EXERCISES III

Exercise 8 (Example 3.1 in the notes revisited)**.** *Let X be an n-dimensional manifold and let $v \in \mathfrak{X}(X)$. The* **momentum function** *of the vector field v is given by*

$$\begin{aligned} \mu_v \colon T^*X &\longrightarrow \mathbb{R} \\ (x, \xi) &\longmapsto \xi(v(x)) \end{aligned} \tag{1}$$

Let $H_{\mu_v} \in \mathfrak{X}(T^*X)$ denote the Hamiltonian vector field of μ_v with respect to the canonical symplectic structure $d\tau \in \Omega^2(T^*X)$. We want to show that its flow $e^{tH_{\mu_v}}$, which is known to preserve the symplectic form $d\tau$, obeys the following formula:

$$e^{tH_{\mu_v}}(x, \xi) = (e^{tv}(x), ((T_x e^{tv})^*)^{-1}(\xi)) \tag{2}$$

In particular, we have

$$\pi \circ e^{tH_{\mu_v}} = e^{tv} \circ \pi, \tag{3}$$

where $\pi \colon T^*X \to X$ is the projection.

We proceed to prove it in the following way:

For each $t \in \mathbb{R}$, we have a diffeomorphism $e^{tv} \colon X \to X$ (defined on an open subset of X). According to exercise 2.1, we have an induced diffeomorphism

$$\begin{aligned} \Phi^t \colon T^*X &\longrightarrow T^*X \\ (x, \xi) &\longmapsto (e^{tv}(x), ((T_x e^{tv})^*)^{-1}(\xi)), \end{aligned}$$

i.e. it is given by the formula in equation 2 (notice also that this formula makes sense for all $(t, x, \xi) \in \mathbb{R} \times T^*M$ for which $e^{tv}(x)$ is defined), and such that

$$\Phi^{t*}\tau = \tau$$

Out of Φ^t one can define the following vector field in T^*X:

$$w(x, \xi) := \frac{d}{dt}\Phi^t(x, \xi)_{|t=0} \tag{4}$$

i. *Show that Φ^t coincides with the flow of the vector field $w \in \mathfrak{X}(T^*X)$ in equation 4. In other words, you must show that for all $(x, \xi) \in T^*X$, $s \mapsto \Phi^s(x, \xi)$ is an integral curve for w, which is equivalent to showing*

$$\frac{d}{ds}\Phi^s(x, \xi) = w(\Phi^s(x, \xi)),$$

and by equation 4

$$\frac{d}{ds}\Phi^s(x, \xi) = \frac{d}{dt}\Phi^t(\Phi^s(x, \xi))_{|t=0} \tag{5}$$

This last equation is equivalent to

$$\Phi^t \circ \Phi^s = \Phi^{t+s}$$

So we conclude that e^{tw} preserves τ (and hence also $d\tau$, the symplectic form). Equivalently,

$$\mathcal{L}_w \tau = 0 \tag{6}$$

ii. *Show that w is a Hamiltonian vector field.*

Hint: Rewrite equation 6 using Cartan's homotopy formula.

iii. *Show that the Hamiltonian function of w coming from Cartan's formula is the momentum function $\mu_v \colon T^*X \to \mathbb{R}$.*

We will study conditions under which the **Legendre transform** is a global diffeomorphism from TX to T^*X.

Exercise 9. *Let E be a vector space and $F\colon E \to \mathbb{R}$ a smooth function. The Legendre transform is by definition*

$$\begin{aligned} L_F\colon E &\longrightarrow E^* \\ p &\longmapsto dF_p \end{aligned}$$

The function F is said to be **strictly convex** *if for every $p, v \in E$ the real function*

$$t \mapsto F(p + tv)$$

is strictly convex (i.e. its second derivative is strictly positive everywhere).

The **Hessian** *of F is the quadratic form*

$$d^2F_p : v \mapsto \frac{d^2}{dt^2}F(p + tv)$$

Recall that F is strictly convex if and only if its Hessian is positive definite for all points p in E (i.e. if and only if the **Legendre condition** *holds).*

i. *Show that if F is strictly convex then the Legendre transform is a local diffeomorphism.*
ii. *Show that the following four conditions are equivalent:*
   (a) *$dF_{p_0} = 0$ for some $p_0 \in E$.*
   (b) *F has a local minimum at some point p_0.*
   (c) *F has a unique (global) minimum at some p_0.*
   (d) *$lim_{|p|\to\infty} F(p) = +\infty$.*

A strictly convex function is called **stable** *if either of the previous conditions holds.*

iii. *Find an example of an strictly convex function $F\colon \mathbb{R} \to \mathbb{R}$ which is not stable.*

Given F strictly convex we denote by S_F the set $l \in E^*$ for which the function

$$\begin{aligned} F_l\colon E &\longrightarrow \mathbb{R} \\ p &\longmapsto F(p) - l(p) \end{aligned}$$

is stable.

iv. *Show that the subset $S_F \subset E^*$ is open and convex.*
v. *Show that L_F maps diffeomorphically onto S_F.*

Let F be strictly convex. It is said to have **quadratic growth at infinity** *if there exists a positive definite quadratic form Q and a constant K such that*

$$F(p) \geq Q(p) - K$$

vi. *Show that if F has quadratic growth at infinity then $S_F = E^*$, and hence the Legendre transform is an isomorphism onto E^*.*

When we have $L\colon TX \to \mathbb{R}$ such that L restricted to each fiber has quadratic growth, then exercise 9 implies that the Legendre transform

$$\begin{aligned} TX &\longrightarrow T^*X \\ (x, v) &\longmapsto (x, \frac{\partial L}{\partial v}) \end{aligned}$$

is a diffeomorphism onto its image.

Exercise 10. *In $\mathbb{R}^2$ with coordinates x, y consider*

(1) *the canonical symplectic structure $\sigma_0 = dx \wedge dy$, and*
(2) *σ another symplectic form.*

i. *Show that there exists a diffeomorphism* $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *onto its image such that* $\phi^*\sigma_0 = \sigma$.

Hint: Define ϕ *by rescaling the vertical lines, say.*

ii. *Find a sufficient condition on* σ *(or rather in* h *if* $\sigma = h\sigma_0$*) such that* $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is onto. In other words, that we have a* **global** *Darboux theorem.*

Exercise 11.

i. *Let* $(D^2\backslash\{0\}, \sigma_0)$ *be the punctured open unit disk in* $\mathbb{R}^2$ *equipped with the restriction the canonical symplectic structure. Find a diffeomorphism*

$$\phi\colon D^2\backslash\{0\} \to D^2\backslash\{0\}$$

such that

(a) ϕ *sends circles with center the origin to circles with center the origin.*

(b) ϕ *sends circles approaching the outer boundary component to circles approaching the puncture.*

(c) $\phi^*\sigma_0 = \sigma_0$.

Hint: Use polar coordinates and the previous exercise.

ii. *Let* $B^{2n}\backslash\{0\}$ *be the punctured unit ball in* $\mathbb{R}^{2n}$. *Let* ϕ *be any self-diffeomorphism of* $B^{2n}\backslash\{0\}$ *which sends points approaching the outer boundary component to points approaching the puncture. Then the manifold*

$$B^{2n} \coprod B^{2n} / \sim \phi,$$

where the equivalence relation amounts to identifying $x \in B^{2n}\backslash\{0\}$ *in the first ball with* $\phi(x)$ *in the second, is known to be homeomorphic to the sphere* S^n.

Let σ_0 *be the restriction to* $B^{2n}\backslash\{0\}$ *of the canonical symplectic form in* $\mathbb{R}^{2n}$. *Show that for* $n > 1$ *one cannot find* ϕ *as above so that* $\phi^*\sigma_0 = \sigma_0$.

Exercise 12. *Let* (M, σ), (N, σ') *be two symplectic manifolds, and* $\phi\colon M \to N$ *a diffeomorphism. Show that the following three statements are equivalent*

(1) $\phi^*\sigma' = \sigma$ *(i.e.* ϕ *is a canonical transformation).*

(2) *For all* $f \in C^\infty(N)$,

$$T\phi H_{\phi^* f} = H_f,$$

where $H_{\phi^* f}$ *is the Hamiltonian vector field of* $\phi^* f$ *w.r.t.* σ, *and* H_f *is the Hamiltonian vector field of* f *w.r.t.* σ'.

(3) *For all* $f, g \in C^\infty(N)$,

$$\phi^*\{f, g\}_{\sigma'} = \{\phi^* f, \phi^* g\}_\sigma,$$

where $\{\cdot,\cdot\}_\sigma$ *(resp.* $\{\cdot,\cdot\}_{\sigma'}$*) denote the Poisson bracket on functions induced by* σ *(resp.* σ'*). In other words,* ϕ *is a* **Poisson map**.

Exercise 13. *Let* (M, σ) *be a symplectic manifold. Suppose that we have an action of a Lie group* G *on* M *with the following properties:*

- *For all* $g \in M$, *the corresponding diffeomorphism* $g\colon M \to M$ *is such that* $g^*\sigma = \sigma$ *(i.e. the action is symplectic).*
- *The action is free, meaning that for every* $m \in M$ *and* $g \in G$ *not the identity,* $gm \neq m$.
- *The action is proper meaning that the map*

$$\begin{array}{ccc} G \times M & \to & M \times M \\ (g, m) & \longmapsto & (m, gm) \end{array}$$

is proper.

The second and third condition condition imply that M is a principal fiber bundle with group G. Let B denote its base, i.e. the space of orbits -which is a manifold- and let $p\colon M \to B$ be the projection.

At each point $m \in M$ we have $T^v_m M$ the **vertical tangent bundle**. *It is by definition the kernel of $T_m p$. It is easy to see that this is also the subspace of $T_p M$ generated by the* **symplectic vector fields** X_M, $X \in \mathfrak{g}$.

i. *Show that the symplectic annihilator of the subbundle $T^v M$ defines a foliation $\mathcal{F}'$ on M.*

Let us assume that the orbits of the action of G are coisotropic.

ii. *Show that the foliation $\mathcal{F}'$ descends to a foliation $\mathcal{F}$ in B (i.e. the pullback of $\mathcal{F}$ by the submersion $p\colon M \to B$ is $\mathcal{F}'$).*
iii. *Show that there is a (unique) Poisson structure Λ on B for which the submersion $p\colon M \to B$ is a Poisson map (Hint: there is a unique possibility if p is to be a Poisson map).*
iv. *Show that the symplectic leaves of Λ are the leaves of $\mathcal{F}$.*

Let us assume that we have a free symplectic action of the Lie group $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ on (M, σ).

v. *Find $\alpha' \in \Omega^2(M)$, $\alpha \in \Omega^2(B)$ such that*
   - $ker\alpha' = \mathcal{F}'$, $ker\alpha = \mathcal{F}$.
   - $d\alpha' = 0$, $d\alpha = 0$.

   Hint: Use X_M.
vi. *Suppose further that the symplectic form σ is integral. Then show that both $\mathcal{F}'$ and $\mathcal{F}$ are the fibers of surjective submersions $B \stackrel{[\mathcal{P}]}{\to} S^1$, $M \stackrel{p}{\to} B \stackrel{[\mathcal{P}]}{\to} S^1$. To do that consider the* **period map** *defined as follows: fix $b_0 \in B$ and consider the manifold $\mathcal{P}(B, b_0)$ of homotopy classes of paths starting at b_0. Then*

$$\begin{array}{rcl} t\colon \mathcal{P}(B,b_0) & \longrightarrow & B \\ \gamma/\sim & \longmapsto & \gamma(1) \end{array}$$

Then we have the following map

$$\begin{array}{rcl} \mathcal{P}\colon \mathcal{P}(B,b_0) & \longrightarrow & \mathbb{R} \\ \gamma/\sim & \longmapsto & \displaystyle\int_\gamma \alpha \end{array}$$

One has to show that it is a well defined map (and it is a surjective submersion).

One cannot make it descend to a map defined on B, but since if γ, γ' are paths from b_0 to b in different homotopy classes, we have

$$\mathcal{P}(\gamma) - \mathcal{P}(\gamma') \in \frac{1}{2\pi}\mathbb{Z},$$

then it induces a surjective submersion

$$[\mathcal{P}]\colon B \longrightarrow \mathbb{R}/(\mathbb{Z}/2\pi),$$

whose fibers are the leaves of $\mathcal{F}$ (actually, when B is compact, B with its Poisson structure is what is called a **symplectic mapping torus***).*

Exercise 14.

i. *Let (M, σ) be a symplectic manifold, and let G be a Lie group acting on M in a Hamiltonian fashion on M*

$$G \times M \to M$$

Let $\mu\colon M \to \mathfrak{g}^*$ *be the momentum map. For each* H *a closed subgroup of* G *(with the inclusion denoted by* $i\colon H \hookrightarrow G$*), consider the restriction of the action to* H*, i.e.*

$$H \times M \stackrel{i\times Id}{\longrightarrow} G \times M \to M$$

Show that this is also a Hamiltonian action (observe that there is a natural candidate for the momentum map, which is the composition of μ *with the obvious map from* $\mathfrak{g}^*$ *to* $\mathfrak{h}^*$*, where* $\mathfrak{h}$ *is the Lie algebra of* H*).*

ii. *Let* G_j*,* $j = 1, \ldots, s$ *be Lie groups acting in a Hamiltonian fashion on* (M_j, σ_j)*,* $j = 1, \ldots, s$*, with momentum maps* $\mu_j\colon M_j \to \mathfrak{g}_j$*. Show that the product action of* $G_1 \times \cdots \times G_s$ *on* $(M_1 \times \cdots \times M_s, pr_1^*\sigma_1 + \cdots + pr_s^*\sigma_s)$ *(*pr_j *the projection onto the* j*-th factor) is Hamiltonian, with momentum map*

$$\mu_1 \times \cdots \times \mu_s\colon M_1 \times \cdots \times M_s \to \mathfrak{g}_1^* \times \cdots \times \mathfrak{g}_s^*$$

iii. *Let* G *be a Lie group acting in a Hamiltonian fashion on manifolds* (M_j, σ_j)*,* $j = 1, \ldots, s$*, with momentum maps* $\mu_j\colon M_j \to \mathfrak{g}$*. Show that the diagonal action of* G *on* $M_1 \times \cdots \times M_s$ *is hamiltonian with moment map* $\mu_1 + \cdots + \mu_s$*.*

Exercise 15. *Let* $T^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n | |t_j| = 1\}$ *be the* n*-torus acting on* $\mathbb{C}^n$ *via the formula*

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1^{k_1} z_1, \ldots, t_n^{k_n} z_n),$$

whre $k_1, \ldots, k_n$ *are fixed integers. Show that the action is Hamiltonian w.r.t. the standard symplectic form (the imaginary part of the standard hermitian inner product), with momentum map*

$$\mu(z_1, \ldots, z_n) = -\frac{1}{2}(k_1|z_1|^2, \ldots, k_n|z_n|^2) + constant$$