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Explicit normal form coefficients for all codim 2 singularities of equilibria

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Content:

1. Introduction
2. List of codim 2 normal forms
3. Normalization technique
4. Cusp
5. Bogdanov-Takens
6. Generalized Hopf (Bautin)
7. Fold-Hopf
8. Hopf-Hopf
9. Numerical implementation

1. Introduction

Consider a system of ODEs

$$\dot{x} = f(x, \alpha), \quad x \in \mathbf{R}^n, \ \alpha \in \mathbf{R}^m,$$

where f is smooth, and suppose it has at $\alpha = 0$ a **nonhyperbolic equilibrium** $x = 0$ with n_0 eigenvalues of $A = f_x(0,0)$ with Re $\lambda = 0$.

Write

$$\dot{x} = f(x, 0) = Ax + R(x), \quad x \in \mathbf{R}^n,$$

with

$$\begin{aligned} R(x) &= \frac{1}{2}B(x,x) + \frac{1}{3!}C(x,x,x) + \frac{1}{4!}D(x,x,x,x) \\ &+ \frac{1}{5!}E(x,x,x,x,x) + O(\|x\|^6). \end{aligned}$$

There exists an n_0-dimensional invariant **center manifold** tangent at $x = 0$ to the (generalized) critical eigenspace of A.

Problem: Compute normalized equations on CM up to a certain order.

Smooth normal forms on CM for codim 1 cases:

- **Fold:** $\lambda_1 = 0$

$$\dot{w} = aw^2 + O(w^3),\ w \in \mathbf{R}^1.$$

Let $Aq = 0,\ A^T p = 0,\ \langle p, q\rangle = \langle q, q\rangle = 1.$

$$a = \frac{1}{2}\langle p, B(q, q)\rangle$$

- **Andronov-Hopf:** $\lambda_{1,2} = \pm i\omega_0,\ \omega_0 > 0$

$$\dot{w} = i\omega_0 w + \frac{1}{2}G_{21} w|w|^2 + O(|w|^4),\ w \in \mathbf{C}^1,$$

where

$$l_1 = \frac{1}{2\omega_0}\text{Re } G_{21}$$

is the **first Lyapunov coefficient**. Let

$$Aq = i\omega_0 q,\ \ A^T p = -i\omega_0 p,\ \ \langle p, q\rangle = \langle q, q\rangle = 1.$$

$$\begin{aligned} G_{21} &= \langle p, C(q, q, \bar{q})\rangle - 2\langle p, B(q, A^{-1}B(q, \bar{q}))\rangle \\ &\quad + \langle p, B(\bar{q}, (2i\omega_0 I_n - A)^{-1}B(q, q))\end{aligned}$$

[Kopell & Howard, 1976; van Gils, 1982; Kuznetsov, 1995].

For $p, q \in \mathbf{C}^n$, we use

$$\langle p, q\rangle := \bar{p}^T q, \quad \|q\| := \sqrt{\langle q, q\rangle}.$$

2. List of codim 2 normal forms

Smooth normal forms for five generic codim 2 equilibrium singularities:

- **Cusp:** $\lambda_1 = 0, \ a = 0$

$$\dot{w} = cw^3 + O(w^4), \quad w \in \mathbf{R}^1.$$

- **Bogdanov-Takens:** $\lambda_{1,2} = 0$

$$\begin{cases} \dot{w}_0 & = & w_1, \\ \dot{w}_1 & = & aw_0^2 + bw_0w_1 + O(\|w\|^3), \end{cases}$$

where $w = (w_0, w_1) \in \mathbf{R}^2$.

- **Generalized Hopf:** $\lambda_{1,2} = \pm i\omega_0, \ l_1 = 0$

$$\dot{w} = i\omega_0 w + \frac{1}{2}G_{21}w|w|^2 + \frac{1}{12}G_{32}w|w|^4 + O(|w|^6),$$

where $w \in \mathbf{C}^1$ and

$$l_2 = \frac{1}{12\omega_0}\text{Re } G_{32}$$

is the **second Lyapunov coefficient**.

- **Fold-Hopf:** $\lambda_1 = 0,\ \lambda_{2,3} = \pm i\omega_0$

$$
\begin{cases}
\dot{w}_0 & = & \frac{1}{2}G_{200}w_0^2 + G_{011}|w_1|^2 \\
 & + & \frac{1}{6}G_{300}w_0^3 + G_{111}w_0|w_1|^2 \\
 & + & O(\|(w_0, w_1, \bar{w}_1)\|^4), \\
\dot{w}_1 & = & i\omega_0 w_1 + G_{110}w_0 w_1 \\
 & + & \frac{1}{2}G_{210}w_0^2 w_1 + \frac{1}{2}G_{021}w_1|w_1|^2 \\
 & + & O(\|(w_0, w_1, \bar{w}_1)\|^4),
\end{cases}
$$

where $w_0 \in \mathbf{R}^1$ and $w_1 \in \mathbf{C}^1$.

- **Hopf-Hopf:** $\lambda_{1,2} = \pm i\omega_1,\ \lambda_{3,4} = \pm i\omega_2$

$$k\omega_1 \neq j\omega_2,\ k,j > 0,\ k+j \leq 5$$

$$\begin{cases} \dot{w}_1 &= i\omega_1 w_1 + \frac{1}{2}G_{2100}w_1|w_1|^2 + G_{1011}w_1|w_2|^2 \\ &+ \frac{1}{12}G_{3200}w_1|w_1|^4 + \frac{1}{2}G_{2111}w_1|w_1|^2|w_2|^2 \\ &+ \frac{1}{4}G_{1022}w_1|w_2|^4 \\ &+ O(\|(w_1,\overline{w}_1,w_2,\overline{w}_2)\|^6), \\ \dot{w}_2 &= i\omega_2 w_2 + G_{1110}w_2|w_1|^2 + \frac{1}{2}G_{0021}w_2|w_2|^2 \\ &+ \frac{1}{4}G_{2210}w_2|w_1|^4 + \frac{1}{2}G_{1121}w_2|w_1|^2|w_2|^2 \\ &+ \frac{1}{12}G_{0032}w_2|w_2|^4 \\ &+ O(\|(w_1,\overline{w}_1,w_2,\overline{w}_2)\|^6), \end{cases}$$

where $(w_1, w_2) \in \mathbf{C}^2$.

3. Normalization technique

[Coullet & Spiegel, 1983; Kurakin & Judovich, 1986]

Write the system at $\alpha = 0$ as

$$\dot{x} = F(x), \quad x \in \mathbf{R}^n, \tag{1}$$

and restrict it to its n_0-dimensional CM:

$$x = H(w), \quad H : \mathbf{R}^{n_0} \to \mathbf{R}^n, \tag{2}$$

Then the restricted equation

$$\dot{w} = G(w), \quad G : \mathbf{R}^{n_0} \to \mathbf{R}^{n_0}. \tag{3}$$

Substitution of (2) and (3) into (1) gives the following **homological equation**:

$$H_w(w)G(w) = F(H(w)). \tag{4}$$

Now expand the functions G, H into multivariant Taylor series,

$$G(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} g_\nu w^\nu, \quad H(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} h_\nu w^\nu,$$

and assume that the restricted equation (3) is put into the **normal form** up to a certain order.

Collecting the coefficients of the w^ν-terms in (4) gives a linear system for h_ν

$$Lh_\nu = R_\nu. \tag{5}$$

When R_ν involves only known quantities, the linear system has a solution because either L is nonsingular, or R_ν satisfies the **Fredholm solvability condition**

$$\langle p, R_\nu \rangle = 0,$$

where p is a null-vector of the adjoint matrix $\overline{L}^T$. When R_ν depends on the unknown coefficient g_ν of the normal form, L is singular and the solvability condition gives the expression for g_ν.

If zero eigenvalue of $L(\overline{L}^T)$ is simple, the unique solution $h_\nu = L^{INV} R_\nu$ to (5) satisfying $\langle p, h_\nu \rangle = 0$ can be obtained by solving the $(n+1)$-dimensional **bordered system**:

$$\begin{pmatrix} L & q \\ \overline{p}^T & 0 \end{pmatrix} \begin{pmatrix} h_\nu \\ s \end{pmatrix} = \begin{pmatrix} R_\nu \\ 0 \end{pmatrix}$$

where

$$Lq = 0, \quad \overline{L}^T p = 0, \quad \langle p, q \rangle = 1.$$

4. Cusp

Let $q, p \in \mathbf{R}^n$ satisfy

$$Aq = 0, \quad A^T p = 0, \quad \langle p, q \rangle = 1.$$

The homological equation has the form

$$H_w \dot{w} = F(H(w)),$$

where

$$F(H) = AH + \frac{1}{2}B(H,H) + \frac{1}{6}C(H,H,H) + O(\|H\|^4),$$

the parametrization of the CM:

$$H(w) = wq + \frac{1}{2}h_2 w^2 + \frac{1}{6}h_3 w^3 + O(w^4),$$

$h_i \in \mathbf{R}^n$, and

$$\dot{w} = bw^2 + cw^3 + O(w^4)$$

with unknown b and c.
The w^2-terms give the equation for h_2:

$$Ah_2 = -B(q,q) + 2bq. \tag{6}$$

The solvability of this system implies

$$\langle p, -B(q,q) + 2bq \rangle = -\langle p, B(q,q) \rangle + 2b\langle p, q \rangle = 0$$

and allows to find b:

$$b = \frac{1}{2}\langle p, B(q,q) \rangle$$

+ +

The linear system (6) becomes

$$Ah_2 = -B(q,q) + \langle p, B(q,q)\rangle q$$

and its unique solution $h_2 = -A^{INV}[B(q,q) - \langle p, B(q,q)\rangle q]$ satisfying $\langle p, h_2\rangle = 0$ can be computed by solving the $(n+1)$-dimensional bordered system

$$\begin{pmatrix} A & q \\ p^T & 0 \end{pmatrix} \begin{pmatrix} h_2 \\ s \end{pmatrix} = \begin{pmatrix} -B(q,q) + \langle p, B(q,q)\rangle q \\ 0 \end{pmatrix}.$$

Collecting the w^3-terms yields

$$cq + bh_2 = \frac{1}{6}Ah_3 + \frac{1}{2}B(q,h_2) + \frac{1}{6}C(q,q,q)$$

or another singular system

$$Ah_3 = cq + bh_2 - \frac{1}{6}[C(q,q,q) + 3B(q,h_2)].$$

Its solvability implies

$$c\langle p,q\rangle + b\langle p,h_2\rangle - \frac{1}{6}\langle p, C(q,q,q) + 3B(q,h_2)\rangle = 0.$$

Since $\langle p, h_2\rangle = 0$,

$$c = \frac{1}{6}\langle p, C(q,q,q) + 3B(q,h_2)\rangle.$$

Since $b = 0$ at the cusp bifurcation,

$$c = \frac{1}{6}\langle p, C(q,q,q) - 3B(q, A^{INV}B(q,q))\rangle$$

5. Bogdanov-Takens

There exist $q_{0,1} \in \mathbf{R}^n$, such that

$$Aq_0 = 0, \quad Aq_1 = q_0,$$

and $p_{1,0} \in \mathbf{R}^n$ satisfying

$$A^T p_1 = 0, \quad Ap_0 = p_1.$$

One can assume

$$\langle q_0, p_0 \rangle = \langle q_1, p_1 \rangle = 1, \; \langle q_1, p_0 \rangle = \langle q_0, p_1 \rangle = 0.$$

The homological equation has the form

$$H_{w_0} \dot{w}_0 + H_{w_1} \dot{w}_1 = F(H(w_0, w_1)),$$

where

$$F(H) = AH + \frac{1}{2} B(H, H) + O(\|H\|^3),$$

$$\begin{aligned} H(w_0, w_1) &= w_0 q_0 + w_1 q_1 \\ &+ \frac{1}{2} h_{20} w_0^2 + h_{11} w_0 w_1 + \frac{1}{2} h_{02} w_1^2 \\ &+ O(\|(w_0, w_1)\|^3) \end{aligned}$$

with $h_{jk} \in \mathbf{R}^n$, and $\dot{w}_0, \dot{w}_1$ defined by the normal form with unknown a and b.

Collecting the w_0^2-terms, gives

$$Ah_{20} = 2aq_1 - B(q_0, q_0). \tag{7}$$

Its solvability condition is

$$a = \frac{1}{2}\langle p_1, B(q_0, q_0)\rangle.$$

Taking the scalar product of both sides of (7) with p_0 yields

$$\langle p_1, h_{20}\rangle = -\langle p_0, B(q_0, q_0)\rangle. \tag{8}$$

The $w_0 w_1$-terms give the linear system

$$Ah_{11} = h_{20} + bq_1 - B(q_0, q_1).$$

Its solvability implies

$$\langle p_1, h_{20}\rangle + b\langle p_1, q_1\rangle - \langle p_1, B(q_0, q_1)\rangle = 0.$$

Taking into account (8), we get

$$b = \langle p_0, B(q_0, q_0)\rangle + \langle p_1, B(q_0, q_1)\rangle$$

6. Generalized Hopf (Bautin)

Introduce two complex eigenvectors:

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p,$$

and normalize

$$\langle p, q \rangle = 1.$$

The homological equation takes the form

$$H_w \dot{w} + H_{\overline{w}} \dot{\overline{w}} = F(H(w, \overline{w})),$$

where

$$H(w,\overline{w}) = wq + \overline{w}\ \overline{q} + \sum_{1 \le j+k \le 5} \frac{1}{j!k!} h_{jk} w^j \overline{w}^k + O(|w|^6),$$

$$\begin{aligned} F(H) \;\; = \;\; & AH + \frac{1}{2}B(H,H) + \frac{1}{3!}C(H,H,H) + \\ & \frac{1}{4!}D(H,H,H,H) + \frac{1}{5!}E(H,H,H,H,H) \\ & + \; O(\|H\|^6). \end{aligned}$$

and

$$\dot{w} = i\omega_0 w + \frac{1}{2}G_{21} w|w|^2 + \frac{1}{12}G_{32} w|w|^4 + O(|w|^6).$$

Quadratic terms give

$$\begin{aligned} h_{20} &= (2i\omega_0 I_n - A)^{-1} B(q,q), \\ h_{11} &= -A^{-1} B(q,\overline{q}). \end{aligned}$$

The w^3-term leads to

$$h_{30} = (3i\omega_0 I_n - A)^{-1}[C(q,q,q) + 3B(q,h_{20})],$$

while the $w^2\overline{w}$-terms give the singular system

$$\begin{aligned} (i\omega_0 I_n - A)h_{21} &= C(q,q,\overline{q}) + B(\overline{q},h_{20}) \\ &+ 2B(q,h_{11}) - G_{21}q. \end{aligned}$$

The solvability of this system is equivalent to

$$\langle p, C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11}) - G_{21}q\rangle = 0,$$

so the cubic normal form coefficient can be expressed as

$$\begin{aligned} G_{21} &= \langle p, C(q,q,\overline{q}) + B(\overline{q},(2i\omega_0 I_n - A)^{-1}B(q,q)) \\ &- 2B(q,A^{-1}B(q,\overline{q}))\rangle, \end{aligned}$$

which gives the formula for $l_1 = \mathrm{Re}\ G_{21}$ from the Introduction.

+ +

Then,

$$
\begin{aligned}
h_{21} &= (i\omega_0 I_n - A)^{INV}[C(q,q,\overline{q}) + B(\overline{q},h_{20}) \\
&\qquad + 2B(q,h_{11}) - G_{21}q].
\end{aligned}
$$

As before, h_{21} satisfying $\langle p, h_{21}\rangle = 0$ can be found by solving the bordered system

$$
\begin{pmatrix} i\omega_0 I_n - A & q \\ \overline{p}^T & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} R_{21} \\ 0 \end{pmatrix},
$$

where

$$
R_{21} = C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11}) - G_{21}q
$$

For the fourth-order coefficients, we get

$$
\begin{aligned}
h_{40} &= (4i\omega_0 I_n - A)^{-1}[D(q,q,q,q) + 6C(q,q,h_{20}) \\
&\qquad + 4B(q,h_{30}) + 3B(h_{20},h_{20})], \\
h_{31} &= (2i\omega_0 I_n - A)^{-1}[D(q,q,q,\overline{q}) + 3C(q,q,h_{11}) \\
&\qquad + 3C(q,\overline{q},h_{20}) + 3B(h_{20},h_{11}) \\
&\qquad + B(\overline{q},h_{30}) + 3B(q,h_{21}) - 3G_{21}h_{20}], \\
h_{22} &= -A^{-1}[D(q,q,\overline{q},\overline{q}) + 4C(q,\overline{q},h_{11}) \\
&\qquad + C(\overline{q},\overline{q},h_{20}) + C(q,q,\overline{h}_{20}) \\
&\qquad + 2B(h_{11},h_{11}) + 2B(q,\overline{h}_{21}) + 2B(\overline{q},h_{21}) \\
&\qquad + B(\overline{h}_{20},h_{20}) - 2h_{11}(G_{21} + \overline{G}_{21})].
\end{aligned}
$$

At the Bautin bifurcation $l_1 = 0$, i.e.

$$G_{21} + \overline{G}_{21} = 0.$$

The solvability condition of the linear system for h_{32} gives the following expression for G_{32}:

$$\begin{aligned} G_{32} &= \langle p, E(q,q,q,\overline{q},\overline{q}) + D(q,q,q,\overline{h}_{20}) \\ &+ 3D(q,\overline{q},\overline{q},h_{20}) + 6D(q,q,\overline{q},h_{11}) \\ &+ C(\overline{q},\overline{q},h_{30}) + 3C(q,q,\overline{h}_{21}) \\ &+ 6C(q,\overline{q},h_{21}) + 3C(q,\overline{h}_{20},h_{20}) \\ &+ 6C(q,h_{11},h_{11}) + 6C(\overline{q},h_{20},h_{11}) \\ &+ 2B(\overline{q},h_{31}) + 3B(q,h_{22}) + B(\overline{h}_{20},h_{30}) \\ &+ 3B(\overline{h}_{21},h_{20}) + 6B(h_{11},h_{21})\rangle. \end{aligned}$$

Notice that h_{40} does not enter the formula.

Then

$$l_2 = \frac{1}{12\omega_0} \text{Re } G_{32}.$$

7. Fold-Hopf

Introduce

$$Aq_0 = A^T p_0 = 0, \; Aq_1 = i\omega_0 q_1, \; A^T p_1 = -i\omega_0 p_1,$$

satisfying

$$\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 1.$$

The homological equation is

$$H_{w_0}\dot{w}_0 + H_{w_1}\dot{w}_1 + H_{\overline{w}_1}\dot{\overline{w}}_1 = F(H(w_0, w_1, \overline{w}_1)),$$

where

$$F(H) = AH + \frac{1}{2}B(H,H) + \frac{1}{6}C(H,H,H) + O(\|H\|^4),$$

$$\begin{aligned} H(w_0, w_1, \overline{w}_1) &= w_0 q_0 + w_1 q_1 + \overline{w}_1 \bar{q}_1 \\ &+ \sum_{2 \le j+k+l \le 3} \frac{1}{j!k!l!} h_{jkl} w_0^j w_1^k \overline{w}_1^l \\ &+ O(\|(w_0, w_1, \overline{w}_1)\|^4), \end{aligned}$$

and $(\dot{w}_0, \dot{w}_1)$ are defined by the normal form. Then, the quadratic coefficients:

$$G_{200} = \langle p_0, B(q_0, q_0) \rangle,$$
$$G_{011} = \langle p_0, B(q_1, \bar{q}_1) \rangle,$$
$$G_{110} = \langle p_1, B(q_0, q_1) \rangle.$$

Solve for

$$
\begin{aligned}
h_{200} &= -A^{INV} \\
&\quad [B(q_0, q_0) - \langle p_0, B(q_0, q_0)\rangle q_0], \\
h_{020} &= (2i\omega_0 I_n - A)^{-1} B(q_1, q_1), \\
h_{110} &= (i\omega_0 I_n - A)^{INV} \\
&\quad [B(q_0, q_1) - \langle p_1, B(q_0, q_1)\rangle q_1], \\
h_{011} &= -A^{INV} \\
&\quad [B(q_1, \bar{q}_1) - \langle p_0, B(q_1, \bar{q}_1)\rangle q_0],
\end{aligned}
$$

then the cubic coefficients:

$$
\begin{aligned}
G_{300} &= \langle p_0, C(q_0, q_0, q_0) + 3B(q_0, h_{200})\rangle, \\
G_{111} &= \langle p_0, C(q_0, q_1, \bar{q}_1) + \ B(q_1, h_{101}) \\
&\quad + B(\bar{q}_1, h_{110}) + B(q_0, h_{011})\rangle, \\
G_{210} &= \langle p_1, C(q_0, q_0, q_1) \\
&\quad + \ 2B(q_0, h_{110}) + B(q_1, h_{200})\rangle, \\
G_{021} &= \langle p_1, C(q_1, q_1, , \bar{q}_1) \\
&\quad + \ 2B(q_1, h_{011}) + B(\bar{q}_1, h_{020})\rangle.
\end{aligned}
$$

Example: Lorenz-84 system

$$\begin{cases} \dot{x} &= -y^2 - z^2 - ax + aF, \\ \dot{y} &= xy - bxz - y + G, \\ \dot{z} &= bxy + xz - z, \end{cases}$$

where $a = \frac{1}{4}, b = 4$. At

$$F_0 = \frac{3907}{2320}, \quad G_0 = \frac{1297}{9280}\sqrt{145},$$

the system has the equilibrium

$$(x_0, y_0, z_0) = \left(\frac{9}{8},\ -\frac{1}{1160}\sqrt{145},\ \frac{9}{290}\sqrt{145}\right)$$

with the eigenvalues

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm i\omega_0, \quad \omega_0 = \frac{1}{1160}\sqrt{27561455}.$$

The following vectors in $\mathbf{C}^3$

$$\begin{aligned} q_0 &= \left(1,\ -\frac{1007}{188065}\ \sqrt{145}, -\frac{5252}{188065}\sqrt{145}\right) \\ q_1 &= \left(\frac{2}{145}\sqrt{145},\ 1,\ -\frac{1}{36} - \frac{i}{5220}\sqrt{27561455}\right) \\ p_0 &= \left(\frac{188065}{190079},\ -\frac{2594}{190079}\ \sqrt{145}, 0\right) \\ p_1 &= \left(\frac{1007}{380158}\sqrt{145} - \frac{i}{380158}\sqrt{145}\sqrt{27561455}, \right. \end{aligned}$$

$$\left. \frac{188065}{380158} + \frac{i}{380158}\sqrt{27561455}, -\frac{18\ i}{190079}\sqrt{27561455}\right)$$

satisfy

$$Aq_0 = A^T p_0 = 0, \quad Aq_1 = i\omega_0 q_1, \quad A^T p_1 = -i\omega_0 p_1,$$

with the normalization conditions

$$\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 1.$$

The described procedure gives:

$$\begin{aligned}
G_{200} &= -\frac{124102}{190079}, \\
G_{110} &= \frac{252130}{190079} + \frac{141\ i}{190079}\sqrt{145}\sqrt{190079}, \\
G_{011} &= -\frac{6915604}{1710711}, \\
G_{300} &= \frac{76808444160}{36130026241}, \\
G_{111} &= \frac{9729482240}{325170236169}, \\
G_{210} &= -\frac{36640288960}{36130026241} + \\
&\quad \frac{959598417280\ i}{6867559257863039}\sqrt{145}\sqrt{190079}, \\
G_{021} &= -\frac{4426799104}{325170236169} \\
&\quad + \frac{51158402858528\ i}{268864944945337976885}\sqrt{145}\sqrt{190079}.
\end{aligned}$$

This leads to the numerical values

$$b = -\frac{62051}{190079} = -0.3264484767\ldots,$$

$$c = -\frac{6915604}{1710711} = -4.042532023\ldots,$$

$$d = \frac{252130}{190079} + \frac{19784887\ i}{11794592029}\sqrt{27561455},$$

while

$$\theta = \frac{\mathrm{R}e\ d}{G_{200}} = -\frac{126065}{62051} = -2.031635268\ldots < 0$$

and

$$e = \frac{33652980958948512}{20391681953530129} = 1.650328847\ldots > 0$$

in the **smooth orbital normal form**

$$\begin{cases} \dot{u} &= bu^2 + c|z|^2 + O(\|(u, z, \bar{z})\|^4), \\ \dot{z} &= i\omega_0 z + duz + eu^2 z + O(\|(u, z, \bar{z})\|^4). \end{cases}$$

Thus, the case $s = \text{sign }(bc) = 1,\ \theta < 0$ occurs without time reversing (see, Kuznetsov [1995]). Therefore, a **nontrivial invariant set** bifurcates from the critical equilibrium under parameter variations.

8. Hopf-Hopf

Introduce $q_{1,2} \in \mathbf{C}^n$:

$$Aq_1 = i\omega_1 q_1, \;\; Aq_2 = i\omega_2 q_2.$$

and $p_{1,2} \in \mathbf{C}^n$

$$A^T p_1 = -i\omega_1 p_1, \;\; A^T p_2 = -i\omega_2 p_2.$$

Normalize in $\mathbf{C}^n$,

$$\langle p_1, q_1\rangle = \langle p_2, q_2\rangle = 1.$$

The homological equation is

$$H_{w_1}\dot{w}_1 + H_{\overline{w}_1}\dot{\overline{w}}_1 + H_{w_2}\dot{w}_2 + H_{\overline{w}_2}\dot{\overline{w}}_2 = F(H(w_1, \overline{w}_1, w_2, \overline{w}_2)),$$

where

$$\begin{aligned} H &= w_1 q_1 + \overline{w}_1 \overline{q}_1 + w_2 q_2 + \overline{w}_2 \overline{q}_2 \\ &+ \sum_{j+k+l+m \geq 2} \frac{1}{j!k!l!m!} h_{jklm} w_1^j \overline{w}_1^k w_2^l \overline{w}_2^m \end{aligned}$$

and $(\dot{w}_1, \dot{\overline{w}}_2)$ are specified by the normal form.

Quadratic coefficients give:

$$
\begin{aligned}
h_{1100} &= -A^{-1}B(q_1,\bar{q}_1),\\
h_{2000} &= (2i\omega_1 I_n - A)^{-1}B(q_1,q_1),\\
h_{1010} &= [i(\omega_1+\omega_2)I_n - A]^{-1}B(q_1,q_2),\\
h_{1001} &= [i(\omega_1-\omega_2)I_n - A]^{-1}B(q_1,\bar{q}_2),\\
h_{0020} &= (2i\omega_2 I_n - A)^{-1}B(q_2,q_2),\\
h_{0011} &= -A^{-1}B(q_2,\bar{q}_2).
\end{aligned}
$$

Cubic terms give:

$$
\begin{aligned}
h_{3000} &= (3i\omega_1 I_n - A)^{-1}[C(q_1,q_1,q_1)+3B(h_{2000},q_1)],\\
h_{2010} &= [i(2\omega_1+\omega_2)I_n - A]^{-1}\\
&\quad [C(q_1,q_1,q_2)+B(h_{2000},q_2)+2B(h_{1010},q_1)],\\
h_{2001} &= [i(2\omega_1-\omega_2)I_n - A]^{-1}\\
&\quad [C(q_1,q_1,\bar{q}_2)+B(h_{2000},\bar{q}_2)+2B(h_{1001},q_1)],\\
h_{1020} &= [i(\omega_1+2\omega_2)I_n - A]^{-1}\\
&\quad [C(q_1,q_2,q_2)+B(h_{0020},q_1)+2B(h_{1010},q_2)],\\
h_{1002} &= [i(\omega_1-2\omega_2)I_n - A]^{-1}\\
&\quad [C(q_1,\bar{q}_2,\bar{q}_2)+B(\bar{h}_{0020},q_1)+2B(h_{1001},\bar{q}_2)],\\
h_{0030} &= (3i\omega_2 I_n - A)^{-1}[C(q_2,q_2,q_2)+3B(h_{0020},q_2)].
\end{aligned}
$$

Collecting the coefficients of the resonant cubic terms, one obtains the cubic coefficients in the normal form

$$
\begin{aligned}
G_{2100} &= \langle p_1, C(q_1, q_1, \bar{q}_1) \\
&+ B(h_{2000}, \bar{q}_1) + 2B(h_{1100}, q_1)\rangle, \\
G_{1011} &= \langle p_1, C(q_1, q_2, \bar{q}_2) \\
&+ B(h_{1010}, \bar{q}_2) + B(h_{1001}, q_2) + B(h_{0011}, q_1)\rangle, \\
G_{1110} &= \langle p_2, C(q_1, \bar{q}_1, q_2) \\
&+ B(h_{1100}, q_2) + B(h_{1010}, \bar{q}_1) + B(\bar{h}_{1001}, q_1)\rangle, \\
G_{0021} &= \langle p_2, C(q_2, q_2, \bar{q}_2) \\
&+ B(h_{0020}, \bar{q}_2) + 2B(h_{0011}, q_2)\rangle,
\end{aligned}
$$

The 4th- and 5th-order coefficients can be found in Kuznetsov [1999] SIAM J. Numer. Anal. **36**: 1104-1124.

9. Numerical implementation

Consider the computation of l_1 for Hopf bifurcation. Using

$$r, \quad q = q_R + iq_I, \quad p = p_R + ip_I, \quad s = s_R + is_I,$$

the linear systems to solve become

$$Ar = B(q_R, q_R) + B(q_I, q_I),$$

and

$$\begin{cases} -As_R - 2\omega_0 s_I & = & B(q_R, q_R) - B(q_I, q_I) \\ 2\omega_0 s_R - As_I & = & 2B(q_R, q_I). \end{cases}$$

Then

$$\begin{aligned} \text{Re}\ \langle p, B(q, r)\rangle &= \langle p_R, B(q_R, r)\rangle + \langle p_I, B(q_I, r)\rangle, \\ \text{Re}\ \langle p, B(\bar{q}, s)\rangle &= \langle p_R, B(q_R, s_R)\rangle + \langle p_R, B(q_I, s_I)\rangle \\ &+ \langle p_I, B(q_R, s_I)\rangle - \langle p_I, B(q_I, s_R)\rangle, \end{aligned}$$

$$\begin{aligned} \text{Re}\ \langle p, C(q, q, \bar{q})\rangle = \\ &+ \frac{2}{3}\langle p_R, C(q_R, q_R, q_R)\rangle + \frac{2}{3}\langle p_I, C(q_I, q_I, q_I)\rangle \\ &+ \frac{1}{6}\langle p_R + p_I, C(q_R + q_I, q_R + q_I, q_R + q_I)\rangle \\ &+ \frac{1}{6}\langle p_R - p_I, C(q_R - q_I, q_R - q_I, q_R - q_I)\rangle \end{aligned}$$

The quadratic forms can be approximated by

$$B(q,q) \;\approx\; \frac{1}{\tau^2}\left[f(x_0+\tau q,\alpha_0)+f(x_0-\tau q,\alpha_0)\right],$$

$$\begin{aligned}B(q,r) \;\approx\; \frac{1}{\tau^2}\Big[& f(x_0+\frac{\tau}{2}(q+r),\alpha_0)\\ - \;& f(x_0+\frac{\tau}{2}(q-r),\alpha_0)\\ + \;& f(x_0-\frac{\tau}{2}(q+r),\alpha_0)\\ - \;& f(x_0-\frac{\tau}{2}(q-r),\alpha_0)\Big].\end{aligned}$$

$$\begin{aligned}\langle p, B(q,q)\rangle &= \left.\frac{d^2}{d\tau^2}\right|_{\tau=0}\langle p, f(x_0+\tau q,\alpha_0)\rangle\\ &\approx \frac{1}{\tau^2}\langle p, f(x_0+\tau q,\alpha_0)+f(x_0-\tau q,\alpha_0)\rangle.\end{aligned}$$

For the cubic form

$$\begin{aligned}C(r,r,r) &= \left.\frac{d^3}{d\tau^3}\right|_{\tau=0} f(x_0+\tau r,\alpha_0)\\ &\approx \frac{1}{\tau^3}\Big[f(x_0+\frac{3\tau}{2}r,\alpha_0)-3f(x_0+\frac{\tau}{2}r,\alpha_0)\\ &\quad + \; 3f(x_0-\frac{\tau}{2}r,\alpha_0)-f(x_0-\frac{3\tau}{2}r,\alpha_0)\Big]\end{aligned}$$