EXERCISES IV

Exercise 4.3. *Let X be a smooth n-dimensional manifold and $f\colon X \to \mathbb{R}$ a smooth function. Let $S \subset X$ be an (n-1)-dimensional submanifold of X, and $\psi\colon S \to \mathbb{R}$ a smooth function. Let $x_0 \in S$ and $\xi_0 \in T^*_{x_0}X$ such that the following conditions hold:*

a. $\xi_{0|T_{x_0}S} = d\psi(x_0)$.
b. $f(x_0, \xi_0) = 0$.
c. $\frac{\partial f}{\partial \xi}(x_0, \xi)_{|\xi=\xi_0} \notin T_{x_0}S$.

Then Hamilton-Jacobi theory implies that the initial value problem

$$f(x, \phi(x)) = 0,\ \phi_{|S} = \psi \tag{1}$$

has a unique solution in a small enough neighborhood of x_0 in X. The aim of this exercise is giving a proof of part of this fact using local coordinates.

i. *Fix any coordinates $x_1, \dots, x_n$ about x_0 so that*

$$S = \{x \in \mathbb{R}^n \mid x_n = 0\},$$

and complete them with dual coordinates to obtain $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ coordinates on T^*X. Show that condition c above (the "transversality condition") is equivalent to

$$\frac{\partial f}{\partial \xi_n}(0, \xi_0) \neq 0$$

ii. *Show that there exists*

$$\xi_n = \xi_n(x_1, \dots, x_{n-1})$$

smooth and defined in U a neighborhood of zero in $\mathbb{R}^{n-1}$, uniquely determined by the conditions

$$f(x_1, \cdots, x_{n-1}, 0, \frac{\partial \psi}{\partial x_1}, \dots, \frac{\partial \psi}{\partial x_{n-1}}, \xi_n) = 0\ ,\ \xi_n(0) = \xi_{0,n},$$

where $\xi_{0,n}$ denotes the n-th coordinate of ξ_0.

iii. *Show that*

$$I_S := \{(x_1, \dots, x_{n-1}, 0, \frac{\partial \psi}{\partial x_1}, \dots, \frac{\partial \psi}{\partial x_{n-1}}, \xi_n) \mid (x_1, \dots, x_{n-1}) \in U\} \subset (T^*\mathbb{R}^n, d\tau)$$

is an isotropic submanifold.

Exercise 4.4. *Let $x_1, \dots, x_n$ be fixed coordinates in $\mathbb{R}^n$, and consider $T^*\mathbb{R}^n$ equipped with the canonical symplectic structure $d\tau$. Let f be the Hamiltonian*

$$f(x, \xi) = -\frac{1}{2}\sum_{j=1}^{n}(x_j^2 + \xi_j^2),$$

where $\xi_1, \dots, \xi_n$ are coordinates dual to $x_1, \dots, x_n$.

Any $c < 0$ is a regular value for f, and the corresponding level hypersurface $f^{-1}(c)$ is nothing but the sphere of radius $\sqrt{-2c} \subset \mathbb{R}^{2n} = T^*\mathbb{R}^n$.

Let Z be the vector subspace of $T^*\mathbb{R}^n$ defined by the equations $\xi_j = 0,\ j = 1, \dots, n$ (the zero section of $T^*\mathbb{R}^n$). Fix any $c < 0$.

i. *Show that $I := Z \cap f^{-1}(c)$ is an isotropic submanifold of $(T^*\mathbb{R}^n, d\tau)$.*

ii. *Show that I is tranversal to the flow lines of the Hamiltonian vector field H_f (exercises 3.14 and 3.15 may be helpful at this point).*

Let $L(I)$ be the germ of of Lagrange submanifold defined by I and the flow of H_f for small time.

iii. *Show that for any point $x \in I \subset \mathbb{R}^n$, there is no local solution $\phi\colon U_x \to \mathbb{R}$ about x of the Hamilton-Jacobi equation*

$$f(x, \phi(x)) - c = 0, \tag{2}$$

subject to the additional condition $(x, d\phi(x)) \subset L(I)$.

iv. *Show that the germ of Lagrange submanifold extends to a connected closed submanifold $L \subset f^{-1}(c)$. In order to do that, check that the quotient of $f^{-1}(c)$ by the orbits of the Hamiltonian vector field H_f is a manifold (again exercises 3.14 and 3.15 are helpful to identify the orbits of the Hamiltonian flow). The reduced space is a familiar symplectic manifold (M, σ). Let $p\colon f^{-1}(c) \to M$ be the projection, which is a submersion with connected fibers. Then prove that $p(I_c) \subset M$ is a submanifold. General theory of reduction tells us that $p(I_c) \subset (M, \sigma)$ is a Lagrange submanifold, and so $p^{-1}(p(I_c)) \subset (T^*\mathbb{R}^n, d\tau)$ is, and this is our closed Lagrange submanifold L.*

Recall that for open subset $U \subset \mathbb{R}^n$ which is diffeomorphic to a ball, the Poincarè's lemma asserts that its De Rham cohomology is trivial. In other words, for any $\alpha \in \Omega^d(U)$ such that $d\alpha = 0$, there exists $\beta \in \Omega^{d-1}(U)$ so that $d\beta = \alpha$.

v. *Show that for any x_0 with $|x_0|^2 < 2c$, $x_0 \neq 0$, the Hamilton-Jacobi equation 2 subject to the condition $(x, d\phi(x)) \subset L$ has solution in a suitable neighborhood of x_0, and that if the aforementioned neighborhood is diffeomorphic to a ball and does not contain the origin, the solution is unique up to an additive constant.*

vi. *Describe the ray bundle associated to L (the projection to $\mathbb{R}^n$ of the trajectories of H_f contained in L).*