot m : G \to M$, not much in the spirit of Proposition 7.15 can be concluded.

For instance, if the vector field v is complete, then $(t,\, m) \mapsto \mathrm{e}^{t\, v}(m) : \mathbf{R} \times M \to M$ defines an action of the Lie group $G = (\mathbf{R},\, +)$ on M, and it follows from the commutativity of $(\mathbf{R},\, +)$ that (7.2) holds. Clearly every solution of (1.1) is a relative equilibrium, but it is certainly not true that for every dynamical system every solution is either quasi-periodic or running out of every compact subset. ⊘

Example 7.17 Let $G = \mathbf{R} \times \mathbf{C}$, with the multiplication

$$(\xi,\, \zeta)\,(\xi',\, \zeta') = \left(\xi + \xi',\, \mathrm{e}^{\mathrm{i}\,\xi}\, \zeta' + \zeta\right).$$

G can be identified with the universal covering of the group $\mathrm{E}(2)$ of motions in the plane, cf. Example 9.5. The only elliptic element in the Lie algebra of G is the zero element.

Let $M = \mathbf{R} \times \mathbf{R} \times \mathbf{C}$, on which we have the proper and free G-action defined by

$$((\xi,\, \zeta)\, ,\, (a,\, x,\, z)) \mapsto \left(a,\, \xi + x,\, \mathrm{e}^{\mathrm{i}\,\xi}\, z + \zeta\right).$$

The G-orbit space $G\backslash M$ can be identified with $\mathbf{R}$ and the canonical projection $\pi : M \to G\backslash M$ can be identified with the projection $(a,\, x,\, z) \mapsto a$.

On M we take the vector field v defined by

$$v(a,\, x,\, z) = \left(0,\, a,\, \mathrm{e}^{\mathrm{i}\,x}\right), \quad a \in \mathbf{R}, \quad x \in \mathbf{R}, \quad z \in \mathbf{C}.$$

It is readily verified that v is G-invariant, i.e. it satisfies (5.1). The solutions of (1.1) are given by

$$\begin{aligned}
a(t) &= a(0), \\
x(t) &= x(0) + t\, a(0), \\
z(t) &= \begin{cases} z(0) + \mathrm{e}^{\mathrm{i}\,x(0)} \left(\mathrm{e}^{\mathrm{i}\,t\,a(0)} - 1\right) / \,\mathrm{i}\, a(0) & \text{when} \quad a(0) \neq 0, \\ z(0) + t\, \mathrm{e}^{\mathrm{i}\,x(0)} & \text{when} \quad a(0) = 0. \end{cases}
\end{aligned}$$

All solutions are relative equilibria, and for each $m \in M$ the mapping $t \mapsto \mathrm{e}^{t\, v}(m)$ is proper. Nevertheless the action of $\mathbf{R}$ on M defined by the v-flow is *not* proper. Indeed, if $a(0) \neq 0$, and $t = 2\pi / a(0)$, then $a(t) = a(0)$, $x(t) = x(0) + 2\pi$ and $z(t) = z(0)$. If we now let $a(0)$ converge to 0 and keep $x(0)$ and $z(0)$ bounded, which implies that $(a(0),\, x(0),\, z(0))$ remains in a compact subset of M, then also $(a(t),\, x(t),\, z(t))$ remains in a compact subset of M, whereas t does not remain in any compact subset of $\mathbf{R}$. ⊘

7.3 When the Action is Not Free

Remark 7.5 and Proposition 7.11 are not quite optimal if the action is not free at m. In order to obtain stronger statements, we begin with the following observation.

Lemma 7.18 *If $m \in M$, $t \in I_m$, $g \in G$, and $\mathrm{e}^{t\,v}(m) = g \cdot m$, then $g \in \mathrm{N}(G_m)$.*

Proof Write $H = G_m$, i.e. $m \in M_H$. Lemma 5.5 implies that $g \cdot m = \mathrm{e}^{t\,v}(m) \in M_H$. It now follows from iii) $\Longrightarrow$ (i) in Lemma 3.3 that $g \in \mathrm{N}(H)$. $\square$

Write $H = G_m$ and $\mathfrak{h} = \mathfrak{g}_m$. It follows from (7.2) and Lemma 7.18 with $g = \mathrm{e}^{t\,X}$ that $\mathrm{e}^{t\,X} \in \mathrm{N}(H)$ for every $t \in \mathbf{R}$, and therefore $X + \mathfrak{h}$ belongs to the Lie algebra of the Lie group $\mathrm{N}(H)/H$. Because the Lie group $\mathrm{N}(H)/H$ acts freely on the isotropy type M_H of m, cf. Lemma 3.3, the following proposition is obtained from Proposition 7.11, by replacing the Lie group G by the Lie group $\mathrm{N}(H)/H$.

Proposition 7.19 *Assume that m is a relative equilibrium, with X as in* (7.1), (7.2)*. Write $H = G_m$ and $\mathfrak{h} = \mathfrak{g}_m$. Assume that $X + \mathfrak{h}$ is an elliptic element of the Lie algebra of the Lie group $\mathrm{N}(H)/H$, which implies that the closure of the one-parameter subgroup of $\mathrm{N}(H)/H$ generated by $X + \mathfrak{h}$ is a torus subgroup T of $\mathrm{N}(H)/H$. Then the solution of* (1.1) *which starts at m is quasi-periodic.*

More precisely, the same conclusions hold as in Proposition 7.11, with G and X replaced by $\mathrm{N}(H)/H$ and $X + \mathfrak{h}$, respectively. Moreover, Γ is a smooth embedding of $\mathbf{R}^k/\mathbf{Z}^k$ into M, the closure of the v-orbit through m is equal to $\Gamma(\mathbf{R}^k/\mathbf{Z}^k)$, which is a smooth submanifold of M, diffeomorphic to a k-dimensional torus, and k is equal to the minimal number of frequencies of the quasi-periodic solution.

The solution of (1.1) *starting at m is an equilibrium or a periodic solution if and only if $\dim T = 0$ or $\dim T = 1$, respectively.*

Remark 7.20 X is an elliptic element of $\mathfrak{g}$ $\Longleftrightarrow$ the set of all $\mathrm{e}^{t\,X}$, $t \in \mathbf{R}$, is contained in a compact subset of G, hence of the closed subgroup $\mathrm{N}(H)$ of H $\Longrightarrow$ the set of all $\mathrm{e}^{t\,(X+\mathfrak{h})}$, $t \in \mathbf{R}$, is contained in a compact subset of $\mathrm{N}(H)/H$ $\Longleftrightarrow$ $X + \mathfrak{h}$ is an elliptic element of the Lie algebra of $\mathrm{N}(H)/H$. Here the middle implication follows from the observation that the mapping $g \mapsto g\,H : \mathrm{N}(H) \to \mathrm{N}(H)/H$ is continuous.

If the mapping $g \mapsto g \cdot m : G \to M$ is proper, which is certainly the case if the G-action is proper, then $H = G_m$ is a compact subgroup of G and then the middle implication is an equivalence, i.e. $X + \mathfrak{h}$ is an elliptic element of the Lie algebra of $\mathrm{N}(H)/H$ if and only if X is an elliptic element of $\mathfrak{g}$. However, if H is not compact, then the condition that $X + \mathfrak{h}$ is an elliptic element of the Lie algebra of $\mathrm{N}(H)/H$ is weaker than the condition that X is an elliptic element of $\mathfrak{g}$, which makes Proposition 7.19 more general than Proposition 7.11.

In the case of non-free actions, Proposition 7.19 is somewhat simpler than Proposition 7.11, because the torus subgroup is defined in the group $\mathrm{N}(H)/H$ which acts freely on m. $\oslash$

Remark 7.21 If the identity component of $\mathrm{N}(H)/H$ is compact, then every relative equilibrium in the orbit type $M_{[H]}$ is quasi-periodic. $\oslash$

Remark 7.22 When G is compact, Proposition 7.19 follows from applying Proposition B1 of Field [11] to the G-orbit $G \cdot m \simeq G/G_m$ through the point m. $\oslash$

7.4 Other Relative Equilibria in the G-orbit

If m is a relative equilibrium, X is as in (7.1), (7.2), and $g \in G$, then it follows from (5.2) that

$$\mathrm{e}^{t\,v}(g \cdot m) = g \cdot \mathrm{e}^{t\,v}(m) = g \cdot \mathrm{e}^{t\,X} \cdot m = \left(g\, \mathrm{e}^{t\,X}\, g^{-1}\right) \cdot g \cdot m = \mathrm{e}^{t\,(\mathrm{Ad}\,g)(X)} \cdot (g \cdot m). \tag{7.6}$$

Here we have used the notation $\mathrm{Ad}\,g$ for the adjoint transformation or the infinitesimal conjugation in the Lie algebra $\mathfrak{g}$, which is defined by the element g of the Lie group, cf. [10, 1.1.9 and Th. 1.5.2,(b)].

This shows that $g \cdot m$ is also a relative equilibrium, and that (7.2) holds with m and X replaced by $g \cdot m$ and $(\mathrm{Ad}\,g)(X)$, respectively,

The solution of (1.1) starting at $g \cdot m$ is constant, periodic, quasi-periodic and runaway if and only if the solution of (1.1) starting at m has these properties. Moreover, these solutions define a smooth fibration of the orbit $G \cdot m$ through m.

For instance, if we are in the situation of Proposition 7.11 and the action is free at m, then the mapping (2.5) induces a diffeomorphism A_m from G onto $G \cdot m$, where in $G \cdot m$ we use the orbit topology. The action $(h,\, g) \mapsto g\,h : T \times G \to G$ of T on G by means of right multiplications is proper and free, making G into a principal T-bundle over G/T. This fibration is invariant under the action of G on itself defined by multiplications from the left. On G we also have the vector field X^{L} which is invariant under multiplications from the left by elements of G and which is equal to X at the identity element. The flow of X^{L} is given by $(t,\, g) \mapsto g\, \mathrm{e}^{t\,X}$, cf. [10, Lemma 1.3.1], and the closures of its orbits are the fibers of the T-bundle which we just have introduced. The diffeomorphism A_m intertwines X^{L} with v, and A_m maps the fibers of the principal T-bundle onto the closures of the v-orbits in $G \cdot m$.

If the action is not free, we write $H = G_m$ and first apply the above to the $\mathrm{N}(H)$-orbit through m, which according to Lemma 3.3 is equal to the intersection of $G \cdot m$ with the isotropy type M_H of m. If T is as in Proposition 7.19, then we obtain that the closures of the v-orbits through the points of $\mathrm{N}(H) \cdot m$ define a smooth $\mathrm{N}(H)/H$-invariant principal T-fibration of $\mathrm{N}(H) \cdot m$, with base space diffeomorphic to the homogeneous space $(\mathrm{N}(H)/H)/T$.

If $g,\, g' \in G$ and $h,\, h' \in \mathrm{N}(H)/H$, then $g \cdot h \cdot m = g' \cdot h' \cdot m$ if and only if there exists a $k \in \mathrm{N}(H)$ such that $h' = (k\,H)\,h$ and $g' = g\,k^{-1}$. In other words, if we let $k \in \mathrm{N}(H)$ act on $G \times (\mathrm{N}(H)/H)$ by sending $(g,\, h)$ to $(g\,k^{-1},\, (k\,H)\,h)$, then the mapping $(g,\, h) \mapsto g \cdot h \cdot m$ induces a diffeomorphism Φ from $G \times_{\mathrm{N}(H)} (\mathrm{N}(H)/H)$ onto $G \cdot m$.

The action of $\mathrm{N}(H)$ on $G \times (\mathrm{N}(H)/H)$ commutes with the G-action and the T-action on $G \times (\mathrm{N}(H)/H)$ defined by multiplication from the left on the first factor and multiplication from the right on the second factor, respectively. In order to conclude that the T-action on $G \times (\mathrm{N}(H)/H)$ induces a proper, free and G-invariant T-action on $G \times_{\mathrm{N}(H)} (\mathrm{N}(H)/H)$, we formulate the following lemma.

Lemma 7.23 *Let G be a Lie group with closed Lie subgroup H, and let K be another Lie group. Let V be a smooth manifold on which we have commuting smooth actions of H and K. Let $h \in H$ act on $G \times V$ by sending $(g,\, v)$ to $(g\,h^{-1},\, h \cdot v)$, as usual we denote the corresponding H-orbit space by $G \times_H V$.*

Then the canonical projection from $G \times V$ onto the orbit space $G \times_H V$ intertwines the G-action on $G \times V$ by left multiplications on the first factor and the K-action on $G \times V$ on the second factor with a uniquely defined smooth G-action and K-action on $G \times_H V$.

The K-action on $G \times_H V$ commutes with the G-action on $G \times_H V$, and the K-action on $G \times_H V$ is proper and free if the K-action on V is proper and free, respectively.

Proof For $h \in H$, $k \in K$, $(g, v) \in G \times V$, we have

$$h \cdot k(g, v) = h \cdot (g, k \cdot v) = (g\, h^{-1}, h \cdot k \cdot v) = (g\, h^{-1}, k \cdot h \cdot v) = k \cdot (g\, h^{-1}, h \cdot v) = k \cdot h \cdot (g, v).$$

This shows that the H-action on $G \times V$ commutes with the K-action on $G \times V$, and therefore the canonical projection from $G \times V$ onto $G \times_H V$ intertwines the K-action on $G \times V$ with a uniquely defined smooth K-action on $G \times_H V$.

Now suppose that the K-action on V is free. If $k \in K$, $(g, v) \in G \times V$ and there exists $h \in H$ such that $k \cdot (g, v) = (g\, h^{-1}, h \cdot v)$, then $h = 1$ and $k \cdot v = v$, which in turn implies that $k = 1$. This shows that the K-axtion on $G \times_H V$ is free.

Now assume that the K-action on V is proper. Let $[g_j, v_j]$ be an infinite sequence in $G \times_H V$ which converges to $[g, v]$, this sequence can be represented by a sequence (g_j, v_j) in $G \times V$ which converges to (g, v). Suppose that k_j is a sequence in K such that $k_j \cdot [g_j, v_j]$ converges to $[g', v']$. This means that there exists a sequence $h_j \in H$ such that $k_j \cdot h_j \cdot (g_j, v_j)$ converges to (g', v'), i.e. $g_j\, h_j^{-1} \to g'$ and $k_j \cdot h_j \cdot v_j \to v'$. Because $g_j \to g$ it follows from $g_j\, h_j^{-1} \to g'$ that $h_j \to h := (g')^{-1}\, g$, where $h \in H$ because H is a closed subset of G. The continuity of the H-action on V then implies that $h_j \cdot v \to h \cdot v$ in V, and it now follows from $k_j \cdot h_j \cdot v_j \to v'$ and the properness of the K-action on V that a subsequence of the k_j converges to some element of K. This proves that the K-action on $G \times_H V$ is proper.

Finally the K-action on $G \times V$ obviously commutes with the G-action on $G \times V$ defined by left multiplications on the first factor, which implies that the K-action on $G \times_H V$ commutes with the G-action on $G \times_H V$. □

We apply Lemma 7.23 with V, H and K replaced by $\mathrm{N}(H)/H$, $\mathrm{N}(H)$ and T, respectively. It follows that the canonical projection from $G \times (\mathrm{N}(H)/H)$ onto its $\mathrm{N}(H)$-orbit space $G \times_{\mathrm{N}(H)} (\mathrm{N}(H)/H)$ intertwines the G-action and the T-action on $G \times (\mathrm{N}(H)/H)$ with a uniquely defined Gaction and T-action on $G \times_{\mathrm{N}(H)} (\mathrm{N}(H)/H)$. Moreover, the T-action is proper and free, and commutes with the G-action.

The mapping Φ intertwines the G-action on $G \times_{\mathrm{N}(G_m)} (\mathrm{N}(G_m)/G_m)$ with the G-action on $G \cdot m$, and Φ intertwines the T-action on $G \times_{\mathrm{N}(G_m)} (\mathrm{N}(G_m)/G_m)$, with a proper and free action of T on $G \cdot m$, of which the orbits are the closures of the v-orbits in $G \cdot m$. In this way we arrive at the following conclusion.

Proposition 7.24 *Under the assumptions of Proposition 7.19, the closures of the v-orbits in $G \cdot m$ define a smooth G-invariant principal T-fibration of $G \cdot m$. The vector field v on $G \cdot m$ is equal to the infinitesimal T-action on $G \cdot m$ of the element $X + \mathfrak{g}_m$ in the Lie algebra $\mathfrak{t}$ of T.*

Similar observations can be made in the other cases that the relative equilibrium is an equilibrium, a periodic solution, or a runaway solution.

8 Relative Periodic Solutions

Recall that I_m is the interval of definition of the maximal solution of (1.1) which starts at m.

Lemma 8.1 *Let $\tau \in I_m$, $\tau \neq 0$. Then the following conditions are equivalent.*

i) *There exists an element $s \in G$ such that*

$$\mathrm{e}^{\tau\, v}(m) = s \cdot m. \tag{8.1}$$

ii) *$I_m = \mathbf{R}$ and there exists an element $s \in G$ such that*

$$\mathrm{e}^{t\, v}(m) = s^p \cdot \mathrm{e}^{(t-p\,\tau)\, v}(m) = \mathrm{e}^{(t-p\,\tau)\, v}(s^p \cdot m), \quad t \in [p\,\tau,\, (p+1)\,\tau] \tag{8.2}$$

for every $p \in \mathbf{Z}$. Here we have assumed that $\tau > 0$. If $\tau < 0$ then we have the same statement with τ and s replaced by $-\tau$ and s^{-1}, respectively.

iii) *$G \cdot m$ is a periodic point of period τ of the induced flow in $G\backslash M$, i.e. $\Phi^\tau(G \cdot m) = G \cdot m$.*

The element s in ii) *can be chosen as the one in* i)*, and is unique up to muliplication to the right by an element of G_m.*

Proof We only need to prove i) $\Longrightarrow$ ii).

Assume that i) holds and that $\tau > 0$. Note that $t \in [p\,\tau,\, (p+1)\,\tau]$ if and only if $t - p\,\tau \in [0,\, \tau]$. Therefore the right hand side of (8.2) defines is a solution γ of (1.1) on $[p\,\tau,\, (p+1)\,\tau]$, with $\gamma(p\,\tau) = s^p \cdot m$ and $\gamma((p+1)\,\tau) = s^p \cdot \mathrm{e}^{\tau\, v}(m) = s^p \cdot s \cdot m = s^{p+1} \cdot m$. These curves, when p ranges over $\mathbf{Z}$, piece together to a solution of (1.1) on $\mathbf{R}$ such that $\gamma(0) = m$. This proves ii). Note that (8.1) implies that $m = s^{-1} \cdot \mathrm{e}^{\tau\, v}(m) = \mathrm{e}^{\tau\, v}(s^{-1} \cdot m)$, hence $\mathrm{e}^{-\tau\, v}(m) = s^{-1} \cdot m$. Therefore, if $\tau < 0$, we have (8.2) with τ and s replaced by $-\tau$ and s^{-1}, respectively. □

Definition 8.2 The solution of (1.1) starting at m is called a *relative periodic solution* with *relative period* τ, if one (each) of the conditions i) – iii) in Lemma 8.1 holds. The element $s \in G$ is called the corresponding *shift element* in the symmetry group G. The element $m \in M$ is called a *relative periodic point* and τ a *relative period.* ⊘

8.1 Quasi-periodic Relative Periodic Solutions

Let G be a Lie group and $s \in G$. Let $\langle s \rangle$ denote the set of all integral powers s^p, $p \in \mathbf{Z}$, which is the smallest subgroup of G which contains s. The closure S of $\langle s \rangle$ in G is the smallest closed subgroup of G which contains s, and is called the *closed subgroup of G which is generated by s.*

Lemma 8.3 *Then either the mapping $p \mapsto s^p : \mathbf{Z} \to G$ is proper, or the closed subgroup S of G which is generated by s is a compact subgroup of G.*

In the first case, $S = \langle s \rangle \simeq \mathbf{Z}$. In the second case, we have:

i) *If $T := S^\circ$ denotes the identity component of S, then T is a torus subgroup of G.*

ii) *There is a unique smallest positive integer p_0 such that $s^{p_0} \in T$, and the mapping $\varphi : p \mapsto s^p\, T$ induces an isomorphism from $\mathbf{Z}/p_0\,\mathbf{Z}$ onto S/T. The dimension of T (which is equal to the dimension of S) is equal to zero, if and only if $s^{p_0} = 1$, in which case $S = \langle s \rangle \simeq \mathbf{Z}/p_0\,\mathbf{Z}$.*

iii) *If $\mathfrak{t}$ denotes the Lie algebra of T, then $s^{p_0} = \mathrm{e}^X$ for an element $X \in \mathfrak{g}$ which is uniquely determined module the integral lattice $\Lambda = \ker \exp$ in $\mathfrak{t}$. If λ_j, $1 \leq j \leq k$ denotes a Z-basis of Λ, and $X = \sum_{j=1}^k X_j\, \lambda_j$, then the real numbers $X_1, \ldots, X_k$ and 1 are linearly independent over $\mathbf{Q}$.*

Proof The closure S in G of the commutative subgroup $\langle s \rangle$ of G is a commutative subgroup of G. As a closed subgroup of the Lie group G, S is a Lie subgroup of G, cf. [10, Cor. 1.10.7].

Let $T = S^\circ$ denote the identity component of S, which is an open and closed subgroup of S, cf. [10, Th. 1.9.1], and therefore is a connected and commutative Lie subgroup of G. T is closed in G because S is closed in G.

Suppose that the homomorphism $\varphi : p \mapsto s^p\, T : \mathbf{Z} \to S/T$ is injective, and that we have an unbounded sequence of integers p_i such that the s^{p_i} converge to some element g of G. Because S is closed in G, we have $g \in S$ and $s^{p_i} \to g$ in S. Because $g\,T$ is an open neighborhood of g in S, we have $s^{p_i} \in g\,T$ for all sufficiently large i, which implies that $\varphi(p_i) = \varphi(p_j)$ for all sufficiently large i, j, in contradiction with the injectivity of φ. This shows that the mapping $p \mapsto s^p : \mathbf{Z} \to G$ is proper if φ is injective.

If φ is not injective, then there is a unique positive integer p_0 such that $\ker \varphi = p_0\,\mathbf{Z}$, which means that there is a unique smallest positive integer p_0 such that $t := s^{p_0} \in T$. Because $\langle s \rangle$ is dense in S, each connected component of S contains an s^p for some $p \in \mathbf{Z}$. Therefore the homomorphism $\varphi : \mathbf{Z} \to S/T$ is surjective, and induces an isomorphism from $\mathbf{Z}/p_0\,\mathbf{Z}$ onto S/T.

We have $s^p \in T$ if and only if $p \in p_0\,\mathbf{Z}$. Because $\langle s \rangle$ is dense in S, $\langle t \rangle$ is dense in T. Because T is a connected and commutative Lie group, there exist $k,\, l \in \mathbf{Z}_{\geq 0}$, such that T is isomorphic to $(\mathbf{R}/\mathbf{Z})^k \times\ R^l$, cf. [10, Cor. 1.12.4]. If $(a + \mathbf{Z}^k,\, b) \in (\mathbf{R}^k/\mathbf{Z}^k) \times \mathbf{R}^l$ corresponds to the element $t \in T$, then for each integer q the element $(q\,a + \mathbf{Z}^k,\, q\,b)$ corresponds to t^q. If $b \neq 0$, then the density of $\langle t \rangle$ in T leads to a contradiction. Therefore $b = 0$, and the density of $\langle t \rangle$ in T implies that $l = 0$, which proves that T is a torus subgroup of G.

Because the exponential mapping from $\mathfrak{t}$ to T is a surjective homomorphism, there exists an $X \in \mathfrak{t}$, uniquely determined modulo adding an element of Λ, such that $t = s^{p_0} = \mathrm{e}^X$. In view of the isomorphism from $\mathfrak{t}/\Lambda$ onto T induced by the exponential mapping $\exp : \mathfrak{t} \to T$, it follows that the $q\,X + \Lambda$, $q \in \mathbf{Z}$, are dense in $\mathfrak{t}/\Lambda$. In view of the isomorphism from $\mathbf{R}^k/\mathbf{Z}^k$

onto $\mathfrak{t}/\Lambda$ induced by the mapping $\theta \mapsto \sum_{j=1}^{k} \theta_j \, \lambda_j$, this leads to the linear independence of $X_1, \ldots, X_k, 1$ over $\mathbf{Q}$, cf. [10, p. 61]. $\square$

Definition 8.4 Let G be a Lie group and $s \in G$. Then s is called an *elliptic element* of G if the closed subgroup of G which is generated by s is a compact subgroup of G. In view of Lemma 8.3, this condition is equivalent to the condition that there exists a compact subset K of G and an unbounded sequence of integers p_i, such that $s^{p_i} \in K$ for every i. $\oslash$

Proposition 8.5 *Assume that the solution of* (1.1) *which starts at m is a relative periodic solution, and suppose that the shift element s in* (8.1), (8.2) *is an elliptic element of G. Then the solution of* (1.1) *which starts at m is quasi-periodic.*

More precisely, in the notation of Lemma 8.3, the mapping Γ in Definition 7.6, with k replaced by $k+1$, can be defined by

$$\Gamma : (\theta_1, \ldots, \theta_k, \theta_{k+1}) \to \exp\left(\sum_{j=1}^{k} \theta_j \, \lambda_j - \theta_{k+1} \, X\right) \cdot \mathrm{e}^{\theta_{k+1} \, p_0 \, \tau \, v}(m), \tag{8.3}$$

with frequencies $\nu_j = X_j / p_0 \, \tau$, $1 \leq j \leq k$, and $\nu_{k+1} = 1/p_0 \, \tau$.

The closure of v-orbit through m is equal to $\Gamma(\mathbf{R}^{k+1}/\mathbf{Z}^{k+1})$. The pre-image $\Gamma^{-1}(\{m\})$ of m is a closed sugroup of $\mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$ and Γ induces a smooth embedding of the torus $T_0 := (\mathbf{R}^{k+1}/\mathbf{Z}^{k+1})/\Gamma^{-1}(\{m\})$ into M, with image equal to the closure of v-orbit through m, making the latter diffeomorphic to T_0. The minimal number of frequencies is equal to the dimension of T_0 = the dimension of the closure of the v-orbit through m.

If τ is the smallest positive real number t such that $\mathrm{e}^{t\,v}(m) \in G \cdot m$, and the G-action is free at m, then the mapping Γ itself is a smooth embedding of $\mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$ into M, and the minimal number of frequencies is equal to $k+1$.

Proof If $\theta'_j = \theta_j + n_j$ with $n_j \in \mathbf{Z}$, then

$$\begin{aligned}
\Gamma(\theta') &= \exp\left(\sum_{j=1}^{k} \theta_j \, \lambda_j - \theta_{k+1} \, X\right) \mathrm{e}^{-n_{k+1} \, X} \cdot \mathrm{e}^{\theta_{k+1} \, p_0 \, \tau \, v} \circ \mathrm{e}^{n_{k+1} \, p_0 \, \tau \, v}(m) \\
&= \exp\left(\sum_{j=1}^{k} \theta_j \, \lambda_j - \theta_{k+1} \, X\right) \cdot \mathrm{e}^{\theta_{k+1} \, p_0 \, \tau \, v} \left(s^{-n_{k+1} \, p_0} \cdot \mathrm{e}^{n_{k+1} \, p_0 \, \tau \, v}(m)\right) \\
&= \exp\left(\sum_{j=1}^{k} \theta_j \, \lambda_j - \theta_{k+1} \, X\right) \cdot \mathrm{e}^{\theta_{k+1} \, p_0 \, \tau \, v}(m) = \Gamma(\theta).
\end{aligned}$$

Here we have used, in the second identity, (5.2) and $s^{p_0} = \mathrm{e}^X$, cf. iii) in Lemma 8.3. In the third identity we have used (8.2) with $t = p, \tau$ and $p = n_{k+1} \, p_0$. It follows that Γ is a well-defined smooth mapping from $\mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$ to M.

If $\theta_j = t\,X_j/p_0\,\tau$, $1 \leq j \leq k$, and $\theta_{k+1} = t/p_0\,\tau$, then

$$\sum_{j=1}^{k} \theta_j\,\lambda_j - \theta_{k+1}\,X = (t/p_0\,\tau)\left(\sum_{j=1}^{k} X_j\,\lambda_j - X\right) = 0$$

and $\theta_{k+1}\,p_0\,\tau = t$. In view of the definition of Γ this implies that the solution curve γ of (1.1) which starts at m is given by $\gamma(t) = \Gamma(t\,\nu + \mathbf{Z}^{k+1})$, $t \in \mathbf{R}$, if $\nu_j = X_j/p_0\,\tau$, $1 \leq j \leq k$ and $\nu_{k+1} = 1/p_0\,\tau$.

Because the numbers $X_1, \ldots, X_k$ and 1 are linearly independent over $\mathbf{Q}$, the numbers ν_j, $1 \leq j \leq k+1$ are linearly independent over $\mathbf{Q}$, and it follows in view of iii) in Lemma 7.10 that the set of all $t\,\nu + \mathbf{Z}^{k+1}$, $t \in \mathbf{R}$ is dense in $\mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$. Applying the continuous mapping Γ, we obtain that the v-orbit through m is dense in $\Gamma(\mathbf{R}^{k+1}/\mathbf{Z}^{k+1})$.

It follows from (5.2) and $\gamma(m) = m$ that

$$((g,\,t),\,m) \mapsto g \cdot \mathrm{e}^{t\,v}(m) = \mathrm{e}^{t\,v}(g \cdot m)$$

is a local smooth action of $T \times \mathbf{R}$ in M, and it follows that $(g,\,t) \mapsto g \cdot \mathrm{e}^{t\,v}(m)$ defines a smooth immersion from $(T \times \mathbf{R})/(T \times \mathbf{R})_m$ into M. This proves that Γ induces a smooth embedding of T_0 into M and that the minimal number of frequencies is equal to $\dim T_0$, in the same way as for the corresponding statements in Proposition 7.11.

Now suppose that τ is the smallest positive number t such that $\mathrm{e}^{t\,v}(m) \in G \cdot m$, and that the T-action is free at m, which certainly is the case if the G-action is free at m. Then $\Gamma(\theta) = m$ implies that $\mathrm{e}^{\theta_{k+1}\,p_0\,\tau\,v}(m) \in T \cdot m \subset G \cdot m$, hence there exists a $p \in \mathbf{Z}$ such that $\theta_{k+1}\,p_0\,\tau = p\,\tau$, which in view of the freeness of the T-action at m implies that $\mathrm{e}^{\theta_{k+1}\,p_0\,\tau\,v}(m) = s^p \cdot m \in T \cdot m$. Again using the freeness of the T-action at m, it follows that there exists a $q \in \mathbf{Z}$ such that $p = q\,p_0$, and we conclude that $\theta_{k+1} = q \in \mathbf{Z}$. However, $\mathrm{e}^X = s^{p_0}$, and therefore $\mathrm{e}^{-\theta_{k+1}\,X} = s^{-q\,p_0}$, and because $\mathrm{e}^{\theta_{k+1}\,p_0\,\tau\,v}(m) = s^p \cdot m = s^{q\,p_0}$, it follows from (8.3) that

$$\exp\left(\sum_{j=1}^{k} \theta_j\,\lambda_j\right) \cdot m = m,$$

which in view of the freeness of the T-action at m implies that $\theta_j \in \mathbf{Z}$ for every $1 \leq j \leq k$. This proves that $\Gamma^{-1}(\{m\}) = \{1\}$, and therefore $T_0 = \mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$. □

Remark 8.6 If X is an elliptic element of $\mathfrak{g}$, then $\mathrm{e}^{t\,X}$ is an elliptic element of G for every $t \in \mathbf{R}$. This shows that Proposition 8.5 is a generalization of Proposition 7.11. ⊘

Remark 8.7 Suppose that m is not a relative equilibrium. If the G-orbit $G \cdot m$ through m is a closed subset of M then there is a minimal positive τ such that $\mathrm{e}^{\tau\,v}(m) \in G \cdot m$. Note that $G \cdot m$ is closed in M if and only if the one point set $\{G \cdot m\}$ is a closed subset of the orbit space $G\backslash M$, and that this certainly is the case if the G action on M is proper.

If there *exists* such a minimal positive relative period τ, then $\mathrm{e}^{t\,v}(m) \in G \cdot m$ if and only if $t = p\,\tau$ for some integer p. Assuming also that the action is free at m, let s be the unique element of G such that $e^{\tau v}(m) = s \cdot m$. Then $e^{p \tau v}(m) = s^p \cdot m$ for every $p \in \mathbf{Z}$. Let $q \in \mathbf{Z}$, $q \neq 0$ and assume that s^q is an elliptic element of G. Then also s^{-q} is an elliptic element of G and we may assume that $q > 0$. The closure A in G of the set of all integral powers of s^q is compact. Because the closure B of the set of all integral powers of s is equal to the union of the finitely many $s^j A$, $0 \leq j \leq q-1$, it follows that B is compact as well, i.e. s is elliptic, and we have the conclusion of Proposition 8.5 that Γ is a smooth embedding and that the minimal number of frequencies is equal to $k+1$.

The following two examples illustrate what can happen if we drop the condition that the G-orbit through m is closed.

a) Let $M = \mathbf{R}/\mathbf{Z}$, $G = \mathbf{Q}/\mathbf{Z}$ acting freely on M by translations and v equal to the constant vector field equal to 1. Take $m = 0$. Then $e^{t\, v}(m) \in G \cdot m$ if and only if $t \in \mathbf{Q}$, in which case $e^{t\, v}(m) = s \cdot m$, in which $s = t + \mathbf{Z}$. If $t \in \mathbf{Q}$, then the set of all $p\, t + \mathbf{Z}$, $p \in \mathbf{Z}$, is a finite subgroup of $G = \mathbf{Q}/\mathbf{Z}$, and therefore s is elliptic. However, there is no minimal positive τ such that $e^{\tau\, v}(m) \in G \cdot m$.

b) In the example a) the group G is discrete and countably infinite, and therefore it has a countably infinite set of connected components. Now let $M = \mathbf{R}^2/\mathbf{Z}^2$ and consider the connected Lie subgroup

$$G := \{a\,(1,\, y) + \mathbf{Z}^2 \mid a \in \mathbf{R}\}$$

of $\mathbf{R}^2/\mathbf{Z}^2$, acting freely on M by translations. Let v be equal to the constant vector field equal to $(1,\, 0)$. We take $y \in \mathbf{R} \setminus \mathbf{Q}$, which implies that $a \mapsto a\,(1,\, y) + \mathbf{Z}^2$ defines an isomorphism from $\mathbf{R}$ onto G. $a\,(1,\, y) + \mathbf{Z}^2$ is an elliptic element of G, if and only if $a \in \mathbf{Q}$.

We have $t\, v + \mathbf{Z}^2 \in G$, if and only if there exists an $a \in \mathbf{R}$ such that $t \in a + \mathbf{Z}$ and $a\, y \in \mathbf{Z}$. This means that there exist $p,\, q \in \mathbf{Z}$ such that $a = p/y$ and $t = q + p/y$. The set of these t does not have a smallest positive element.

On the other hand, $a\,(1,\, y) + \mathbf{Z}^2$, with a as above, is an elliptic element of G if and only if $p = 0$, in which case $t = q$. Therefore the conditions in Proposition 8.5 are satified with $\tau = 1$ and $s = (0,\, 0) + \mathbf{Z}^2$. Also, Γ is an embedding.

$\oslash$

Remark 8.8 If G is compact, then every relative periodic solution is quasi-periodic. $\oslash$

Remark 8.9 For actions of compact Lie groups, Proposition 8.5 follows from applying Proposition B2 of Field [11] to the pre-image in M of the periodic orbit in $G\backslash M$ under the canonical projection $\pi : M \to G\backslash M$. $\oslash$

Remark 8.10 One may wonder whether a result analogous to Proposition 8.5 holds for quasi-periodic motions in $G\backslash M$.

Assume that $M = (\mathbf{R}/\mathbf{Z})^k \times (\mathbf{R}/\mathbf{Z})^l$, on which we have a vector field $v : (a,\, b) \mapsto (\dot{a},\, \dot{b}(a))$, in which $\dot{a} \in \mathbf{R}^k$ is constant and $\dot{b}(a) \in \mathbf{R}^l$ only depends on $a \in (\mathbf{R}/\mathbf{Z})^k$, in a smooth way. The differential equation (1.1) then is equivalent to the system

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \dot{a}, \quad \frac{\mathrm{d}b}{\mathrm{d}t} = \dot{b}(a). \tag{8.4}$$

In this situation v is invariant under the action of the torus group $G = (\mathbf{R}/\mathbf{Z})^l$ by multiplication on the second factor, the orbit space $G\backslash M$ can be identified with $(\mathbf{R}/\mathbf{Z})^k$ on which the flow is given by $a(0) \mapsto a(0) + t\,\dot{a} + \mathbf{Z}^k$, which is quasi-periodic. The minimal number of frequencies of this flow is equal to k if and only if the coordinates $\dot{a}_j$, $1 \leq j \leq k$ are linearly independent over $\mathbf{Q}$, cf. Lemma 7.10, iii).

If we substitute the solution $t\,\dot{a}$ modulo $\mathbf{Z}^k$ in the second equation in (8.4), we see that $b(t)$ is obtained by integration of the function $t \mapsto \dot{b}(t\, \cdot a)$. For the function $a \mapsto \dot{b}(a)$ we have the Fourier series

$$\dot{b}(a) = \sum_{n \in \mathbf{Z}^k} \mathrm{e}^{2\pi\,\mathrm{i}\,\langle n,\, a\rangle}\; \dot{b}_n, \tag{8.5}$$

in which the inner product

$$\langle n,\, a\rangle = \sum_{j=1}^{k} n_j\, a_j \in \mathbf{R}/\mathbf{Z}$$

defines a homomorphism $a \mapsto \langle n,\, a\rangle$ from $(\mathbf{R}/\mathbf{Z})^k$ to $\mathbf{R}/\mathbf{Z}$. If in the right hand side of (8.5) we substitute $a = t\,\dot{a}$ and perform a formal termwise integration, then we arrive at

$$b(t) = b(0) + t \sum_{n \in \mathbf{Z}^k,\, \langle n,\, \dot{a}\rangle = 0} \dot{b}_n + \sum_{n \in \mathbf{Z}^k,\, \langle n,\, \dot{a}\rangle \neq 0} \mathrm{e}^{2\pi\,\mathrm{i}\,t\,\langle n,,\, \dot{a}\rangle}\, (2\pi\,\mathrm{i}\,\langle n,\, \dot{a}\rangle)^{-1}\, \dot{b}_n \tag{8.6}$$

modulo $\mathbf{Z}^l$.

The assumption that the function $a \mapsto \dot{b}(a)$ is smooth is equivalent to the condition that for every $N > 0$ there is a constant C_N such that $\|\dot{b}_n\| \leq C_N\,(1 + \|n\|)^{-N}$ for every $n \in \mathbf{Z}^k$. This shows that the coefficient of t in (8.6) is given by a nicely convergent Fourier series. Note that this coefficient is equal to $\dot{b}_0$, if and only if the minimal number of frequencies of the quasi-periodic function $a(t)$ is equal to k.

However, the factors $\langle n, ,\, \dot{a}\rangle^{-1}$ in the last sum in (8.6) might become so small for a suitable infinite sequence of $n \in \mathbf{Z}^k$, that the series does not converge and we cannot use the series in order to conclude that the function $t \mapsto b(t)$ is quasi-periodic.

Also, if we vary $\dot{a}$ continuously in $\mathbf{R}^k$, then the coefficient of t in (8.6) may vary wildly. These problems caused by the denominators $\langle n, ,\, \dot{a}\rangle$ becoming small or equal to zero show that, already in this simple looking example, the reconstruction of quasi-periodic solutions in $G\backslash M$ can have subtle aspects. $\oslash$

8.2 Runaway Relative Periodic Solutions

Proposition 8.11 *Let m be a relative periodic point with shift element s as in* (8.1), (8.2). *Assume that the mapping $g \mapsto g \cdot m : G \to M$ is proper, which is certainly the case if the G-action on M is proper. Then the following conditions are equivalent.*

i) *The element s of G is not elliptic.*

ii) *The solution of* (1.1) *starting at m is a runaway curve in M.*

iii) *The solution of* (1.1) *starting at m is not quasi-periodic.*

Proof If i) holds, then it follows from Lemma 8.3 that the mapping $p \mapsto s^p \cdot m : \mathbf{Z} \to M$ is proper. Let K be a compact subset of M, and assume that $\tau > 0$. Then

$$K' := \{\mathrm{e}^{-s\,v}(x) \;\mid\; x \in K,\; s \in [0,\, \tau]\}$$

is equal to the image of the compact set $[0,\, \tau] \times K$ under the continuous mapping $(s,\, x) \mapsto \mathrm{e}^{-s\,v}(x)$, and therefore K' is a compact subset of M. Because the mapping $p \mapsto s^p \cdot m : \mathbf{Z} \to M$ is proper, there exists an $N \in \mathbf{Z}_{>0}$ such that $|p| \leq N$ whenever $s^p \cdot m \in K'$. Therefore, if $t \in [p\,\tau,\, (p+1)\,\tau]$ and $\mathrm{e}^{t\,v}(m) \in K$, then it follows from (8.2) that

$$s^p \cdot m = \mathrm{e}^{-(t - p\,\tau)\,v} \circ \mathrm{e}^{t\,v}(m) \in K',$$

hence $|p| \leq N$ and therefore $|t| = |(t - p\,\tau) + p\,\tau| \leq \tau + N\,\tau$. This proves that the mapping $t \mapsto \mathrm{e}^{t\,v}(m) : \mathbf{R} \to M$ is proper, i.e. we have proved ii). For $\tau < 0$ we replace τ and s by $-\tau$ and s^{-1}, respectively.

If the solution of (1.1) starting at m is quasi-periodic, then its image is contained in the image in M of a torus under a continuous map, which is a compact subset of M. This is in contradiction with ii) and we have proved ii) $\Longrightarrow$ iii).

The implication iii) $\Longrightarrow$ i) follows from Proposition 8.5. $\square$

Remark 8.12 Without any additional assumption on the action, like the assumption of properness of the mapping $g \mapsto g \cdot m : G \to M$, not much in the spirit of Proposition 8.11 can be concluded.

For instance, suppose that the vector field v is complete, and let τ be any nonzero real number. Then $(p,\, m) \mapsto \mathrm{e}^{p\,\tau\,v}(m) : \mathbf{Z} \times M \to M$ defines an action of the discrete Lie group $G = (\mathbf{Z},\, +)$ on M for which (7.2) holds. Clearly every solution of (1.1) is a relative periodic solution with relative period τ, but it is certainly not true that for every dynamical system every solution is either quasi-periodic or running away out of every compact subset.

$\oslash$

8.3 When the Action is Not Free

Proposition 8.5 is not quite optimal if the action is not free at m.

It follows from (8.1) and Lemma 7.18 with $t = \tau$ and $g = s$ that $s \in \mathrm{N}(H)$, where $H := G_m$. Because the Lie group $\mathrm{N}(H)/H$ acts freely on the isotropy type M_H of m, cf. Lemma 3.3, the following proposition is obtained from Proposition 8.5, by replacing the Lie group G by the Lie group $\mathrm{N}(H)/H$.

Proposition 8.13 *Assume that m is a relative periodic point with shift element $s \in G$ as in* (8.1), (8.2)*. Write $H := G_m$ and sssume that $s\,H$ is an elliptic element of the Lie group $\mathrm{N}(H)/H$. Then the solution of* (1.1) *which starts at m is quasi-periodic.*

More precisely, we have the same conclusions as in Proposition 8.5, with G and s replaced by $\mathrm{N}(H)/H$ and $s\,H$, respectively.

If τ is the smallest positive real number t such that $\mathrm{e}^{t\,v}(m) \in G \cdot m$, then Γ is a smooth embedding of $\mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$ into M, and the minimal number of frequencies is equal to $k+1$.

Remark 8.14 s is an elliptic element of $G \Longleftrightarrow$ the closed subgroup of G generated by s is a compact subset of G, hence of the closed subgroup $\mathrm{N}(H)$ of $G \Longrightarrow$ the closed subgroup of $\mathrm{N}(H)/H$ generated by $s\,H$ is a compact subset of $\mathrm{N}(H)/H \Longleftrightarrow s\,H$ is an elliptic element of $\mathrm{N}(H)/H$. Here the middle implication follows from the observation that the mapping $g \mapsto g\,H : \mathrm{N}(H) \to \mathrm{N}(H)/H$ is continuous.

If the mapping $g \mapsto g \cdot m : G \to M$ is proper, which is certainly the case if the G-action is proper, then $H = G_m$ is a compact subgroup of G and then the middle implication is an equivalence, i.e. $s\,H$ is an elliptic element of $\mathrm{N}(H)/H$ if and only if s is an elliptic element of G. However, if H is not compact, then the condition that $s\,H$ is an elliptic element of $\mathrm{N}(H)/H$ is weaker than the condition that s is an elliptic element of G, which makes Proposition 8.13 more general than Proposition 8.5.

In the case of non-free actions, Proposition 8.13 is somewhat simpler than Proposition 8.5, because the torus T is a subgroup of $\mathrm{N}(H)/H$, which group acts freely on m. $\oslash$

Remark 8.15 If $\mathrm{N}(H)/H$ is compact, then every relative periodic solution in the orbit type $M_{[H]}$ is quasi-periodic. $\oslash$

Remark 8.16 For actions of compact Lie groups, Proposition 8.13 follows from applying Proposition B2 of Field [11] to the pre-image in M of the periodic orbit in $G\backslash M$ under the canonical projection $\pi : M \to G\backslash M$. $\oslash$

8.4 Other Relative Periodic Solutions in the $(G \times \mathbf{R})$-orbit

If m is a relative periodic point, s is as in (8.1), (8.2), and $g \in G$, then it follows from (5.2) that

$$\mathrm{e}^{\tau\,v}(g \cdot m) = g \cdot \mathrm{e}^{\tau\,v}(m) = g \cdot s \cdot m = \left(g\,s\,g^{-1}\right) \cdot g \cdot m. \tag{8.7}$$

This shows that $g \cdot m$ is also a relative periodic point, with the same relative period, and with the conjugate $g\, s\, g^{-1}$ of s by means of g as the shift element.

The solution of (1.1) starting at $g \cdot m$ is periodic, quasi-periodic and runaway if and only if the solution of (1.1) starting at m has these properties. Moreover, the closures of the v-orbits through the points of $G \cdot m$ define a smooth fibration of the v-flowout of $G \cdot m$.

Let us explain this in more detail in the case that we are in the situation of Proposition 8.5, that τ is the smallest positive real number t such that $\mathrm{e}^{t\, v}(m) \in G \cdot m$, and that the G-action is free at m. The situation is a bit more complicated than in Subsection 7.4, because the closures of the v-orbits are not generated by a torus subgroup of G, because the v-orbits are not contained in the G-orbits.

For $g,\, g' \in G$ and $t,\, t' \in \mathbf{R}$ we have $g \cdot \mathrm{e}^{t\, v}(m) = g' \cdot \mathrm{e}^{t'\, v}(m)$ if and only if $\mathrm{e}^{(t'-t)\, v}(m) = (g')^{-1} \cdot g \cdot m \in G \cdot m$ if and only if there exists a $p \in \mathbf{Z}$ such that $t' - t = p\, \tau$ and $(g')^{-1}\, g = s^p$. If we let $p \in \mathbf{Z}$ act on $G \times \mathbf{R}$ by sending $(g,\, t)$ to $(g\, s^{-p},\, t + p\, \tau)$, then we obtain that the mapping

$$(g,\, t) \mapsto g \cdot \mathrm{e}^{t\, v}(m) \tag{8.8}$$

from $G \times \mathbf{R}$ to M induces a smooth embedding Φ from $(G \times \mathbf{R})/\mathbf{Z}$ onto the v-flowout $(G \times \mathbf{R}) \cdot m$ of the G-orbit $G \cdot m$ through the point m. The canonical projection π from $G \times \mathbf{R}$ onto $(G \times \mathbf{R})/\mathbf{Z}$ intertwines the G-action on $G \times \mathbf{R}$, by multiplications from the left on the first factor, with a smooth G-action on $(G \times \mathbf{R})/\mathbf{Z}$. It also intertwines the $\mathbf{R}$-action on $G \times \mathbf{R}$, by translations on the second factor, with a smooth $\mathbf{R}$-action on $(G \times \mathbf{R})/\mathbf{Z}$. Φ intertwines the G-action and the $\mathbf{R}$-action on $(G \times \mathbf{R})/\mathbf{Z}$ with the G-action and the v-flow on $(G \times \mathbf{R}) \cdot m$, respectively.

Note that the projection from $G \times \mathbf{R}$ onto the second factor $\mathbf{R}$ exhibits $(G \times \mathbf{R})/\mathbf{Z}$ as a principal G-bundle over the circle $\mathbf{R}/\mathbf{Z}\, \tau$. This bundle is identified by means of Φ with the preimage under $\pi : M \to G\backslash M$ of the periodic orbit through $G \cdot m$ in the orbit space $G\backslash M$.

Let $\theta \in \mathbf{R}^{k+1}$ act on $G \times \mathbf{R}$ by sending $(g,\, t) \in G \times \mathbf{R}$ to

$$\left(g\, \exp\left(\sum_{j=1}^{k} \theta_j\, \lambda_j - \theta_{k+1}\, X \right),\, t + \theta_{k+1}\, p_0\, \tau \right). \tag{8.9}$$

If $\theta \in \mathbf{Z}^{k+1}$, then it follows from $\mathrm{e}^X = s^{p_0}$ that (8.9) is equal to $\left(g\, s^{-\theta_{k+1}\, p_0},\, t + \theta_{k+1}\, p_0\, \tau \right)$, which is equal to the action of the integer $p = \theta_{k+1}\, p_0$ on $(g,\, t)$. We therefore have an induced action of $(\mathbf{R}/\mathbf{Z})^{k+1} = \mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$ on $(G \times \mathbf{R})/\mathbf{Z}$. Moreover, the $(\mathbf{R}/\mathbf{Z})^{k+1}$-action on $(G \times \mathbf{R})/\mathbf{Z}$ commutes with the G-action on $(G \times \mathbf{R})/\mathbf{Z}$ defined by multiplications from the left on the first factor.

If (8.9) is equal to $(g\, s^{-p},\, t + p\, \tau)$ for some $p \in \mathbf{Z}$, then

$$s^p = \exp\left(-\sum_{j=1}^{k} \theta_j\, \lambda_j + \theta_{k+1}\, X \right) \in T, \tag{8.10}$$

which implies that $p = q\, p_0$ for some $q \in \mathbf{Z}$. But then $p\, \tau = \theta_{k+1}\, p_0\, \tau$ implies that $\theta_{k+1} = q$ and now (8.10) in combination with $\mathrm{e}^X = s^{p_0}$ implies that $\sum_{j=1}^{k} \theta_j\, \lambda_j$ belongs to the integral lattice in $\mathfrak{t}$, which in turn implies that $\theta_j \in \mathbf{Z}$ for every $1 \leq j \leq k$. This shows that the action of $(\mathbf{R}/\mathbf{Z})^{k+1}$ on $(G \times \mathbf{R})/\mathbf{Z}$ is free, and it is automatically proper because the group $(\mathbf{R}/\mathbf{Z})^{k+1}$ is compact.

The diffeomorphism Φ intertwines the action of $(\mathbf{R}/\mathbf{Z})^{k+1}$ on $(G \times \mathbf{R})/\mathbf{Z}$ with a uniquely defined proper and free smooth action of $(\mathbf{R}/\mathbf{Z})^{k+1}$ on the v-flowout $(G \times \mathbf{R}) \cdot m$ of the G-orbit $G \cdot m$ through m. This action commutes with the G-action on $(G \times \mathbf{R}) \cdot m$.

Let X_j, $1 \leq j \leq k$, denote the coordinates of $X \in \mathfrak{t}$ with respect to the $\mathbf{Z}$-basis λ_j, $1 \leq j \leq k$, of the integral lattice of $\mathfrak{t}$. Then

$$\sum_{j=1}^{k} \theta_j \, \lambda_j - \theta_{k+1} \, X = \sum_{j=1}^{k} \left(\theta_j - \theta_{k+1} \, X_j\right) \, \lambda_j,$$

and it follows that on $(G \times \mathbf{R}) \cdot m$ the vector field v is equal to the inifinitesimal action of the element $\dot{\theta}$ of the Lie algebra $\mathbf{R}^{k+1}$ of $\mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$, which is defined by

$$\dot{\theta}_j = \dot{\theta}_{k+1} \, X_j, \quad 1 \leq j \leq k, \quad \text{and} \quad \dot{\theta}_{k+1} = 1/p_0 \, \tau. \tag{8.11}$$

Because the $k+1$ real numbers $(X_1, \ldots, X_k, 1)$ are linearly independent over $\mathbf{Q}$, it follows from Kronecker's lemma 7.10, iii) that the (8.11)-orbits are dense in $(\mathbf{R}/\mathbf{Z})^{k+1}$, which implies that the fibers of the $(\mathbf{R}/\mathbf{Z})^{k+1}$-orbits in $(G \times \mathbf{R}) \cdot m$ are equal to the closures of the v-orbits in $(G \times \mathbf{R}) \cdot m$.

Now assume that τ is the minimal positive relative period, the G-action is not free at m, i.e. $H := G_m \neq \{1\}$, and $s\,H$ is an elliptic element of $\mathrm{N}(H)/H$. Then we can write the v-flowout of $G \cdot m$ as a fiber bundle over $G/\,\mathrm{N}(H)$ using Lemma 3.7, a) and Lemma 5.5, and typical fiber equal to the v-flowout of $\mathrm{N}(H) \cdot m$. Because the action of $\mathrm{N}(H)/H$ is free at m, we can apply the above construction in order to obtain that the closures of the v-orbits define a $\mathrm{N}(H)/H$-invariant smooth principal $(\mathbf{R}/\mathbf{Z})^{k+1}$-torus fibration of the v-flowout of $\mathrm{N}(H) \cdot m$. Using Lemma 7.23 with $H = \mathrm{N}(H)$, $K = (\mathbf{R}/\mathbf{Z})^{k+1}$ and $V = ((\mathrm{N}(H)/H) \times \mathbf{R})/\mathbf{Z}$, where V is diffeomorphic to the v-flowout of $\mathrm{N}(H) \cdot m$, we arrive at the following conclusion.

Proposition 8.17 *Under the assumptions of Proposition 8.13, the closures of the v-orbits in the v-flowout $(G \times \mathbf{R}) \cdot m$ of $G \cdot m$ define a smooth G-invariant principal $(\mathbf{R}/\mathbf{Z})^{k+1}$-fibration of $(G \times \mathbf{R}) \cdot m$. The vector field v on $(G \times \mathbf{R}) \cdot m$ is equal to the infinitesimal action on $(G \times \mathbf{R}) \cdot m$ of the element $\dot{\theta}$ in the Lie algebra $\mathbf{R}^{k+1}$ of $(\mathbf{R}/\mathbf{Z})^{k+1}$ which is defined by* (8.11).

9 Smooth Dependence on Parameters

In this section we consider the situation that we have a smooth family of equibrium points or periodic solutions in the orbit space $G \backslash M$, and we will investigate whether we have a corresponding smooth family of quasi-periodic solutions of (1.1) in M.

Let T be a torus with Lie algebra $\mathfrak{t}$. For every $X \in \mathfrak{t}$, let $T(X)$ be the closure in T of the set of all $\mathrm{e}^{t\,X}$, $t \in \mathbf{R}$. Note that $T(X)$ is the smallest closed Lie subgroup of T such that

X belongs to the Lie algebra of $T(X)$. Assume that $\dim T > 1$. For every $1 \leq k \leq \dim T$, the set of $X \in \mathfrak{t}$ such that $\dim T(X) = k$ is dense in $\mathfrak{t}$. This shows that the dependence of $T(X)$ on X is highly discontinuous, even its dimension is an everywhere discontinuous function on $\mathfrak{t}$. As a consequence, if the group G contains tori of dimension > 1, then one needs very detailed information on the dependence on the parameters of the element $X \in \mathfrak{t}$ in (7.1), (7.2), if one wants that the closures of the v-orbits in Proposition 7.11 depend smoothly on the parameters. However, when we drop the condition that we always have the minimal number of frequencies, then under reasonably general conditions we can obtain smooth families of tori on which the motion is quasi-periodic.

Similar observations can be made for relative periodic orbits. For any $h \in T$, let $T(h)$ denote the closure of the set of all h^p, $p \in \mathbf{Z}$. Note that $T(h)$ is the closed subgroup of T generated by h. For every $0 \leq k \leq \dim T$, the set of $h \in T$ such that $\dim T(h) = k$ is dense in T. Therefore, if the group G contains tori of positive dimension, then one needs very detailed information on the dependence on the parameters of the shift element s in (8.1), (8.2), if one wants that the closures of the v-orbits in Proposition 8.5 depend smoothly on the parameters. And again, if we drop the condition that we always have the minimal number of frequencies, then under reasonably general conditions we can obtain smooth families of tori on which the motion is quasi-periodic.

9.1 Families of Quasi-periodic Relative Equilibria

Definition 9.1 Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$, the *centralizer* $\mathfrak{g}_X$ of X in $\mathfrak{g}$ is defined as the set of all $Y \in \mathfrak{g}$ such that $[X, Y] = 0$. It is equal to the Lie algebra of the centralizer G_X of X in G, the set of all $g \in G$ such that $(\mathrm{Ad}\, g)(X) = X$.

X is called a *regular element* of $\mathfrak{g}$ if there is a neighborhood U of X in $\mathfrak{g}$ such that $\dim \mathfrak{g}_X \leq \dim \mathfrak{g}_Y$ for every $Y \in U$. Because always $\dim \mathfrak{g}_Y \leq \dim \mathfrak{g}_X$ for all Y near X, X is regular if and only if $\dim \mathfrak{g}_Y$ is constant for all Y near X.

X is called a *stably elliptic* element of $\mathfrak{g}$ if there is a neighborhood U of X in $\mathfrak{g}$ such that every $Y \in U$ is an elliptic element of $\mathfrak{g}$. $\oslash$

Lemma 9.2 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$. Then the following conditions are equivalent.*

i) *X is a regular and stably elliptic element of $\mathfrak{g}$.*

ii) *The centralizer $\mathfrak{g}_X$ of X in $\mathfrak{g}$ is equal to the Lie algebra of a torus subgroup T of G.*

The set $\mathfrak{g}^{\mathrm{se}}$ of stably elliptic elements of $\mathfrak{g}$ is an open subset of $\mathfrak{g}$ and the set $\mathfrak{g}^{\mathrm{rse}}$ of regular and stably elliptic elements of $\mathfrak{g}$ is a dense open subset of $\mathfrak{g}^{\mathrm{se}}$. For every $X \in \mathfrak{g}^{\mathrm{rse}}$ there exists an open neighborhood U of X in $\mathfrak{g}$ and an analytic mapping $\theta : U \to G$ with $\theta(X) = 1$, such that if $X' \in U$, $X'' = (\mathrm{Ad}\, \theta(X'))(X')$, then $\mathfrak{g}_{X''} = \mathfrak{g}_X$.

It follows that if C is a connected component of $\mathfrak{g}^{\mathrm{rse}}$, then for any X, $X' \in C$ there exists $g \in G^\circ$ such that $\mathfrak{g}_{X'} = (\mathrm{Ad}\, g)(\mathfrak{g}_X)$. In particular $\dim \mathfrak{g}_X$ is constant for $X \in C$, say equal to

k. The mapping $X \mapsto \mathfrak{g}_X$ is analytic from C to the Grassmann manifold of all k-dimensional linear subspaces of $\mathfrak{g}$.

Proof If $\operatorname{ad} X$ denote the linear mapping from $\mathfrak{g}$ to $\mathfrak{g}$ defined by $(\operatorname{ad} X)(Y) = [X,\, Y]$, $Y \in \mathfrak{g}$, then

$$\mathrm{e}^{t\, \operatorname{ad} X} = \operatorname{Ad}\left(\mathrm{e}^{t\, X}\right), \quad t \in \mathbf{R},$$

cf. [10, Th. 1.5.2,a)]. If X is an elliptic element of the Lie algebra of G, then the closure of the $\mathrm{e}^{t\, X}$, $t \in \mathbf{R}$ is a torus subgroup of G, and because the adjoint representation $\operatorname{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a continuous homomorphism, it follows that the $\mathrm{e}^{t\, \operatorname{ad} X}$ belong to a compact subgroup K of $\mathrm{GL}(\mathfrak{g})$. Averaging an arbitrary inner product on $\mathfrak{g}$ over K, we obtain a K-invariant inner product β on $\mathfrak{g}$, cf. [10, Cor. 4.2.2]. Then $t \mapsto \mathrm{e}^{t\, \operatorname{ad} X}$ is a one-parameter group of β-orthogonal transformations, and therefore $\operatorname{ad} X$ is β-anti-symmetric. It follows that that the range $(\operatorname{ad} X)(\mathfrak{g})$ of $\operatorname{ad} X$ is equal to the β-orthogonal complement of the kernel $\ker(\operatorname{ad} X) = \mathfrak{g}_X$ of $\operatorname{ad} X$. This implies that

$$\mathfrak{g} = (\operatorname{ad} X)(\mathfrak{g}) \oplus \mathfrak{g}_X, \tag{9.1}$$

and $\operatorname{ad} X$ is a bijective linear mapping from $(\operatorname{ad} X)(\mathfrak{g})$ onto itself.

The tangent mapping at $(1,\, X)$ of the mapping

$$G \times \mathfrak{g}_X \ni (g,\, Z) \mapsto (\operatorname{Ad} g)(Z) \in \mathfrak{g} \tag{9.2}$$

is equal to $(A,\, B) \mapsto [A,\, X] + B$, which is surjective in view of (9.1). It follows that there is an open neighborhood U of X in $\mathfrak{g}$ and there are analytic mappings $\varphi : U \to G$, $\zeta : U \to \mathfrak{g}_X$, such that $\varphi(X) = 1$, $\zeta(X) = X$ and $X' = (\operatorname{Ad} \varphi(X'))(\zeta(X'))$ for every $X' \in U$. This implies that all elements near X are conjugate to elements of $\mathfrak{g}_X$.

Now let $X'' \in \mathfrak{g}_X$. Then the linear mappings $\operatorname{ad} X$ and $\operatorname{ad} X''$ commute, cf. [10, (1.1.22)]. It follows that $\operatorname{ad} X''$ leaves both $\mathfrak{g}_X = \ker(\operatorname{ad} X)$ and $(\operatorname{ad} X)(\mathfrak{g})$ invariant. If X'' is sufficiently close to X in $\mathfrak{g}_X$, then the restriction of $\operatorname{ad} X''$ to $(\operatorname{ad} X)(\mathfrak{g})$ will still be bijective, and we have that $\mathfrak{g}_{X''} = \ker(\operatorname{ad} X'') \subset \mathfrak{g}_X$.

If X is regular, then we conclude that $\mathfrak{g}_{X''} = \mathfrak{g}_X$ for all $X'' \in \mathfrak{g}_X$ near X. This implies that if $A \in \mathfrak{g}_X$ is sufficiently close to zero, then $[A,\, B] = [X + A,\, B] = 0$ for every $B \in \mathfrak{g}_X$, and because $A \mapsto [A,\, B]$ is linear, it follows that $\mathfrak{g}_X$ is commutative.

Because $\exp(\mathfrak{g}_X)$ is equal to the identity component of the closed subgroup G_X of G, $T := \exp(\mathfrak{g}_X)$ is a closed subgroup of G. Therefore, if for a given $X'' \in \mathfrak{g}_X$ the closure of the $\mathrm{e}^{t\, X''}$, $t \in \mathbf{R}$ is a torus subgroup of G, then it is a torus subgroup of T. It follows that if X is a regular and stably elliptic element in the Lie algebra of G then it is stably elliptic as an element of the Lie algebra of T.

Because T is a connected and commutative Lie group, it is isomorphic to $(\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$, cf. [10, Cor. 1.12.4], of which the Lie algebra contains stably elliptic elements if and only if $l = 0$. This concludes the proof of i) $\Longrightarrow$ ii).

By definition $\mathfrak{g}^{\mathrm{se}}$ and the set of regular elements are open subsets of $\mathfrak{g}$. If $X \in \mathfrak{g}^{\mathrm{se}}$ is not regular, then we can find X' arbitrarily close to X with $\dim \mathfrak{g}_{X'} < \mathfrak{g}_X$. Because of the finiteness of the dimensions we can repeat this only finitely times with X replaced by X'. It follows that arbitrarily close to X we can find X' for which $\mathfrak{g}_{X'}$ has locally minimal dimension, i.e. X' is regular. This shows that $\mathfrak{g}^{\mathrm{rse}}$ is a dense open subset of $\mathfrak{g}^{\mathrm{se}}$.

Now assume that ii) holds. For every $X'' \in \mathfrak{g}_X$, we have $\mathrm{e}^{t\,X''} \in T$, $t \in \mathbf{R}$, hence the closure of these elements is a connected commutative closed subgroup of the torus group T, hence a torus subgroup of T, hence of G, which means that X'' is an elliptic element of $\mathfrak{g}$.

Because $X \in \mathfrak{g}_X$, we have in particular that X is an elliptic element of $\mathfrak{g}$. As we have seen after (9.2), this implies that all elements in $\mathfrak{g}$ near X are conjugate to elements in $\mathfrak{g}_X$, and therefore are elliptic, and we conclude that X is stably elliptic.

Furthermore, because T is commutative, $\mathfrak{g}_X$ is commutative, which implies that $\mathfrak{g}_X \subset \mathfrak{g}_{X''}$ for every $X'' \in \mathfrak{g}_X$, and we conclude that X is regular. This concludes the proof of ii) $\Longrightarrow$ i).

If we write $\theta(X') = \varphi(X')^{-1}$, then it follows from $X' = (\mathrm{Ad}\,\varphi(X'))(\zeta(X'))$ that $X'' := (\mathrm{Ad}\,\theta(X'))(X') = \zeta(X') \in \mathfrak{g}_X$. Furthermore, if X'' is sufficently close to X in $\mathfrak{g}_X$, then $\mathfrak{g}_X = \mathfrak{g}_{X''} = (\mathrm{Ad}\,\theta(X'))(\mathfrak{g}_{X'})$.

The condition for X, $X' \in \mathfrak{g}$ that $\mathfrak{g}_{X'}$ is conjugate to $\mathfrak{g}_X$ by elements of G° is an equivalence relation. Because nearby elements of C are equivalent and C is connected, it follows that $\mathfrak{g}_{X'}$ is conjugate to $\mathfrak{g}_X$ for all X, $X' \in C$, which implies that the dimension of $\mathfrak{g}_X$ is constant for all $X \in C$. Because for X' near X we have that $\mathfrak{g}_{X'}$ is conjugate to $\mathfrak{g}_X$ by means of an element of G° which depends analytically on X', the mapping $X' \mapsto \mathfrak{g}_{X'}$ is analytic on a neighborhood of X in C, and because this holds for any $X \in C$, the mapping $X \mapsto \mathfrak{g}_X$ is analytic on C. □

Remark 9.3 In general, the torus T in Lemma 9.2 is different from the torus T in Lemma 7.10. Because $X \in \mathfrak{g}_X$, hence $t\,X \in \mathfrak{g}_X$, the one-parameter subgroup $\mathrm{e}^{t\,X}$, $t \in \mathbf{R}$, is contained in the torus $\exp(\mathfrak{g}_X)$. Therefore the closure of the $\mathrm{e}^{t\,X}$, $t \in \mathbf{R}$, is also contained in $\mathrm{e}^{t\,X}$, $t \in \mathbf{R}$.

Note that any torus has a dense one-parameter subgroup, which implies that if T is a torus as in Lemma 9.2 with Lie algebra $\mathfrak{t}$, then for any subtorus T_0 of T there exists a $Y \in \mathfrak{t}$ such that the closure of the $\mathrm{e}^{t\,Y}$, $t \in \mathbf{R}$, is equal to T_0. In this sense the tori in Lemma 7.10 can be arbitrary subtori of the tori in Lemma 9.2. ⊘

Example 9.4 If the identity component G° of G is compact, then every $X \in \mathfrak{g}$ is stably elliptic, and the regular and stably elliptic elements of $\mathfrak{g}$ are just the regular elements of $\mathfrak{g}$ as in [10, (g) on p. 141]. Furthermore, for regular $X \in \mathfrak{g}$, $\mathfrak{g}_X$ is a maximal Abelian subspace of $\mathfrak{g}$, and $\exp(\mathfrak{g}_X)$ is a maximal torus in G°. All maximal tori in G° are conjugate to each other by means of elements of G°, and in particular have the same dimension, which is called the *rank* of G°. See [10, Th. 3.7.1 and p. 153].

If conversely every element of $\mathfrak{g}$ is elliptic, then the identity component G° of G is compact. This follows from Hochschild [14, Ch. XV, Th. 3.1], which result in this generality has been obtained before by Malcev [21]. ⊘

Example 9.5 Let G be equal to the group $\mathrm{E}(2)$ of the *Euclidean motions in the plane* $\mathbf{R}^2$, the group of transfomations $(A,\, a) : x \mapsto A\,x + a$, in which $A \in \mathrm{SO}(2)$ and $a \in \mathbf{R}^2$. Its Lie algebra $\mathfrak{e}(2)$ consists of the infinitesimal transformations $X = (B,\, b)$ such that $B \in \mathfrak{so}(2)$ is an antisymmetric 2×2 -matrix and $b \in \mathbf{R}^2$.

If $B \in \mathfrak{so}(2)$ is nonzero, then B is bijective, and $X = (B,\, b)$ is equal to an infinitesimal rotation about the unique fixed point $p := -B^{-1}\, b$. Therefore any element of the isotropy subgroup $\mathrm{E}(2)_X$ of X in $\mathrm{E}(2)$ leaves p fixed, which shows that $\mathrm{E}(2)_X$ is equal to group of rotations about the point p, which is a circle subgroup of $\mathrm{E}(2)$. In view of Lemma 9.2 we conclude that $(B,\, b)$ is a regular and stably elliptic element of $\mathfrak{e}(2)$ if $B \neq 0$.

On the other hand, $(0,\, b)$ is an elliptic element of $\mathfrak{e}(2)$ if and only if $b = 0$. The conclusion is that $(B,\, b)$ is a regular and stably elliptic element of $\mathfrak{e}(2)$ if and only if $B \neq 0$, which shows that $\mathfrak{e}(2)^{\mathrm{rse}}$ is a dense open subset of $\mathfrak{e}(2)$.

Note that $\mathrm{E}(2)$ is a noncompact Lie group which is (two step) nilpotent. $\oslash$

Example 9.6 The Lie algebra of $G = \mathrm{SL}(2,\, \mathbf{R})$ is equal to the set of all 2×2-matrices

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

such that $\alpha,\, \beta,\, \gamma \in \mathbf{R}$. The regular and stably elliptic elements of the Lie algebra are the matrices X such that $\alpha^2 + \beta\,\gamma < 0$. These form two disjoint open convex cones in the Lie algebra.

The matrices X such that $\alpha^2 + \beta\,\gamma > 0$ are non-elliptic, these also form a nonvoid open subset of the Lie algebra of $\mathrm{SL}(2,\, \mathbf{R})$.

If $\alpha^2 + \beta\,\gamma = 0$, then X is elliptic if and only if $X = 0$. $\oslash$

Suppose that the G-action on M is free and proper, and let F be a smooth submanifold of $G\backslash M$ consisting of fixed points of the flow Φ^t in $G\backslash M$ defined by (5.3). It follows that $\pi^{-1}(F)$ is a locally closed smooth submanifold of M which is G-invariant, hence $\pi : \pi^{-1}(F) \to F$ is a principal G-bundle. Because F consists of relative equilibria, it follows from Lemma 7.1 that $\pi^{-1}(F)$ is invariant under the v-flow and that the restriction of v to $\pi^{-1}(F)$ is a complete vector field on $\pi^{-1}(F)$.

Lemma 9.7 *For $m \in \pi^{-1}(F)$, the unique element $X = X(m) \in \mathfrak{g}$ in* (7.1), (7.2) *depends smoothly on m.*

Proof Write $N = \pi^{-1}(F)$. The $\mathfrak{g}_m = \alpha_m(\mathfrak{g})$, $m \in N$, form a smooth vector subbundle $\mathfrak{g}_N$ of the tangent bundle $\mathrm{T}\,N$ of N. The restriction to N of the vector field v is a smooth mapping from N to $\mathrm{T}\,N$ which takes its values in $\mathfrak{g}_N$, and therefore defines a smooth mapping $v : N \to \mathfrak{g}_N$ On the other hand, the mapping

$$\alpha : (m,\, X) \mapsto X_M(m) : N \times \mathfrak{g} \to \mathfrak{g}_N$$

is smooth, bijective and has bijective tangent mapping at each point, and therefore is a diffeomorphism from $N \times \mathfrak{g}$ onto $\mathfrak{g}_N$. The mapping $m \mapsto X(m)$ is equal to $v : N \to \mathfrak{g}_N$, followed by $\alpha^{-1} : \mathfrak{g}_N \to N \times \mathfrak{g}$, and concluded by the projection $N \times \mathfrak{g} \to \mathfrak{g}$ onto the second factor. Therefore $m \mapsto X(m)$ is smooth as the composition of three smooth mappings. $\square$

Let F^{rse} denote the set of all $f \in F$, such that for some (every) $m \in \pi^{-1}(\{f\})$ the element $X(m)$ is a regular and stably elliptic element of $\mathfrak{g}$. F^{rse} is an open subset of F.

Lemma 9.8 *Let $m_0 \in M$ and $f_0 := \pi(m_0) \in F^{\mathrm{rse}}$. Write $\mathfrak{t} = \mathfrak{g}_{X(m_0)}$ and $T = \exp \mathfrak{t}$ for the torus subgroup of G of which $\mathfrak{t}$ is the Lie algebra. Then there exists an open neighborhood F_0 of f_0 in F^{rse} and a smooth section $\mu : F_0 \to \pi^{-1}(F_0)$ of $\pi : \pi^{-1}(F_0) \to F_0$, such that $X(\mu(f)) \in \mathfrak{t}$ for every $f \in F_0$.*

Proof Because $\pi : \pi^{-1}(F) \to F$ is a smooth fibration, there exists an open neighborhood F_1 of f_0 in in F^{rse} and a smooth section $\nu : F_1 \to \pi^{-1}(F_1)$ of $\pi : \pi^{-1}(F_1) \to F_1$, i.e. ν is smooth and $\pi \circ \nu$ is equal to the identity in F_1. It follows from Lemma 9.2 that there exists an open neighborhood U of $X(m_0)$ in $\mathfrak{g}$ and an analytic mapping $\theta : U \to G$, such that $(\mathrm{Ad}\, \theta(X'))(X') \in \mathfrak{t}$ for every $X' \in U$. It follows from (7.6) with $X = X(m)$ that $X(g \cdot m) = (\mathrm{Ad}\, g)(X(m))$. If we write $m = \nu(f)$, $X' = X(\nu(f))$, $g = \theta(X')$, and $\mu(f) := \theta(X(\nu(f))) \cdot \nu(f)$, then we obtain that $X(\mu(f)) \in \mathfrak{t}$ for every $f \in F_1$ such that $X(\nu(f)) \in U$. These f form an open neighborhood F_0 of f_0 in F^{rse}, and because $\pi(g \cdot m) = \pi(m) = f$, $\mu : F_0 \to \pi^{-1}(F_0)$ is a smooth section of $\pi : \pi^{-1}(F_0) \to F_0$. $\square$

The mapping $\Phi : (f,\, g) \mapsto g \cdot \mu(f)$ is a diffeomorphism from $F_0 \times G$ onto $\pi^{-1}(F_0)$. On $F_0 \times G$ we have the free and proper action of T by multiplication on the second factor *from the right*. Φ intertwines this action with a free and proper T-action on $\pi^{-1}(F_0)$, which defines a principal T-fibration of $\pi^{-1}(F_0)$. The right T-action on $F_0 \times G$ commutes with the action of G on $F_0 \times G$ by multiplication on the second factor from the left. Because Φ intertwines this left G-action on $F_0 \times G$ with the G-action on $\pi^{-1}(F_0)$, it follows that the T-action on $\pi^{-1}(F_0)$ commutes with the G-action on $\pi^{-1}(F_0)$. In other words, the principal T-fibration on $\pi^{-1}(F_0)$ is invariant under the G-action.

On $F_0 \times G$ we finally have the action of $\mathbf{R}$ defined by $(t,\, (f,\, g)) \mapsto (f,\, g\, \mathrm{e}^{t\, X(\mu(f))})$, which is a quasi-periodic motion on the T-orbits, "depending smoothly on the parameter f". Φ intertwines the $\mathbf{R}$-action on $F_0 \times G$ with the v-flow in $\pi^{-1}(F_0)$. In other words, the formula $\xi(f) := X(\mu(f))$ defines a smooth mapping $\xi : F_0 \to \mathfrak{t}$ with the property that, for each $m \in \pi^{-1}(F_0)$, $v(m)$ is equal to the infinitesimal action of $\xi(\pi(m)) \in \mathfrak{t}$ for the proper and free action of T on $\pi^{-1}(F_0)$.

If the G-action on M is proper but not free, then the above constructions can be applied with the manifold M and the Lie group G replaced by an isotropy type M_H and the Lie group $\mathrm{N}(H)/H$, respectively, where $\mathrm{N}(H)/H$ acts freely on M_H, cf. Lemma 3.3, and M_H is invariant under the v-flow, cf. Lemma 5.5. The $\mathrm{N}(H)/H$-invariant principal T-fibration in $\pi^{-1}(F_0) \cap M_H$ has a unique extension to a G-invariant principal T-fibration in $\pi^{-1}(F_0)$. For the proof one can use a version with parameters of the constructions at the end of Subsection 7.4. This leads to the following conclusions.

Proposition 9.9 *Suppose that the G-action on M is proper, and let F be a smooth submanifold of the orbit type $G\backslash M_{[H]}$ in the orbit space $G\backslash M$, consisting of fixed points of the flow Φ^t in $G \setminus M$ defined by* (5.3)*.*

Write, for each $m \in \pi^{-1}(F)$, $X(m) = X + \mathfrak{g}_m$, with $X \in \mathfrak{g}$ as in (7.1), (7.2)*. Let F^{rse} denote the set of all $f \in F$ such that for some (every) $m \in \pi^{-1}(\{f\})$ we have that $X(m) + \mathfrak{g}_m$ is a regular and stably elliptic element of the Lie algebra of $\mathrm{N}(G_m)/G_m$. Write $\mathfrak{t}(m)$ for the centralizer of $X(m) + \mathfrak{g}_m$ in the Lie algebra of $\mathrm{N}(G_m)/G_m$, and $T(m)$ for the torus subgroup of $\mathrm{N}(G_m)/G_m$ with Lie algebra equal to $\mathfrak{t}(m)$.*

Then F^{rse} is an open subset of F; let C be a connected component of F^{rse}. If through each $m \in \pi^{-1}(C)$ we draw the $T(m)$-orbit $T(m) \cdot m$, then these subsets of $\pi^{-1}(C)$ define a G-invariant smooth fibration of $\pi^{-1}(C)$ with fibers diffeomorphic to the tori $T(m)$.

Let $m_0 \in M$ be such that $f_0 := \pi(m_0) \in C$. Write $\mathfrak{t} = \mathfrak{t}(m_0)$ and $T = T(m_0)$. Then there is an open neighborhood F_0 of f_0 in C, a smooth mapping $\xi : F_0 \to \mathfrak{t}$, and a proper and free action of T on $\pi^{-1}(F_0)$, called the right T-action, with the following properties.

i) *The right T-action commutes with the G-action and, for each $m \in \pi^{-1}(F_0)$, the right T-orbit through m is equal to the fiber $T(m) \cdot m$ of the toral fibration of $\pi^{-1}(C)$.*

ii) *For each $m \in \pi^{-1}(F_0)$, $v(m)$ is equal to the infinitesimal right action at m of $\xi(\pi(m)) \in \mathfrak{t}$, and the solution of* (1.1) *starting at m is equal to the right action of the one-parameter subgroup of T generated by $\xi(\pi(m))$.*

Because one-parameter subgroups of tori are quasi-periodic, this expresses in a quite strong sense that the v-flow in $\pi^{-1}(C)$ is quasi-periodic on tori, where the tori form a smooth fibration and the velocity vector at m is an element of the Lie algebra of the torus which depends smoothly on $\pi(m)$. Note that the fiber through m of the toral fibration is contained in the G-orbit $G \cdot m = \pi(m)$ through m, which implies that the element in the Lie algebra of the torus does not depend on choice of the point on the fiber of the torus fibration.

Remark 9.10 In general, the torus T in Proposition 9.9 is different from the tori T in Proposition 7.19 and Proposition 7.24. The tori in Proposition 7.19 and Proposition 7.24 can be arbitrary subtori of the torus T in Proposition 9.9, cf. Remark 9.3. $\oslash$

A smooth torus bundle need not be a principal bundle, in the sense that the fibers of the torus bundle need not be equal to the orbits of a proper and free action of a fixed torus. For instance, Klein's bottle is a circle bundle over a circle, but is not a principal bundle.

In order to investigate whether the whole toral fibration of $\pi^{-1}(C)$, introduced in Proposition 9.9, is a principal fibration, we resume the discussion after Example 9.6, of the case of a proper and free action of G on M.

As observed there, we have a covering of C with open subsets C_μ, which are the domains of definitions of smooth sections $\mu : C_\mu \to \pi^{-1}(C_\mu)$ of $\pi : \pi^{-1}(C_\mu) \to C_\mu$, such that $X(\mu(f)) \in \mathfrak{t}$ for every $f \in F_0$. If $f \in C_\mu \cap C_\nu$, we have a unique $g = g_{\nu\,\mu}(f) \in G$, depending smoothly on f, such that $\nu(f) = g \cdot \mu(f)$. It follows from (7.6) that $X(\nu(f)) = (\mathrm{Ad}\, g)(X(\mu(f)))$, which implies that

$$\mathfrak{t} = \mathfrak{g}_{X(\nu(f))} = (\mathrm{Ad}\, g)\left(\mathfrak{g}_{X(\mu(f))}\right) = (\mathrm{Ad}\, g)(\mathfrak{t}),$$

where the first and the last identity follow from the fact that $X(\nu(f))$ and $X(\mu(f))$ are regular elements of $\mathfrak{t}$. Because $\mathfrak{t}$ is equal to the Lie algebra of T, it follows from $g\, \mathrm{e}^X\, g^{-1} = \mathrm{e}^{(\mathrm{Ad}\, g)(X)}$ that $(\mathrm{Ad}\, g)(\mathfrak{t}) = \mathfrak{t}$ if and only if $g\, T\, g^{-1} = T$, i.e. if and only if g belongs to the normalizer $\mathrm{N}(T)$ of T in G.

If we write $\Phi_\mu : (f,\, g) \mapsto g \cdot \mu(f)$, then

$$\Phi_\nu(f,\, g) = g \cdot \nu(f) = g \cdot g_{\nu\,\mu}(f) \cdot \mu(f) = \Phi_\mu(f,\, g \cdot g_{\nu\,\mu}(f)), \quad f \in C_\mu \cap C_\nu$$

shows that the principal G-bundle $\pi^{-1}(C)$ can be obtained by glueing $C_\mu \times G$ to $C_\nu \times G$ along $(C_\mu \cap C_\nu) \times G$ by means of the glueing map

$$\Phi_{\nu\,\mu} : (f,\, g) \mapsto (f,\, g \cdot g_{\nu\,\mu}(f)), \tag{9.3}$$

in which $g_{\nu\,\mu}$ is a smooth mapping from $C_\mu \cap C_\nu$ to $\mathrm{N}(T)$.

The sets $(f,\, g\,T) = \{(f,\, g\,h) \mid h \in T\}$ form a fibration of $C_\mu \times G$ into tori, which fibration is invariant under the glueing map (9.3), because $g_{\mu\,\nu}(f) \in \mathrm{N}(T)$ implies that $T\, g_{\nu\,\mu}(f) = g_{\nu\,\mu}(f)\, T$. It follows that the fibrations glue together to a torus fibration of the bundle $\pi^{-1}(C)$, which is equal to the one in Proposition 9.9.

However, the glueing map (9.3) only commutes with the T-action $(h,\, (f,\, g)) \mapsto (f,\, g\,h)$ on $(C_\mu \cap C_\nu) \times G$ if $g = g_{\nu\,\mu}(f)$ satisfies $h\,g = g\,h$ for every $h \in T$. That is, if g belongs to the *centralizer* $\mathrm{Z}(T)$ of T in G, the set of all $g \in G$ such that $g\,h\,g^{-1} = h$ for every $h \in T$.

The Lie algebra of $\mathrm{N}(T)$ consists of the $Y \in \mathfrak{g}$ such that $[Y,\, X] \in \mathfrak{t}$ for every $X \in \mathfrak{t}$. If X is a regular element of $\mathfrak{t}$, then it follows from (9.1) that $\mathfrak{g} = (\mathrm{ad}\, X)(\mathfrak{g}) \oplus \mathfrak{t}$ and $\mathrm{ad}\, X : (\mathrm{ad}\, X)(\mathfrak{g}) \to (\mathrm{ad}\, X)(\mathfrak{g})$ is bijective, hence $[Y,\, X] = -(\mathrm{ad}\, X)(Y) \in \mathfrak{t}$ if and only if $Y \in \mathfrak{t}$. Therefore the identity component of $\mathrm{N}(T)$ is equal to T, and the group $\mathrm{N}(T)/T$ is discrete. The centralizer $\mathrm{Z}(T)$ of T in G is a closed, hence Lie subgroup of G, and it is a normal subgroup of $\mathrm{N}(T)$ such that $T \subset \mathrm{Z}(T) \subset \mathrm{N}(T)$. The discrete group $\mathrm{W}(T) := \mathrm{N}(T)/\,\mathrm{Z}(T)$ is called the *Weyl group* of T. $\mathrm{N}(T)$ acts by conjugation on T and on $\mathfrak{t}$, this action factors to an effective action of $\mathrm{W}(T)$ on T and on $\mathfrak{t}$. In this way $\mathrm{W}(T)$ is viewed as a group of automorphisms of the torus Lie group T.

Let N be the set of $m \in \pi^{-1}C)$ such that $X(m) \in \mathfrak{t}$. According to the above, $\pi : N \to C$ exhibits N as a principal $\mathrm{N}(T)$-bundle over C, and $\widetilde{C} := \mathrm{Z}(T)\backslash N$ is a principal $\mathrm{W}(T)$-bundle over C. (A principal Γ-bundle with a discrete group Γ is also called a *Galois covering* with group Γ.) Because $\mathrm{W}(T)$ is discrete, we have unique liftings of curves in C, in the sense that for every continuous curve $\gamma : [a,\, b] \to C$ and every $s \in \pi^{-1}(\{\gamma(a)\})$, there is a unique continuous curve $\widetilde{\gamma}_s : [a,\, b] \to \widetilde{C}$ such that $\gamma = \pi \circ \widetilde{\gamma}_s$ and $\widetilde{\gamma}_s(a) = s$. Clearly $w \cdot \widetilde{\gamma}_s(t) = \widetilde{\gamma}_{w \cdot s}(t)$ for every $t \in [a,\, b]$ and $w \in \mathrm{W}(T)$.

If γ is a closed curve in C starting and ending at f, then $\widetilde{\gamma}(b) = w \cdot s$ for a unique $w = w_{\gamma,\, s} \in \mathrm{W}(T)$. If $\delta : [b,\, c] \to C$ is another closed curve in C starting and ending at f, then $w_{\gamma,\, s} \cdot \widetilde{\delta}_s(b) = w_{\gamma,\, s} \cdot s = \widetilde{\gamma}_s(b)$, hence $\widetilde{\gamma}_s$ followed by $w_{\gamma,\, s} \cdot \widetilde{\delta}_s$ is a lift of γ followed by δ, and this lift starts at s and ends at $w_{\gamma,\, s} \cdot w_{\delta, s}$.

Let $\pi_1(C,\, f)$ denote the fundamental group of the manifold C, based at the point f. Then the previous paragraph showed that the mapping $\gamma \mapsto w_{\gamma,\, s}$ induces a homomorphism from $\pi_1(C,\, f)$ to $\mathrm{W}(T)$, which is called the *monodromy* of the Galois covering. In general this homomorphism depends on the choice of the base points $f \in C$ and $s \in \pi^{-1}(\{f\}$.

The point is now that the torus fibration in $\pi^{-1}(C)$ is a G-invariant principal T-bundle $\Longleftrightarrow \widetilde{C}$ admits a global section $\Longleftrightarrow$ the bundle $\widetilde{C}$ over C is trivial $\Longleftrightarrow$ the monodromy is trivial. That is, the monodromy homomorphism : $\pi_1(C,\, f) \to \mathrm{W}(T)$ is the obstruction to the torus fibration being a G-invariant principal T-bundle.

Example 9.11 If G is a compact, connected Lie group, then the tori T with Lie algebra $\mathfrak{g}_X$ are the maximal tori, which all are conjugate to each other, cf. [10, Th.. 3.7.1]. The Weyl group $\mathrm{W}(T)$ is a finite group, generated by the orthogonal reflections in the root hyperplanes, cf. [10, Cor. 3.10.3]. Therefore the Weyl group is nontrivial if and only if G is noncommutative. $\oslash$

Example 9.12 If $G = \mathrm{E}(2)$ is equal to the motion group of the plane, cf. Example 9.5, then the tori T are circle subgroups, all conjugate to each other and have a trivial Weyl group. Therefore in this case the torus fibration in $\pi^{-1}(C)$ is a principal fibration. $\oslash$

9.2 Families of Quasi-periodic Relative Periodic Solutions

Definition 9.13 Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $s \in G$, the *centralizer* $\mathfrak{g}_s$ of s in $\mathfrak{g}$ is defined as the set of all $Y \in \mathfrak{g}$ such that $(\mathrm{Ad}\, s)(Y) = Y$. The centralizer G_s of s in G is defined as the set of all $g \in G$ such that $g\, s = s\, g$, or equivalently $g = s\, g\, s^{-1}$. G_s is a closed, hence Lie subgroup of G, with Lie algebra equal to $\mathfrak{g}_s$, cf. [10, (3.1.3)].

s is called a *regular element* of G if there is a neighborhood U of s in G such that $\dim \mathfrak{g}_s \leq \dim \mathfrak{g}_{s'}$ for every $s' \in U$. Because always $\dim \mathfrak{g}_{s'} \leq \dim \mathfrak{g}_s$ for all s' near s, s is regular if and only if $\dim \mathfrak{g}_{s'}$ is constant for all s' near s.

s is called a *stably elliptic* element of G if there is a neighborhood U of s in G such that every $s' \in U$ is an elliptic element of G. $\oslash$

Lemma 9.14 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $s \in G$. Then the following conditions are equivalent.*

i) *s is a regular and stably elliptic element of $\mathfrak{g}$.*

ii) *The centralizer $\mathfrak{g}_s$ of s in $\mathfrak{g}$ is equal to the Lie algebra of a torus subgroup T of G, and $s^p \in T$ for some nonzero integer p.*

The set G^{se} of stably elliptic elements of G is an open subset of G and the set G^{rse} of regular and stably elliptic elements of G is a dense open subset of G^{se}. For every $s \in G^{\mathrm{rse}}$ there exists an open neighborhood V of s in G and an analytic mapping $\theta : V \to G$, such that if $s' \in V$, $s'' = \theta(s')\, s'\, \theta(s')^{-1}$, then $G^\circ_{s''} = G^\circ_s$ and $s''\, G^\circ_{s''} = s\, G^\circ_s$.

It follows that if C is a connected component of G^{rse} then for any $s,\, s' \in C$ there exists $g \in G^\circ$ such that $\mathfrak{g}_{s'} = (\mathrm{Ad}\, g)(\mathfrak{g}_s)$. The dimension of $\dim \mathfrak{g}_s$ *is constant for $s \in C$, say equal* *to k. The mapping $s \mapsto \mathfrak{g}_s$ is analytic from C to the Grassmann manifold of all k-dimensional linear subspaces of $\mathfrak{g}$. The smallest positive integer p_1 such that $s^{p_1} \in G_s^\circ = \exp(\mathfrak{g}_s)$ is the same for all $s \in C$.*

Proof As Lemma 9.14 is analogous to Lemma 9.2, its proof is analogous to the proof of Lemma 9.2.

If s is an elliptic element of G, then the closure S of the set of integral powers of s is a compact subgroup S of G. Because the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a continuous homomorphism, it follows that the $K := \mathrm{Ad}\, S$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$. Averaging an arbitrary inner product on $\mathfrak{g}$ over K, we obtain a K-invariant inner product β on $\mathfrak{g}$, which implies that $\mathrm{Ad}\, s$ is a β-orthogonal linear transformation of $\mathfrak{g}$. This implies that the range $(1-\mathrm{Ad}\, s)(\mathfrak{g})$ of $1-\mathrm{Ad}\, s$ is equal to the β-orthogonal complement of the centralizer $\mathfrak{g}_s = \ker(1-\mathrm{Ad}\, s)$ of s in $\mathfrak{g}$. It follows that

$$\mathfrak{g} = (1-\mathrm{Ad}\, s)(\mathfrak{g}) \oplus \mathfrak{g}_s, \tag{9.4}$$

and $1-\mathrm{Ad}\, s$ is a bijective linear mapping from $(1-\mathrm{Ad}\, s)(\mathfrak{g})$ onto itself.

The tangent mapping at $(1,\, 0)$ of the mapping

$$G \times \mathfrak{g}_s \ni (g,\, Z) \mapsto g\, (e^Z\, s)\, g^{-1}\, s^{-1} \tag{9.5}$$

is equal to $(A,\, B) \mapsto (1-\mathrm{Ad}\, s)(A) + B$, which is surjective in view of (9.4). It follows that there is an open neighborhood U of 1 in U and that there are analytic mappings $\varphi : U \to G$, $\zeta : U \to \mathfrak{g}_s$, such that $\varphi(1) = 1$, $\zeta(1) = 0$ and $u = \varphi(u)\, (\mathrm{e}^{\zeta(u)}\ s)\, \varphi(u)^{-1}\, s^{-1}$ for every $u \in U$. Writing $s' = u\, s$, this implies that all elements s' near s are conjugate to elements of $\mathrm{G}_s^\circ\ s$, by means of elements of G which depend analytically on s'. Note that $\mathrm{G}_s^\circ\ s \subset G_s$, actually $G_s^\circ\, s = s\, G_s^\circ$ is equal to the connected component of G_s to which s belongs.

Now let $s' \in G_s$. Because $g \mapsto \mathrm{Ad}\, g$ is a homomorphism from G to $\mathrm{GL}(\mathfrak{g})$, cf. [10, (1.1.10)], it follows that the linear mappings $\mathrm{Ad}\, s'$ and $\mathrm{Ad}\, s$ commute. It follows that $\mathrm{Ad}\, s'$ leaves both $\mathfrak{g}_s = \ker(1-\mathrm{Ad}\, s)$ and $(1-\mathrm{Ad}\, s)(\mathfrak{g})$ invariant. If s' is sufficiently close to s, then the restriction of $1-\mathrm{Ad}\, s'$ to $(1-\mathrm{Ad}\, s)(\mathfrak{g})$ will still be bijective, and we have that $\mathfrak{g}_{s'} = \ker(1-\mathrm{Ad}\, s') \subset \mathfrak{g}_s$.

If s is regular, then we conclude that if $s' \in G_s$ is sufficently close to s, then $\mathfrak{g}_{s'} = \mathfrak{g}_s$, which implies that $\mathrm{Ad}(s'\, s^{-1}) = (\mathrm{Ad}\, s') \circ (\mathrm{Ad}\, s)^{-1} = 1$ on $\mathfrak{g}_s$, or $\mathrm{Ad}(s'\, s^{-1})(Y) = Y$ for every $Y \in \mathfrak{g}_s$. Differentiating $s'\, s^{-1}$ in the direction of $X \in \mathfrak{g}_s$, this leads to $[X,\, Y] = 0$ for every $X,\, Y \in \mathfrak{g}_s$, i.e. $\mathfrak{g}_s$ is commutative, which in turn implies that G_s° is commutative and $G_s^\circ = \exp(\mathfrak{g}_s)$, cf. [10, Th. 1.12.1].

Because the identity component G_s° of G_s is an open and closed subgroup of the closed subgroup G_s of G, $T := \exp(\mathfrak{g}_s) = G_s^\circ$ is a closed subgroup of G. Becuase $T \subset G_s$, the union H of all $s^p\, T$, $p \in \mathbf{Z}$, is a commutative subgroup of G, it is the smallest subgroup of G which contains s and T. Furthermore, the mapping $\varphi : p \to s^p\, T$ is a surjective homomorphism from $\mathbf{Z}$ onto H/T. Let S denote the closed subgroup of G which is generated by s. Because s is an elliptic element of G, there exists a smallest positive integer p_0 such that $s^{p_0} \in S^\circ$, cf.

Lemma 8.3. Because $S^\circ \subset G_s$ and S° is connected, we have $S^\circ \subset G_s^\circ = T$, and we conclude that $p_0 \in \ker \varphi$. It follows that $\ker \varphi$ is a nontrivial subgroup of $\mathbf{Z}$, which implies that there is a unique positive integer p_1 such that $\ker \varphi = p_1 \mathbf{Z}$, which implies that $H/T \simeq \mathbf{Z}/p_1 \mathbf{Z}$. Note that p_1 is the smallest positive integer p such that $s^p \in T$, and that p_0 is an integral multiple of p_1. Because H is equal to the union of the $s^p\, T$ such that $0 \leq p < p_1$, and T is closed in G, H is a closed, hence Lie subgroup of G. The $s^p\, T$, $0 \leq p < p_1$ are the connected components of H.

Because H is closed in G, an element $h \in H$ is elliptic as an element in G if and only if it is elliptic as an element in H. Let $h = s\, k$ with $k \in T$. For any $p \in \mathbf{Z}$, we have $p = q\, p_1 + r$ with $q \in \mathbf{Z}$, $0 \leq r < p_1$. Because s commutes with k, it follows that

$$h^p = s^p\, k^p = (s\, k)^r\ (s^{p_1}\, k^{p_1})^q\,,$$

where $s^{p_1} \in T$, and it follows that h is elliptic as an element of H if and only if $h^{p_1} = s^{p_1}\, k^{p_1}$ is elliptic as an element of T. Because $k \mapsto k^{p_1}$ is an open mapping from T onto T, the h^{p_1} for all h near s form a neighborhood of s^{p_1} in T. Therefore, if s is a stably elliptic element of G, hence a stably elliptic element of H, then s^{p_1} is a stably elliptic element of T. Because T is a connected commutative Lie group, it is isomorphic to $(\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$ for some $k,\, l \in \mathbf{Z}_{\geq 0}$, cf. [10, Cor. 1.12.4]. $(\mathbf{R}/\mathbf{Z})^k \times \mathbf{R}^l$ only contains stably elliptic elements if $l = 0$, i.e. T is a torus. This concludes the proof of i) $\Longrightarrow$ ii).

By definition G^{se} and the set of regular elements of G are open subsets of G. If $s \in G^{\mathrm{se}}$ is not regular, then we can find s' arbitrarily close to s with $\dim \mathfrak{g}_{s'} < \dim \mathfrak{g}_s$. Because of the finiteness of the dimensions we can repeat this only finitely times with s replaced by s'. It follows that arbitrarily close to s we can find s' for which $\mathfrak{g}_{s'}$ has locally minimal dimension, i.e. s' is regular. This shows that G^{rse} is a dense open subset of G^{se}.

Now assume that ii) holds. Because $\mathfrak{t} = \mathfrak{g}_s$, we have for every $X \in \mathfrak{t}$ that

$$s\ \mathrm{e}^X\ s^{-1} = \mathrm{e}^{(\mathrm{Ad}\, s)(X)} = \mathrm{e}^X,$$

which shows that s commutes with every element of $T = \exp(\mathfrak{t})$. It follows that the union H of the $s^p\, T$, $p \in \mathbf{Z}$ is a commutative subgroup, which moreover is compact because there is a minimal positive p_1 such that $s^{p_1} \in T$, which implies that $H/T \simeq \mathbf{Z}/p_1 \mathbf{Z}$. It follows that every element of H, and in particular s, is elliptic. As we have seen after (9.5), all elements in G near s are conjugate to elements $s'' \in s\, T \subset H$ near s, and therefore are elliptic, and we conclude that s is stably elliptic.

Because any element $s'' \in H$ near s commutes with s, $\mathrm{Ad}\, s''$ commutes with $\mathrm{Ad}\, s$, which implies that $1 - \mathrm{Ad}\, s''$ maps $\mathfrak{q} := (1 - \mathrm{Ad}\, s)(\mathfrak{g})$ into iself. Because $1 - \mathrm{Ad}\, s$ is bijective on q, $1 - \mathrm{Ad}\, s''$ is bijective on $\mathfrak{q}$ and hence $\dim \mathfrak{g}_{s''} \leq \dim \mathfrak{g}_s$, if $s'' \in H$ is sufficiently close to s. Because every s' in G near s is conjugate to an element $s'' \in H$ near s, we have $\dim \mathfrak{g}_{s'} = \dim \mathfrak{g}_{s''} \leq \dim \mathfrak{g}_s$, and we conclude that s is a regular element of G. This concludes the proof of ii) $\Longrightarrow$ i).

If we write $s' = u\, s$, $\theta(s') = \varphi(u)^{-1}$, $s'' = \theta(s')\, s'\, \theta(s')^{-1}$, then the equation $u = \varphi(u)\, (\mathrm{e}^{\zeta(u)}\ s)\, \varphi(u)^{-1}\, s^{-1}$ is equivalent to $s'' = \mathrm{e}^{\zeta(u)}\ s = s\ \mathrm{e}^{\zeta(u)} \in s\, G_s^\circ$. Furthermore, if s''

is sufficently close to s in G_s, then $\mathfrak{g}_s = \mathfrak{g}_{s''} = (\mathrm{Ad}\,\theta(s'))(\mathfrak{g}_{s'})$, $G_s^\circ = G_{s''}^\circ$ and $s\,G_s^\circ = s''\,G_s^\circ = s''\,G_{s''}^\circ$, the latter because s'' belongs to the connected component $s\,G_s^\circ$ of s in G_s.

The condition for $s,\, s' \in C$ that $\mathfrak{g}_{s'}$ is conjugate to $\mathfrak{g}_s$ by means of an element of G° is an equivalence relation. Because nearby elements of C are equivalent and C is connected, it follows that $\mathfrak{g}_{s'}$ is conjugate to $\mathfrak{g}_s$ for all $s,\, s' \in C$, which implies that the dimension of $\mathfrak{g}_s$ is constant for all $s \in C$. Because for s' near s we have that $\mathfrak{g}_{s'}$ is conjugate to $\mathfrak{g}_s$ by means of an element of G° which depends analytically on s', the mapping $s' \mapsto \mathfrak{g}_{s'}$ is analytic on a neighborhood of s in C, and because this holds for any $s \in C$, the mapping $s \mapsto \mathfrak{g}_s$ is analytic on C. The constancy of p_1 on C is proved in the same way. $\square$

Remark 9.15 In general, the torus T in Lemma 9.14 is different from the torus T in Lemma 8.3. In the same way as in Remark 9.3, the tori in Lemma 8.3 can be equal to arbitrary subtori of the tori in Lemma 9.14. And if $\dim \mathfrak{g}_s > 0$, the number p_0 in Lemma 8.3 can be equal to any positive integral multiple of the number p_1 in Lemma 9.14. $\oslash$

Example 9.16 If G is compact, then every element of G is elliptic, and therefore every element of G is stably elliptic. Therefore in this case G^{rse} is equal to the set of regular element in G as in [10, (g) on p. 137]. Conversely, Djoković [9] proved that a connected locally compact group G is compact if and only if the elliptic elements of G fill up a whole neighborhood of the identity element of G.

If G is connected and s is a regular element of G, then $s \in G_s^\circ$, cf. [10, Prop. 3.1.3], which means that $p_1 = 1$. Because every $s \in G$ belongs to a maximal torus T, for which $T \subset G_s$, hence $T \subset G_s^\circ$, and therefore $T = G_s^\circ$ if s is regular. Therefore, if G is compact and connected, then all tori in Lemma 9.14 are maximal tori, of dimension equal to the rank of G, cf. Example 9.4. $\oslash$

Example 9.17 The element $s = (A,\, a)$ in the the group $G = \mathrm{E}(2)$ of the Euclidean motions in the plane, cf. Example 9.5, is not elliptic if g is a nonzero translation, i.e. if $A = 1$ and $a \neq 0$. If on the other hand $A \neq 1$, then $1 - A$ is invertible and s has $p = (1 - A)^{-1}\, a$ as its unique fixed point. In this case G_s is equal to the group $\mathrm{SO}(2)(p)$ of all rotations about the point p, which is a circle subgroup of $\mathrm{E}(2)$. Therefore G^{rse} is equal to the open dense subset of all $(A,\, a) \in \mathrm{E}(2)$ such that $A \neq 1$. As in the case of a compact connected Lie group, we have $p_1 = 1$ for all elements of $\mathrm{E}(2)^{\mathrm{rse}}$. $\oslash$

Example 9.18 In $G = \mathrm{SL}(2,\, \mathbf{R})$, then the regular and stably elliptic elements are the elements $A \in \mathrm{SL}(2,\, \mathbf{R})$ such that $|\operatorname{trace} A| < 2$. The subgroup $\mathrm{SO}(2)$ meets each conjugacy class in G^{rse} exactly once, which means that in this case G^{rse} is connected and all tori T in Lemma 9.14 are conjugate to each other. Also, $p_1 \equiv 1$ in this case.

The interior of the set of non-elliptic elements consists of all $A \in \mathrm{SL}(2,\, \mathbf{R})$ such that $|\operatorname{trace} A| > 2$.

See [10, Fig. 1.2.3] for a picture of $\mathrm{SL}(2,\, \mathbf{R})$ and its subsets of $A \in \mathrm{SL}(2,\, \mathbf{R})$ such that $|\operatorname{trace} A| < 2$ and $|\operatorname{trace} A| > 2$, respectively. For the set $\mathfrak{g}^{\mathrm{rse}}$ and the interior of the set of non-elliptic elements of $\mathfrak{g}$, see Example 9.6. $\oslash$

Suppose that the G-action on M is free and proper, and let Φ^t denote the flow in $G\backslash M$ defined by (5.3). Suppose that P is a smooth submanifold of $G\backslash M$ with the following properties.

i) For each $p \in P$ the curve $t \mapsto \Phi^t(p)$ is non-constant and periodic.

ii) The function $\tau : P \to \mathbf{R}$, which assigns to each $p \in P$ the minimal positive period $\tau(p)$ of $t \mapsto \Phi^t(p)$, is continuous.

iii) $\Phi^t(p) \in P$ for every $p \in P$ and $t \in \mathbf{R}$.

It follows that $\pi^{-1}(P)$ is a locally closed smooth submanifold of M which is G-invariant, hence $\pi : \pi^{-1}(P) \to P$ is a principal G-bundle. Because $\pi^{-1}(P)$ consists of relative periodic solutions and is G-invariant, it follows from Lemma 8.1 that $\pi^{-1}(P)$ is invariant under the v-flow, and that the restriction of v to $\pi^{-1}(P)$ is a complete vector field on $\pi^{-1}(P)$.

The projection $\pi : M \to G\backslash M$ intertwines v with a unique smooth vector field w on $G\backslash M$, the one for which $\Phi^t = \mathrm{e}^{t\,w}$. It follows from iii) that w is tangent to P, and from i) that w has no zeros in P.

Lemma 9.19 *The orbits of the w-flow define a smooth principal $\mathbf{R}/\mathbf{Z}$-fibration $\psi : P \to Q$ and there is a smooth function $\sigma : Q \to \mathbf{R}$ such that $\tau = \sigma \circ \psi$.*

Proof Let $p \in P$. There exists a smooth codimension one submanifold S of P through p such that $\mathrm{T}_p\, P = \mathrm{T}_p\, S \oplus \mathbf{R}\, w(p)$. Using the implicit function theorem, one can find an open neighborhood S_0 of p in S and an $\epsilon > 0$, such that the mapping $\Phi : (s,\, t) \mapsto \mathrm{e}^{t\,w}(s)$ defines a diffeomorphism from $S_0 \times \,]-\epsilon,\, \epsilon[$ onto an open neighborhood U of p in M. Write $\Phi^{-1}(u) = (s(u),\, t(u))$, in which the mappings $s : U \to S_0$ and $t : U \to \,]-\epsilon,\, \epsilon[$ are smooth.

Because $\mathrm{e}^{\tau(p)\,w}(p) = p \in U$, the set S_1 of all $s \in S_0$ such that $u := \mathrm{e}^{\tau(p)\,w}(s) \in U$ is an open neighborhood of p in S_0, and we have that $s(u) = \mathrm{e}^{(\tau(p)-t(u))\,w}(s) \in S_0$, in which $\rho(s) := \tau(p) - t(u)$ depends smoothly on $s \in S_1$, and $\rho(p) = \tau(p)$. Because $\mathrm{e}^{\rho(s)\,w}(s) \in S_0$ and $\mathrm{e}^{\tau(s)\,w}(s) = s$, we have $\mathrm{e}^{\rho(s)-\tau(s)}(s) \in S_0$, from which we conclude that $\rho(s) - \tau(s) = 0$ if $|\rho(s) - \tau(s)| < \epsilon$. Because $\rho - \tau$ is a continuous function on S_1 which is equal to 0 at p, we conclude that τ is equal to the smooth function ρ on some open neighborhood S_2 of p in S_1.

The mapping $\Psi : (t,\, s) \mapsto \mathrm{e}^{t\,\tau(s)\,w}(s)$ from $(\mathbf{R}/\mathbf{Z}) \times S_2$ to P is well-defined, smooth, injective, and has a bijective tangent mapping at every point. In view of the inverse function theorem, this shows that Ψ is a diffomorphism onto an open neighborhood V of p in P. Moreover, Ψ maps the circles $(\mathbf{R}/\mathbf{Z}) \times \{s\}$, $s \in S_2$, onto w-orbits in P. Because also $\tau(\Psi(t,\, s)) = \tau(s)$ for all $(t,\, s) \in (\mathbf{R}/\mathbf{Z}) \times S_2$, this proves that τ is a smooth function on V which is constant on the w-orbits in V. Because every $p \in P$ has an open neighborhood V which is invariant under the w-flow and on which τ is a smooth function which is constant on the w-orbits in V, the conclusion is that $\tau : P \to \mathbf{R}$ is smooth and is constant on the w-orbits in P. The mapping $\Psi : (t,\, s) \mapsto \mathrm{e}^{t\,\tau(s)\,w}(s)$ from $(\mathbf{R}/\mathbf{Z}) \times P$ to P defines a proper and free $\mathbf{R}/\mathbf{Z}$-action on P, of which the orbits coincide with the w-orbits. The proof of the lemma is complete. $\square$

Example 9.20 Let R be the rotation in the plane about the angle $2\pi\, a/b$, in which $b \in \mathbf{Z}_{\geq 2}$, $a \in \mathbf{Z}$, $0 < a < b$ and a, b without common divisors. On $\mathbf{R}^2 \times \mathbf{R}$ let $p \in \mathbf{Z}$ act by $(x,\, t) \mapsto (R^p\, x,\, t + p)$. Let $P = (\mathbf{R}^2 \times \mathbf{R})/\mathbf{Z}$ and denote the canonical projection from $\mathbf{R}^2 \times \mathbf{R}$ onto P by ψ. Then ψ intertwines the vector field $\partial/\partial t$ with a unique analytic vector field w on P. The w-solution curves are the curves $t \mapsto \psi(x(0),\, t(0) + t)$. All these curves are periodic, where the minimal positive period τ is equal to 1 when $x(0) = 0$ and equal to b when $x(0) \neq 0$. Therefore the minimal period function is discontinuous in this case. The assumption ii) has been made in order to avoid such *subperiodic* solution behaviour. $\oslash$

Lemma 9.21 *For $m \in \pi^{-1}(P)$, the unique element $s = s(m) \in G$ in* (8.1), (8.2) *depends smoothly on m.*

Proof Write $N = \pi^{-1}(P)$. The mapping

$$\mathcal{A} : (g,\, m) \mapsto (m,\, g \cdot m) : G \times N \to N \times N$$

is a injective immersion, proper because the G-action is proper. Therefore its image O is a closed smooth submanifold of $N \times N$, and we have a smooth inverse $\mathcal{A}^{-1} : O \to G \times N$. The mapping

$$m \mapsto (m,\, \mathrm{e}^{\tau(m)\, v}(m)) : N \to N \times N$$

is smooth and satisfies $\psi(N) \subset O$, and therefore it defines a smooth mapping $\mathcal{B} : N \to O$. The mapping $m \mapsto s(m)$ is equal to $\mathcal{B} : N \to O$, followed by $\mathcal{A}^{-1} : O \to G \times N$, and concluded by the projection $N \times G \to G$ onto the first factor. Therefore $m \mapsto s(m)$ is smooth as the composition of three smooth mappings. $\square$

Let P^{rse} denote the set of all $p \in P$, such that for some (every) $m \in \pi^{-1}(\{p\})$ the element $s(m)$ is a regular and stably elliptic element of G. Then P^{rse} is an open subset of P, and $Q^{\mathrm{rse}} := \psi(P^{\mathrm{rse}})$ is an open subset of Q. Because P^{rse} is invariant under the w-flow, we have $P^{\mathrm{rse}} = \psi^{-1}(Q^{\mathrm{rse}})$ and $s(m) \in G^{\mathrm{rse}}$ for every $m \in (\psi \circ \pi)^{-1}(Q^{\mathrm{rse}})$.

Lemma 9.22 *Write $\chi := \psi \circ \pi$. Let $m_0 \in M$ and $q_0 := \chi(m_0) \in Q^{\mathrm{rse}}$. Write $s = s(m_0)$ and $T = G_s^{\circ}$. Then there exists an open neighborhood Q_0 of q_0 in Q^{rse} and a smooth section $\mu : Q_0 \to \chi^{-1}(Q_0)$ of the fibration $\chi : \chi^{-1}(Q_0) \to Q_0$, such that $G^{\circ}_{s(\mu((q))} = T$ and $s(\mu(q))\, T = s\, T$ for every $q \in Q_0$.*

Proof Because $\pi : \chi^{-1}(Q) \to Q$ is a smooth fibration, there exists an open neighborhood Q_1 of q_0 in in Q^{rse} and a smooth section $\nu : Q_1 \to \pi^{-1}(Q_1)$ of $\chi : \chi^{-1}(Q_1) \to Q_1$, i.e. ν is smooth and $\chi \circ \nu$ is equal to the identity in Q_1. It follows from Lemma 9.14 that there exists an open neighborhood V of s in G and an analytic mapping $\theta : V \to G$, such that if $s' \in V$ and $s'' = \theta(s')\, s'\, \theta(s')^{-1}$, then $G^{\circ}_{s''} = G^{\circ}_s = T$ and $s''\, T = s\, T$.

It follows from (8.7) with $s = s(m)$ that $s(g \cdot m) = g\, s(m)\, g^{-1}$. If we write $m = \nu(q)$, $u = s(\nu(q))$, $g = \theta(u)$, and $\mu(q) := \theta(s(\nu(q))) \cdot \nu(q)$, then we obtain that $G^{\circ}_{s(\mu((q))} = T$ and

$s(\mu(q))\,T = s\,T$ for every $q \in Q$ such that $s(\nu(q)) \in V$. These q form an open neighborhood Q_0 of q_0 in Q^{rse}, and μ is a smooth section of $\chi : \chi^{-1}(Q_0) \to Q_0$. $\square$

The mapping $(q,\, g,\, t) \mapsto g \cdot \mathrm{e}^{t\,v}(\mu(q))$ is a surjective local diffeomorphism from $Q_0 \times G \times \mathbf{R}$ onto $\chi^{-1}(Q_0) = \pi^{-1}(P_0)$. If $g \cdot \mathrm{e}^{t\,v}(\mu(q)) = g' \cdot \mathrm{e}^{t'\,v}(\mu(q'))$, then an application of χ shows that $q = q'$, and the equation is equivalent to $(g')^{-1} \cdot g \cdot \mu(q) = \mathrm{e}^{(t'-t)\,v}(\mu(q))$, which in turn is equivalent to the statement that there exists a $p \in \mathbf{Z}$ such that $t' - t = p\,\tau(\pi(\mu(q)))$ and $(g')^{-1}\,g = s(\mu(q))^p$. In view of Lemma 9.19 and the fact that μ is a section of χ, we have $\tau \circ \pi \circ \mu = \sigma \circ \psi \circ \pi \circ \mu = \sigma \circ \chi \circ \mu = \sigma$. It follows that the mapping $(q,\, g,\, t) \mapsto g \cdot \mathrm{e}^{t\,v}(\mu(q))$ induces a diffeomorphism from $(Q_0 \times G \times \mathbf{R})/\mathbf{Z}$ onto $\chi^{-1}(Q_0) = \pi^{-1}(P_0)$, where $p \in \mathbf{Z}$ acts on $Q_0 \times G \times \mathbf{R}$ by sending $(q,\, g,\, t)$ to $(q,\, g\,s(\mu(q))^{-p},\, t + p\,\sigma(q))$.

In the notation of Lemma 9.22, let λ_j, $1 \leq j \leq k$, be a Z-basis of the integral lattice $\ker\exp$ in the Lie algebra $\mathfrak{t}$ of the torus T. In view of Lemma 9.14, we have $s(\mu(q))^{p_1} \in T$ for every $q \in Q_0$. Because the exponential mapping is a local diffeomorphism from $\mathfrak{t}$ onto T, we can arrange, by shrinking Q_0 if necessary, that there is a smooth mapping $X : Q_0 \to \mathfrak{t}$ such that $s(\mu(q))^{p_1} = \mathrm{e}^{X(q)}$ for every $q \in Q_0$.

We now reason as after (8.9), with the torus $T = S^{\circ}$ and the number p_0 replaced by the torus $T = G_s^{\circ}$ and the number p_1, respectively, and adding the parameters q. Let $\theta \in \mathbf{R}^{k+1}$ act on $Q_0 \times G \times \mathbf{R}$ by sending $(q,\, g,\, t) \in G \times \mathbf{R}$ to

$$\left(q,\, g\,\exp\left(\sum_{j=1}^{k} \theta_j\,\lambda_j - \theta_{k+1}\,X(q)\right),\, t + \theta_{k+1}\,p_1\,\sigma(q)\right). \tag{9.6}$$

If $\theta \in \mathbf{Z}^{k+1}$, then it follows from the fact that $\mathrm{e}^{X(q)} = s(\mu(q))^{p_1}$ that (9.6) is equal to

$$\left(q,\, g\,s(\mu(q))^{-\theta_{k+1}\,p_1},\, t + \theta_{k+1}\,p_1\,\sigma(q)\right),$$

which is equal to the action of the integer $p = \theta_{k+1}\,p_1$ on $(q,\, g,\, t)$. It follows that we have an induced action of $(\mathbf{R}/\mathbf{Z})^{k+1} = \mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$ on $(Q_0 \times G \times \mathbf{R})/\mathbf{Z}$. Moreover, the $(\mathbf{R}/\mathbf{Z})^{k+1}$--action on $(Q_0 \times G \times \mathbf{R})/\mathbf{Z}$ commutes with the G-action on $(Q_0 \times G \times \mathbf{R})/\mathbf{Z}$ defined by multiplications from the left on the second factor.

If (9.6) is equal to $(q,\, g\,s(\mu(q))^{-p},\, t + p\,\sigma(q))$ for some $p \in \mathbf{Z}$, then

$$s(\mu(q))^p = \exp\left(-\sum_{j=1}^{k} \theta_j\,\lambda_j + \theta_{k+1}\,X(q)\right) \in T, \tag{9.7}$$

which implies that $p = q\,p_1$ for some $q \in \mathbf{Z}$. But then $p\,\sigma(q) = \theta_{k+1}\,p_1\,\sigma(q)$ implies that $\theta_{k+1} = q$ and now (9.7) in combination with $\mathrm{e}^{X(q)} = s(\mu(q))^{p_1}$ implies that $\sum_{j=1}^{k} \theta_j\,\lambda_j$ belongs to the integral lattice in $\mathfrak{t}$, which in turn implies that $\theta_j \in \mathbf{Z}$ for every $1 \leq j \leq k$. This shows that the action of $(\mathbf{R}/\mathbf{Z})^{k+1}$ on $(Q_0 \times G \times \mathbf{R})/\mathbf{Z}$ is free, and it is automatically proper because the group $(\mathbf{R}/\mathbf{Z})^{k+1}$ is compact.

The diffeomorphism Φ intertwines the action of $(\mathbf{R}/\mathbf{Z})^{k+1}$ on $(Q_0 \times G \times \mathbf{R})/\mathbf{Z}$ with a uniquely defined proper and free smooth action of $(\mathbf{R}/\mathbf{Z})^{k+1}$ on $\chi^{-1}(Q_0) = \pi^{-1}(P_0)$. This action commutes with the G-action on $\pi^{-1}(P_0)$.

Let $X_j(q)$, $1 \leq j \leq k$, denote the coordinates of $X(q) \in \mathfrak{t}$ with respect to the $\mathbf{Z}$-basis λ_j, $1 \leq j \leq k$, of the integral lattice of $\mathfrak{t}$. Then

$$\sum_{j=1}^{k} \theta_j \, \lambda_j - \theta_{k+1} \, X(q) = \sum_{j=1}^{k} \left(\theta_j - \theta_{k+1} \, X_j(q) \right) \, \lambda_j,$$

and it follows that on $\chi^{-1}(\{q\})$ the vector field v is equal to the inifinitesimal action of the element $\dot{\theta}(q)$ of the Lie algebra $\mathbf{R}^{k+1}$ of $\mathbf{R}^{k+1}/\mathbf{Z}^{k+1}$, which is defined by

$$\dot{\theta}_j(q) = \dot{\theta}_{k+1} \, X_j(q), \quad 1 \leq j \leq k, \quad \text{and} \quad \dot{\theta}_{k+1} = 1/p_1 \, \sigma(q). \tag{9.8}$$

If the G-action on M is proper but not free, then the above constructions can be applied with the manifold M and the Lie group G replaced by an isotropy type M_H and the Lie group $\mathrm{N}(H)/H$, respectively, where $\mathrm{N}(H)/H$ acts freely on M_H, cf. Lemma 3.3, and M_H is invariant under the v-flow, cf. Lemma 5.5. The $\mathrm{N}(H)/H$-invariant principal $(\mathbf{R}/\mathbf{Z})^{k+1}$- -fibration in $\pi^{-1}(P_0) \cap M_H$ has a unique extension to a G-invariant principal $(\mathbf{R}/\mathbf{Z})^{k+1}$-fibration in $\pi^{-1}(P_0)$. For the proof one can use a version with parameters of the constructions at the end of Subsection 7.4. This leads to the following conclusions.

Proposition 9.23 *Suppose that the G-action on M is proper, and let P be a smooth submanifold of the orbit type $G\backslash M_{[H]}$ in the orbit space $G\backslash M$, consisting of nonconstant periodic solutions of the flow Φ^t in $G \setminus M$ defined by* (5.3). *Furthermore assume that the function $\tau : P \to \mathbf{R}$, which assigns to each $p \in P$ the minimal positive period $\tau(p)$ of $t \mapsto \Phi^t(p)$, is continuous. It then follows from Lemma 9.19 that the orbits of the flow Φ^t in P define a smooth principal $\mathbf{R}/\mathbf{Z}$-fibration $\psi : P \to Q$ and there is a smooth function $\sigma : Q \to \mathbf{R}$ such that $\tau = \sigma \circ \psi$. Write $\chi = \psi \circ \pi : \pi^{-1}(P) \to Q$.*

Write, for each $m \in \pi^{-1}(P) = \chi^{-1}(Q)$, $s(m) = s\, G_m$, with $s \in G$ as in (8.1), (8.2). *Let Q^{rse} denote the set of all $q \in Q$ such that for some (every) $m \in \chi^{-1}(\{q\})$ we have that $s(m)$ is a regular and stably elliptic element of $\mathrm{N}(G_m)/G_m$. Write $T(m)$ for the identity component of the centralizer of $s(m)$ in $\mathrm{N}(G_m)/G_m$.*

Then Q^{rse} is an open subset of Q; let C be a connected component of Q^{rse}. If through each $m \in \chi^{-1}(C)$ we draw the v-flowout of the $T(m)$-orbit through m, then these subsets of $\chi^{-1}(C)$ define a G-invariant smooth fibration of $\chi^{-1}(C)$, of which the fibers are diffeomorphic to tori of dimension equal to $\dim T(m) + 1$.

Let $m_0 \in M$ be such that $q_0 := \chi(m_0) \in C$. Write $H = G_{m_0}$, $T = T(m_0)$ and $\mathfrak{t}$ for the Lie algebra of the torus subgroup T of $\mathrm{N}(H)/H$. Then there exists an open neighborhood Q_0 of q_0 in C, a smooth section $\mu : Q_0 \to \chi^{-1}(Q_0)$ of the fibration $\chi : \chi^{-1}(Q_0) \to Q_0$, and a smooth mapping $X : Q_0 \to \mathfrak{t}$, such that such that $G^{\circ}_{s(\mu((q))} = T$ and $s(\mu(q))^{p_1} = \mathrm{e}^{X(q)}$ for every $q \in Q_0$. Here, for any $m \in \chi^{-1}(C)$, p_1 denotes the smallest positive integer p such that $s(m)^{p_1} \in T(m)$, cf. the last statement in Lemma 9.14.

Let λ_j, $1 \leq j \leq k$, be a $\mathbf{Z}$-basis of the integral lattice of $\mathfrak{t}$, and let $X_j(q)$, $1 \leq j \leq k$, denote the coordinates of $X(q)$ with respect to this basis. Then (9.6) *defines an action of $\theta \in (\mathbf{R}/\mathbf{Z})^{k+1}$ on $Q_0 \times G \times \mathbf{R}$ which is intertwined by the mapping $(q,\, g,\, t) \mapsto g \cdot \mathrm{e}^{t\,v}(\mu(q))$*

with a G-invariant free and proper action of $(\mathbf{R}/\mathbf{Z})^{k+1}$ on $\chi^{-1}(Q_0)$, the orbits of which are equal to the fibers of the aforementioned toral fibration of $\pi^{-1}(C)$. Furthermore, for each $m \in \chi^{-1}(Q_0)$, $v(m)$ is equal to the infintesimal action of the element $\dot{\theta}(\chi(m))$ in the Lie algebra $\mathbf{R}^{k+1}$ of $(\mathbf{R}/\mathbf{Z})^{k+1}$ which is defined by (9.8).

Because one-parameter subgroups of tori are quasi-periodic, this expresses in a quite strong sense that the v-flow in $\chi^{-1}(C)$ is quasi-periodic on tori, where the tori form a smooth fibration and the velocity vector at m is an element of the Lie algebra of the torus which depends smoothly on the parameters $q = \chi(m)$.

As in the text after Proposition 9.9, the global obstruction to the toral fibration of $\chi^{-1}(C)$ being a G-invariant principal $(\mathbf{R}/\mathbf{Z})^{k+1}$ appears to be the monodromy homomorphism from $\pi_1(C, q_0)$ to the Weyl group of the torus subgroup T of $\mathrm{N}(H)/H$.

Remark 9.24 For free actions of compact and connected Lie groups, Proposition 9.23 has been presented by Hermans [12], who applied it to a spherical ball rolling in a rotationally symmetric bowl. $\oslash$

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