LECTURE NOTES

1. Invertible sheaves and divisors

1.1. **Invertible sheaves and the Picard group.** Let Y be an algebraic variety.

Definition 1.1. A coherent sheaf $\mathscr{F}$ on Y is locally free, if there is an open cover $Y = \bigcup_{i \in I} U_i$ such that $\mathscr{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$. The positive integer r is called the *rank* of $\mathscr{F}$. A *section* s of $\mathscr{F}$ means a global section $s \in \Gamma(Y, \mathscr{F})$. A *rational section* of $\mathscr{F}$ is a section over an nonempty open subset.

An *invertible sheaf* (or *line bundle*) $\mathscr{L}$ is a locally free coherent sheaf of rank one.

Definition 1.2. The *Picard group* $\mathrm{Pic}(Y)$ is the isomorphism classes of invertible sheaves on Y with an abelian group structure given by the tensor product. Namely

$$\mathrm{Pic}(Y) \times \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(Y), \quad ([\mathscr{L}_1], [\mathscr{L}_2]) \mapsto [\mathscr{L}_1 \otimes \mathscr{L}_2].$$

Local description of an invertible sheaf using cocycles. Let $\mathscr{L}$ be an invertible sheaf on Y, and let $Y = \bigcup_{i \in I} U_i$ be an open cover such that

$$\mathscr{L}|_{U_i} = \mathcal{O}_{U_i} s_i,$$

where $s_i \in \Gamma(U_i, \mathcal{O}_Y)$ is a nowhere vanishing section. Then on $U_{ij} = U_i \cap U_j$, we have

$$s_i|_{U_{ij}} = g_{ij}\, s_j|_{U_{ij}}$$

for some $g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_Y^*)$. The collection $\{g_{ij}\}$ satisfies the following *cocycle condition*:

- $g_{ii} = 1$;
- $g_{ij} g_{jk} g_{ki} = 1$.

The construction of $\{g_{ij}\}$ depends on two choices: (1) The local generators s_i and (2) the open covering $\{U_i : i \in I\}$. Now we proceed to get rid of these choices. Let $s_i' \in \Gamma(U_i, \mathscr{L})$ be another set of local generators. Then we have

$$s_i' = h_i s_i, \quad h_i \in \Gamma(U_i, \mathcal{O}_Y^*).$$

Hence we have

$$s_i'|_{U_{ij}} = h_i s_i|_{U_{ij}} = h_i g_{ij} s_j|_{U_{ij}} = h_i g_{ij} h_j^{-1} s_j'|_{U_{ij}}.$$

It follows that

$$g_{ij}' = h_i g_{ij} h_j^{-1}.$$

Definition 1.3. Let $\mathfrak{U} = \{U_i : i \in I\}$ be an open cover of Y. we define

$$\check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{O}_Y^*) = \frac{\{g_{ij} : \text{ satisfying the cocycle condition}\}}{\{h_i h_j^{-1} : h_i \in \Gamma(U_i, \mathcal{O}_Y^*)\}}.$$

To get rid of the choice of the open cover $\mathfrak{U}$, we make the following definition.

Definition 1.4. Let $\mathfrak{V} = \{V_{i'} : i' \in I'\}$ be another open cover of Y. We say that $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$ if there exists a map $\lambda : I' \to I$ such that $V_{i'} \subseteq U_{\lambda(i')}$ for all $i' \in I'$.

There is an induced homomorphism

$$\rho_{\mathfrak{U},\mathfrak{V}} : \check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{O}_Y^*) \longrightarrow \check{\mathrm{H}}^1(\mathfrak{V}, \mathcal{O}_Y^*)$$

by the formula

$$\rho(\{g_{\cdot,\cdot}\})_{i'j'} = g_{\lambda(i'),\lambda(j')}, \quad i'j' \in I'.$$

Definition 1.5.

$$\check{\mathrm{H}}^1(Y, \mathcal{O}_Y^*) = \varinjlim_{\mathfrak{U}} \mathrm{H}^1(\mathfrak{U}, \mathcal{O}_Y^*).$$

The above discussion actually establishes the following.

Theorem 1.6. *There is a natural isomorphism*

$$\mathrm{Pic}(Y) = \check{\mathrm{H}}^1(Y, \mathcal{O}_Y^*).$$

1.2. **Construction of a line bundle from a divisor.** Let Y be an algebraic variety and let $y \in Y$ be a point on Y. Associated to y, we have the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}_Y$. Note that $\mathfrak{m}_y$ is an sheaf locally given by all functions f which vanish at y.

Definition 1.7. The point y is a *smooth point* of Y if

$$\dim \mathfrak{m}_y/\mathfrak{m}_y^2 = \dim Y.$$

Y is said to be smooth if all points of Y are smooth.

Let Y be a smooth algebraic variety.

Definition 1.8. A *prime divisor* on Y is a closed subvariety $D \subseteq Y$ of codimension one. A *divisor* on Y is a linear combination

$$D = \sum_{i=1}^{n} a_i D_i$$

of prime divisors D_i. A divisor D is *effective*, and we write $D \geq 0$, if $a_i \geq 0$ for all $1 \leq i \leq n$.

Important fact: On a smooth variety, every prime divisor is locally defined by one equation. In other words, its sheaf of ideals is locally generated by one element.

Let $f \in \mathbb{C}(Y)$ be a rational function, then we can define

$$\mathrm{div}(f) := (f = 0) - (f = \infty)$$

to be the associated divisor.

Let s be a rational section of an invertible sheaf $\mathscr{L}$. We can find $Y = \bigcup_{i \in I} U_i$ such that $\mathscr{L}_{U_i} = \mathcal{O}_{U_i} s_i$, where $s_i \in \Gamma(U_i, \mathscr{L})$ is a nonwhere vanishing section. Then

$$s|_{U_i} = f_i s_i, \quad f_i \in \mathbb{C}(Y).$$

We define $D_{U_i} = (f_i = 0)$. Note that on $U_i \cap U_j$, the rational functions f_i and f_j differ by an invertible function $g_{ij} \in \mathcal{O}_Y^*$. Thus

$$D_{U_i}|_{U_{ij}} = D_{U_j}|_{U_{ij}}$$

and there exists a unique divisor D on Y such that $D|_{U_i} = D_{U_i}$. We define

$$\mathrm{div}(s) = D.$$

Proposition 1.9. *Let s be a rational section of an invertible sheaf $\mathscr{L}$. Then s is a section if and only if* $\mathrm{div}(s)$ *is effective.*

Let $D \subset Y$ be a prime divisor. Let $Y = \bigcup U_i$ be an affine open cover such that

$$D|_{U_i} = \mathrm{div}(t_i), \quad t_i \in \Gamma(U_i, \mathcal{O}_Y).$$

We define $\mathscr{L}_i$ to be the trivial line bundle on U_i with the formal generator $s_i = \frac{1}{t_i}$. Then we glue these local line bundles together by

$$s_i|_{U_{ij}} = g_{ij} s_j|_{U_{ij}}, \quad g_{ij} = \frac{t_j}{t_i} \in \Gamma(U_{ij}, \mathcal{O}_Y^*).$$

In this way we get a line bundle $\mathscr{L}$ on Y. Note that the local sections $\sigma_i = t_i s_i \in \Gamma(U_i, \mathscr{L})$ satisfies $\sigma_i|_{U_{ij}} = \sigma_j|_{U_{ij}}$. Hence they glue together to give a global section $\sigma \in \Gamma(Y, \mathscr{L})$. We also note that

$$\mathrm{div}(\sigma)|_{U_i} = \mathrm{div}(\sigma_i) = \mathrm{div}(t_i s_i) = \mathrm{div}(t_i) = D|_{U_i}.$$

This implies that $\mathrm{div}(\sigma) = D$.

Theorem 1.10. *Let Y be a smooth variety and let $D = \sum_{i=1}^{n} a_i D_i$ be a divisor. Then there exist a line bundle $\mathscr{L}$ together with a rational section $\sigma \in \Gamma(Y, \mathscr{L})$ such that* $\mathrm{div}(\sigma) = D$.

Proof. The above construction associates a line bundle $\mathscr{L}_i$ and a section σ_i to each D_i. Let

$$\mathscr{L} := \mathscr{L}_1^{\otimes a_1} \otimes \mathscr{L}_2^{\otimes a_2} \otimes \cdots \otimes \mathscr{L}_n^{\otimes a_n}.$$

We see that $\mathscr{L}$ admits a rational section

$$\sigma = \sigma_1^{a_1} \cdot \sigma_2^{a_2} \cdot \cdots \cdot \sigma_n^{a_n}$$

such that $\mathrm{div}(\sigma) = D$. □