EXERCISES I

Let (E, σ) be a symplectic vector space. We have seen that unlike the case for positive non-degenerate symmetric forms, it is not true that for any linear subspace V of E, $(V, \sigma_{|V})$ is a symplectic vector space. Interesting subsets of the grassmannians of linear subspaces are singled out according to the incidence relations between V and its symplectic anhililator V^σ.

- V is said to be **isotropic** if $V \subset V^\sigma$. **Lagrangian** subspaces are defined to be the isotropic subspaces of maximal dimension.
- V is said to be **coisotropic** if $V^\sigma \subset V$.
- V is said to be **symplectic** if $(V, \sigma_{|V})$ is a symplectic vector space.

Exercise 1.6.

i. *Show that symplectic linear subspaces only exist in even dimensions.*
ii. *Characterize symplectic linear subspaces in terms of the incidence relations between the subspace and its symplectic anihilator.*

For any $i \in \{1, \ldots, n\}$, where $2n$ is the dimension of E, the **symplectic grassmannian** *of $2i$ dimensional linear subspaces -denoted by $\mathcal{S}_{2i}(E, \sigma)$- is the subset of $Gr_{2i}(E)$ of symplectic linear subspaces.*

iii. *Show that $\mathcal{S}_{2i}(E, \sigma)$ is an open subset of $Gr_{2i}(E)$. This is in contrast with the Lagrangian Grassmannian $\mathcal{L}(E, \sigma)$.*
iv. *Show that if $V \in \mathcal{S}_{2i}(E, \sigma)$ then $V^\sigma \in \mathcal{S}_{2n-2i}(E, \sigma)$.*

Recall that all lines in a symplectic vector space are isotropic.

Exercise 1.7.

i. *Prove that all hyperplanes $H \subset E$ are coisotropic.*

Now let W be a hyperplane of H. It is a real codimension 2 subspace of E, and hence a candidate to be symplectic.

ii. *Describe all hyperplanes W of H such that $W \in \mathcal{S}_{2n-2}(E, \sigma)$.*

Exercise 1.8. *Show that the following statements are equivalent:*

(1) *L is Lagrangian.*
(2) *L is both isotropic and coisotropic.*
(3) *L is coisotropic of minimal dimension.*

The search for symplectic linear subspaces (motivated for the search of symplectic submanifolds) is an important problem in symplectic geometry. Elucidating the right compatibility relations between symplectic an complex geometry turns out to be a key aspect.

Let J be a complex structure in (E, σ), i.e. $J\colon E \to E$ is an endomorphism such that $J^2 = -Id$. The complex structure J is said to **tame** the linear symplectic form σ if

$$\sigma(u, Ju) > 0 \; \forall u \in E \backslash \{0\}$$

If in addition J belongs to the symplectic linear group $Symp(E, \sigma)$ then it is said to be **compatible** with σ.

Exercise 1.9. *The complex structure J gives rise to the complex grassmannians $Gr_i^{\mathbb{C}}(E, J) \subset Gr_{2i}(E)$, $i = 1, \ldots, n$. Prove that if J tames σ then $Gr_i^{\mathbb{C}}(E, J) \subset \mathcal{S}_{2i}(E, \sigma)$.*

Recall that if (E, J) is a complex vector space and h a complex Hermitian metric on it, then $\sigma := \text{Im}h$ defines a symplectic structure.

Exercise 1.10. *Let (E, σ) be a symplectic vector space and J a complex structure on E.*

i. *Show that σ is the imaginary part of a J-Hermitian metric h if and only J is compatible with σ. Notice that in such a case we can find an isomorphism from (E, J, h) onto $(\mathbb{R}^{2n} = \mathbb{C}^n, J_0, h_0)$, where J_0 is multiplication times i in $\mathbb{C}^n$ and h_0 is the Hermitian basis whose matrix in the fixed complex basis $e_1, \ldots, e_n$ is the identity.*

Recall that when J is compatible with σ, the real part of the complex Hermitian metric defined out of σ and J defines an inner product g.

ii. *Show that with respect to this inner product, for any linear subspace $V \subset E$ we have*
$$JV \perp V^\sigma \tag{1}$$
iii. *Show that if (E, σ) has dimension 2, then a complex structure taming σ is always compatible with it.*
iv. *Let (E, σ) be a symplectic vector space. Describe the manifold of complex structures on E as a homogeneous space of $Gl(E)$ and deduce its dimension. Describe also the manifold of complex structures compatible with σ as a homogeneous space (of a subgroup of $Gl(E)$). Deduce that for symplectic vector spaces of dimension greater that 2, there exists complex structures which tame the symplectic form but are not compatible with it.*

Assume that J is compatible with (E, σ). By exercise 1.9, $Gr_i^{\mathbb{C}}(E, J) \subset \mathcal{S}_{2i}(E, \sigma)$. As we will see complex linear subspaces are in a suitable sense the "most" symplectic ones.

Indeed, if V is a complex subspace then $V = JV$ and therefore by equation 1 we have $V \perp V^\sigma$ (we use g the canonical inner product associated to σ and J) , so the symplectic orthogonal is "as transversal as possible" to the linear subspace V. There is a beautiful way of detecting when V in $Gr(2i, E)$ is symplectic and measuring the "defect" of $V \in \mathcal{S}_{2j}(E, \sigma)$ from being complex.

We will do it just for four dimensional symplectic spaces (E, σ). We consider $Gr_2^{or}(E)$ the grassmannian of oriented planes in E. Notice that since we have a canonical metric g, the choice of orientation of $V \in Gr_2^{or}(E)$ implies that V inherits an area form Ω_V (which is a symplectic form on V!). If V is a symplectic plane, then $\sigma_{|V}$ is another symplectic form on V.

For each $V \in Gr_2^{or}(E)$ the **Kahler angle** is defined
$$\theta(V) := \cos^{-1}(\sigma_{|V}/\Omega_V) \in [0, \pi] \tag{2}$$
By definition,
$$\mathcal{S}_2^{or}(E, \sigma) = \{V \in Gr_2^{or}(E)) | \theta(V) \neq \pi\},$$
and if $\mathcal{L}^{or}(E, \sigma)$ denotes the subset of $Gr_2^{or}(E)$ of Lagrangian planes then we see that
$$\mathcal{L}^{or}(E, \sigma) = \{V \in Gr_2^{or}(E)) | \theta(V) = \pi\}$$
Recall that a compatible complex structure defines a canonical Hermitian metric and hence a group of unitary transformations $U(E, h)$.

Exercise 1.11.

i. *Show that the Kahler angle is well defined, i.e. $\sigma_{|V}/\Omega_V \in [-1, 1]$.*

Recall that $Gr_2^{or}(E)$ is the disjoint union of $\mathcal{S}_2^{or}(E, \sigma)$ and $\mathcal{L}^{or}(E, \sigma)$. The group $U(E, h)$ is contained in $Symp(E, \sigma)$. Hence it acts on $Gr_2^{or}(E)$ preserving each of the two aforementioned subsets.

ii. *Show that* $V, V' \in Gr_2^{or}(E)$ *are in the same* $U(E,h)$*-orbit iff* $\theta(V) = \theta(V')$*.*

iii. *Classify the orbits of the* $U(E,h)$*-action on* $Gr_2^{or}(E)$ *according to its dimension.*

Given V, V' *subspaces of a vector space with inner product* (E,g)*, the maximal angle between* V, V *is defined*

$$\angle_{max}(V, V') = max_{v \in V \setminus \{0\}} angle(v, V')$$

iv. *Show that*

$$|\theta(V)| = \angle_{max}(V, JV)$$

The notion of Kahler (multi)angle extends to higher dimensions. Notice that in higher dimensions the situation is more complicated in the following sense: in dimension 4 a dimension count shows that if $V \in Gr_2(E)$ is not complex, then $V \cap JV = \{0\}$. In dimension eight four example, for a four dimensional linear subspace V we may have either of the following situations

- $V = JV$,
- $dim V \cap JV = 2$,
- $V \cap JV = \{0\}$,

and the dimension $V \cap JV$ is clearly preserved in the $U(E,h)$-orbits.

If $A \in Symp(E,\sigma)$ then we can deduce information about the eigenvalues of A. In particular, if $\lambda \in spec(A)$ (λ is an eigenvalue) then so $1/\lambda$ is. Therefore, one concludes that the determinant of A is ± 1.

Exercise 1.12. *Prove that if* $A \in Symp(E,\sigma)$ *then its is actually equal to 1, (and hence if* $-1 \in spec(A)$ *then it has even multiplicity).*

Exercise 1.13. *Let* E *be an even dimensional vector space. The set of all symplectic forms is an open subset of* $\bigwedge^2 E^*$*, the vector space of all antisymmetric bilinear forms on* E*. Describe it as an homogeneous space of* $Gl(E)$ *and compute its dimension.*