4.5 Planar Hamiltonian and related systems

There is a special class of planar systems, for which a complete characterization of phase portraits is possible and which appear in Classical Mechanics. Moreover, it is possible to draw some conclusions about generic small perturbations of such systems.

4.5.1 Hamiltonian systems with one degree of freedom

Let $H = H(q, p)$ be a smooth (at least $C^2$) function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Define

$$H_q(q, p) = \frac{\partial H(q, p)}{\partial q}, \quad H_p(q, p) = \frac{\partial H(q, p)}{\partial p}.$$

**Definition 4.21** A planar system

$$\begin{cases} \dot{q} = H_p(q, p), \\ \dot{p} = -H_q(q, p), \end{cases}$$

(4.29)

is called a **Hamiltonian system** with one degree of freedom and **Hamiltonian** (or **energy function**) $H$.

**Example 4.22** (Lotka-Volterra system revisited)

Consider once again the Lotka-Volterra system from Section 4.4.1:

$$\begin{cases} \dot{x} = ax - bxy, \\ \dot{y} = cy + dxy. \end{cases}$$

(4.30)

The linear scaling

$$x = \xi_0 \xi, \quad y = \eta_0 \eta, \quad t = \tau_0 \tau$$

with

$$\xi_0 = \frac{a}{d}, \quad \eta_0 = \frac{a}{b}, \quad \tau_0 = \frac{1}{a}$$

reduces (4.30) to

$$\begin{cases} \dot{\xi} = \xi - \xi \eta, \\ \dot{\eta} = -\gamma \eta + \xi \eta, \end{cases}$$

(4.31)

where $\dot{\xi}$ and $\dot{\eta}$ are the derivatives of $\xi$ and $\eta$ with respect to the new time $\tau$, and

$$\gamma = \frac{c}{a}$$

is a new parameter. Introduce new coordinates in the positive quadrant $\mathbb{R}^2_+$ by the formulas:

$$\begin{cases} q = \ln \xi, \\ p = \ln \eta. \end{cases}$$
One has
\[ \dot{q} = \frac{\dot{\xi}}{\xi} = 1 - \eta = 1 - e^p, \]
\[ \dot{p} = \frac{\dot{\eta}}{\eta} = -\gamma + \xi = -\gamma + e^q. \]
Thus, (4.31) is smoothly equivalent in \( \mathbb{R}^2_+ \) to the Hamiltonian system (4.29) with
\[ H(q,p) = q e^q + p e^p. \]

The transformation \((\xi, \eta) \mapsto (q, p)\) maps closed orbits of (4.31) in \( \mathbb{R}^2_+ \) onto closed orbits of (4.29) in \( \mathbb{R}^2 \), preserving their time parameterization (see Figure 4.18).

The following result explains why (4.29) is also called a conservative system (see also Exercise 4.7.12).

**Theorem 4.23 (Conservation of energy)** The Hamiltonian \( H(q, p) \) is a constant of motion for the Hamiltonian system, i.e. \( H(q(t), p(t)) = \text{const along solutions.} \)

**Proof:** \( \dot{H} = H_q \dot{q} + H_p \dot{p} = H_q H_p - H_p H_q = 0. \)

This theorem has as an immediate implication that any orbit of (4.29) belongs to a level set \( \{(q, p) \in \mathbb{R}^2 : H(q, p) = h\} \) of the Hamiltonian function \( H \). Next we can deduce further implications from the geometry of such level sets. As illustrated in the lower half of Figure 4.19, a connected component of a level set is typically either
- a single point (corresponding to a maximum or a minimum of \( H \));
- a closed curve (these occur in families parametrized by the value of \( H \); a strict maximum or minimum is always surrounded by such a family);
- a number of points connected by curves (more on this case below).

Another direct consequence of (4.29) is that equilibria of a Hamiltonian system correspond exactly to critical points of $H$. So on the one hand we can use the classification of nondegenerate critical points into extrema (maxima and minima) and saddle points, and on the other hand we can use the classification of equilibria according to their stability properties. How do these two ways of classification correspond to each other?

A first observation is that (4.29) cannot have asymptotically stable equilibria (nor asymptotically stable cycles). Indeed, if all orbits starting in a neighbourhood of an equilibrium converge towards this equilibrium, then $H$ must be constant on this neighbourhood (since $H$ is continuous and constant along orbits). So the entire neighbourhood must consist of equilibria, a contradiction.

As $H$ is assumed to be $C^2$, the Jacobian matrix at some equilibrium exists and is given by

$$A = \begin{pmatrix} H_{qp}^0 & H_{pp}^0 \\ -H_{qq}^0 & -H_{qp}^0 \end{pmatrix},$$

where the superscript indicates the evaluation of all quantities at the equilibrium. Thus $\text{Tr} A = H_{qp}^0 - H_{pp}^0 = 0$ and the eigenvalues of $A$ are either real and of opposite sign or purely imaginary, depending on the sign of

$$\det A = H_{qq}^0 H_{pp}^0 - (H_{qp}^0)^2 \neq 0.$$

If we consider the equilibrium as a critical point of $H$, we are led to consider the Hessian matrix

$$B = \begin{pmatrix} H_{qq}^0 & H_{qp}^0 \\ H_{qp}^0 & H_{pp}^0 \end{pmatrix},$$

which has exactly the same determinant as $A$. When the determinant is positive, $H$ has an extremum (a maximum if $H_{pp}^0 < 0$ and a minimum if $H_{pp}^0 > 0$). Thus, when $A$ has purely imaginary eigenvalues, the corresponding equilibrium is a (nonlinear) center surrounded by a family of periodic orbits. When the determinant is negative, $H$ has a saddle point and $A$ has one positive and one negative real eigenvalue. Thus, the equilibrium is also a saddle point in the sense of planar dynamical systems (so when locally a level set consists of two intersecting curves, these are exactly the local stable and the local unstable manifolds of the point of intersection, which is a saddle equilibrium point).

In order to formulate our conclusions as a lemma, it is useful to introduce a new term.

**Definition 4.24** An equilibrium is called **simple**, if it has no zero eigenvalue.

**Lemma 4.25** A simple equilibrium of a Hamiltonian system (4.29) with one degree of freedom is either a saddle or a center. The first case applies when

$$H_{qq}^0 H_{pp}^0 - (H_{qp}^0)^2 < 0$$

and the second when the opposite inequality holds. □
4.5. PLANAR HAMILTONIAN AND RELATED SYSTEMS

Remarks:

(1) With any smooth function \( H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) one can also associate the \textit{gradient flow} generated by the system

\[
\begin{aligned}
\dot{q} &= -H_q(q, p), \\
\dot{p} &= -H_p(q, p).
\end{aligned}
\]

(4.32)

For this flow \( \dot{H} = -[(H_q)^2 + (H_p)^2] \leq 0 \), so orbits converge towards minima of \( H \) following steepest descent and crossing level curves of \( H \) orthogonally.

(2) Returning to (4.29), we observe that one might wonder how the period varies as a function of the value of \( H \) when there is a family of closed orbits? Is the period a monotone function? This is a notoriously difficult problem. With considerable effort one can prove monotonicity for some specific systems. But as far as we know, there is no general result and, in fact, the period may not be monotone for other specific systems.

(3) Finally note that the divergence of the vector field \((H_p, -H_q)^T\) is zero. As a consequence, planar Hamiltonian flows preserve area (see Theorem 4.38 in the Appendix to this chapter for a general result).

\[\diamondsuit\]

4.5.2 Potential systems with one degree of freedom

Definition 4.26 A planar system (4.29) with

\[ H(q, p) = \frac{1}{2}p^2 + U(q), \]

where \( U \) is a smooth scalar function, is called a \textit{potential system}.

Here the term \( \frac{1}{2}p^2 \) is called the \textit{kinetic energy}, while \( U(q) \) is called the \textit{potential energy}.

The corresponding Hamiltonian system (4.29) has the form:

\[
\begin{aligned}
\dot{q} &= p, \\
\dot{p} &= -U'(q).
\end{aligned}
\]

(4.33)

Eliminating \( p \) we get Newton's 2nd Law:

\[ \ddot{q} = F(q), \]

with the \textit{potential force} \( F(q) = -U'(q) \). (Side remark for physicists: Notice that the "mass" is put equal to one, which can always be achieved by scaling.)

The phase portrait of (4.33) in the \((q, p)\)-plane is completely determined by the potential energy \( U(q) \) (see Figure 4.19). The portrait is symmetric with respect to the reflection in the \( q \)-axis and time reversal. Equilibrium points of (4.33) have the form \((q^0, 0)\), where \( q^0 \) is a critical point of \( U(q) \), i.e. \( U'(q^0) = 0 \). If \( q^0 \) is a local maximum of \( U \), the equilibrium is a saddle; if \( q^0 \) is a local minimum of \( U \), the equilibrium is a center. A connected subset of any level set \( H(q, p) = h \), where \( h \) is a
regular value, is diffeomorphic to either a circle (closed curve) or a line. The reader is invited to prove that any closed orbit defines a periodic solution with period

$$T = \int_{q_1}^{q_2} \sqrt{\frac{2}{h - U(q)}} \, dq,$$

(4.34)

where \(q_1 < q_2\) are the coordinates of the intersections of the orbit with the \(q\)-axis and, therefore, \(U(q_1) = U(q_2) = h\) (see also Exercise 4.7.11). Critical level sets contain equilibria and hom- or heteroclinic orbits, which are asymptotic to them.

**Example 4.27 (Famous potential systems with \(m = 1\))**

1. **Harmonic oscillator:**
   \[ H = \frac{1}{2}(p^2 + q^2). \]

2. **Ideal pendulum:**
   \[ H = \frac{1}{2}p^2 - \cos q. \]

3. **Duffing oscillator:**
   \[ H = \frac{1}{2}(p^2 - q^2) + \frac{q^4}{4}. \]

You are invited to draw their phase portraits yourself.
4.5.3 Small perturbations of planar Hamiltonian systems

As we have seen in the previous sections, planar Hamiltonian systems have very special phase portraits, which allow for rather detailed characterization. Can we use this information to analyse planar systems which are not Hamiltonian but close to them? One can think of a potential system subject to a small friction, or about a generalized Lotka-Volterra model that takes into account weak competition among prey or predators. As we shall see, the topology of the phase portrait of a Hamiltonian system changes qualitatively under generic (non-Hamiltonian) perturbations: Centers become (stable or unstable) foci, while families of periodic orbits disappear, possible giving rise to (stable or unstable) limit cycles. Homoclinic orbits also disappear. These qualitative changes are examples of bifurcations of dynamical systems, which we will study systematically in Chapters 5, 6, and 7.

First consider the following one-parameter perturbation of (4.29):

\[ \dot{x} = \begin{pmatrix} H_{x_2}(x) \\ -H_{x_1}(x) \end{pmatrix} + \varepsilon f(x), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \]

where \( \varepsilon \in \mathbb{R} \) is a small parameter, and \( H : \mathbb{R}^2 \to \mathbb{R} \) and \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) are smooth functions. Since we are interested in non-Hamiltonian perturbations, \( \text{div} f \) does not vanish.

We begin with a simple result concerning equilibria.

**Theorem 4.28** Consider the system (4.35) and assume that \( f(0) = 0 \).

(i) If \( x = 0 \) is a simple saddle at \( \varepsilon = 0 \), then \( x = 0 \) is a saddle of (4.35) for all \( \varepsilon \) with sufficiently small \( |\varepsilon| \).

(ii) If \( x = 0 \) is a simple center at \( \varepsilon = 0 \), then \( x = 0 \) is a focus of (4.35) for all \( \varepsilon \) with sufficiently small \( |\varepsilon| > 0 \). Moreover, this focus is stable for \( \varepsilon \) \( \text{div} f(0) < 0 \) and unstable for \( \varepsilon \) \( \text{div} f(0) > 0 \).

**Proof:** Write the Jacobian matrix of (4.35) at the equilibrium \( x = 0 \) as

\[ A(\varepsilon) = \begin{pmatrix} H_{x_2}^0(x) & H_{x_2 x_2}^0(x) \\ -H_{x_1}^0(x) & -H_{x_2 x_1}^0(x) \end{pmatrix} + \varepsilon f_x^0. \]

Its eigenvalues \( \lambda_{1,2}(\varepsilon) \) depend smoothly on \( \varepsilon \).

Part (i) is then obvious, since if \( \lambda_1(\varepsilon) \) and \( \lambda_2(\varepsilon) \) have opposite sign at \( \varepsilon = 0 \), then this property will hold for all sufficiently small \( \varepsilon \). Thus, by the Grobman-Hartman Theorem, \( x = 0 \) is a saddle for such parameter values.

When \( x = 0 \) is a center at \( \varepsilon = 0 \), the matrix \( A(\varepsilon) \) has a pair of nonreal eigenvalues \( \lambda_1 = \lambda_2(\varepsilon) \) for all sufficiently small \( \varepsilon \) and

\[ 2\text{Re} \lambda_{1,2}(\varepsilon) = \text{Tr} A(\varepsilon) = \varepsilon \text{Tr} f_x^0 = \varepsilon \text{div} f(0). \]

Applying the Grobman-Hartman Theorem for \( \varepsilon \neq 0 \), we obtain Part (ii) of the theorem. \( \square \)
Example 4.29 (Perturbed Lotka-Volterra system)

Consider the following small perturbation of the (scaled) Lotka-Volterra system (4.31):

\[
\begin{align*}
\dot{\xi} &= \xi - \xi \eta - \varepsilon \xi^2, \\
\dot{\eta} &= -\gamma \eta + \xi \eta, \\
\end{align*}
\]

where \(0 < \varepsilon \ll 1, \gamma > 0\) and the \(\varepsilon \xi^2\)-term describes weak competition among prey. Introducing the same variables as in Example 4.22,

\[
\begin{align*}
q &= \ln \xi, \\
p &= \ln \eta,
\end{align*}
\]
we obtain

\[
\begin{align*}
\dot{q} &= 1 - e^p - \varepsilon e^q, \\
\dot{p} &= -\gamma + e^q.
\end{align*}
\]

This system has for \((1 - \gamma \varepsilon) > 0\) an equilibrium

\[(q^0(\varepsilon), p^0(\varepsilon)) = (\ln \gamma, \ln(1 - \gamma \varepsilon)),\]

which is a center if \(\varepsilon = 0\). Translating the origin of the coordinate system to this equilibrium by the transformation

\[
\begin{align*}
x_1 &= q - \ln \gamma, \\
x_2 &= p - \ln(1 - \varepsilon \gamma),
\end{align*}
\]
we obtain the system

\[
\begin{align*}
\dot{x}_1 &= 1 - e^{x_2} + \varepsilon \gamma (e^{x_2} - e^{x_1}), \\
\dot{x}_2 &= -\gamma(1 - e^{x_1}),
\end{align*}
\]

which has the form (4.35) with \(H(x) = x_2 - e^{x_2} + \gamma (x_1 - e^{x_1})\) and

\[f(x) = \begin{pmatrix} \gamma (e^{x_2} - e^{x_1}) \\ 0 \end{pmatrix}, \quad f(0) = 0.\]

The system (4.37) satisfies the conditions of Theorem 4.28. Since

\[\varepsilon \text{ div } f(0) = -\varepsilon \gamma < 0,\]

the equilibrium \((q^0(\varepsilon), p^0(\varepsilon))\) is a stable focus for sufficiently small \(\varepsilon > 0\).

Note that this result perfectly agrees with our analysis in Section 4.4.3, where it was shown that system (4.13) has a unique positive globally asymptotically stable equilibrium when \(ad - ce > 0\). Indeed, system (4.13) coincides with system (4.36) if we set \(a = b = d = 1, c = \gamma, \) and \(e = \varepsilon, \) so that the above condition turns into \((1 - \varepsilon \gamma) > 0,\) which is obviously true for small \(\varepsilon.\)

Remark: Consider a slightly more general system

\[
\dot{x} = \begin{pmatrix} H_{x_2}(x) \\ -H_{x_1}(x) \end{pmatrix} + \varepsilon f(x, \varepsilon), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \varepsilon \in \mathbb{R},
\]

(4.38)
where $H$ is smooth and has a nondegenerate critical point $x = 0$, while $f$ is a smooth function of $(x, \varepsilon)$. The Implicit Function Theorem assures that (4.38) has a smooth family $x^0(\varepsilon)$ of equilibria for small $\varepsilon \neq 0$, such that $x^0(0) = 0$. Translating the origin to $x^0(\varepsilon)$, we can assume without loss of generality that $f(0, \varepsilon) = 0$ for all $\varepsilon$ with small $|\varepsilon|$. Then, the equilibrium $x = 0$ is either a saddle or, when $\text{div} f(0, 0) \neq 0$, a focus. The stability of the focus is determined by the sign of $\text{div} f(0, 0)$. \( \diamond \)

Now we want to study limit cycles of the perturbed planar Hamiltonian system (4.35).

**Theorem 4.30 (Pontryagin, 1934)** Let $L_0$ be a clockwise-oriented cycle of (4.35) for $\varepsilon = 0$ corresponding to a periodic solution $\varphi(t)$ with the (minimal) period $T_0$ and let $\Omega_0 \subset \mathbb{R}^2$ denote the domain inside the cycle $L_0$. If

$$M_0 = \int_{\Omega_0} \text{div} f(x) \, dx = 0,$$

while

$$M_1 = \int_{T_0} \text{div} f(\varphi(t)) \, dt \neq 0,$$

then

(i) there exists an annulus around $L_0$ in which the system (4.35) has, for all $\varepsilon$ with sufficiently small $|\varepsilon|$, a unique cycle $L_\varepsilon$, such that $L_\varepsilon \to L_0$ as $\varepsilon \to 0$;

(ii) this cycle $L_\varepsilon$ is stable for $\varepsilon M_1 < 0$ and unstable for $\varepsilon M_1 > 0$.

**Proof:**

(i) Consider a cycle $L_0$ for which the assumptions of the theorem are valid. Without loss of generality, suppose that $H(x) = 0$ for $x \in L_0$. Fix a segment $\Sigma$

$$\Sigma$$

$$H(x^h, \varepsilon(T^h, \varepsilon))$$

$$h$$

$L_0$

Figure 4.20: A segment $\Sigma$ orthogonal to $L_0$, together with unperturbed and perturbed orbits.

orthogonal to $L_0$ and consider the value $h$ of the Hamilton function $H$ as a local smooth coordinate along $\Sigma$ (see Figure 4.20). This is possible, since $L_0$ contains no equilibria of the unperturbed system and thus no critical points of $H$. Denote by $x^{h, \varepsilon} = x^{h, \varepsilon}(t)$ the solution of (4.35) starting at a point in $\Sigma$ with a small coordinate
where, and let $T^{h,\varepsilon}$ be the minimal time needed by the solution to return back to $\Sigma$. Due to the smooth dependence of solutions on the initial point and the parameter, $T^{h,\varepsilon}$ is a smooth function of $(h, \varepsilon)$ in a neighbourhood of $(0, 0)$ with $T^{0,0} = T_0$, since $x^{0,0} = \varphi$.

Thus we have

$$\Delta(h, \varepsilon) = H(x^{h,\varepsilon}(T^{h,\varepsilon})) - H(x^{h,\varepsilon}(0)) = H(x^{h,\varepsilon}(T^{h,\varepsilon})) - h,$$

which one calls for obvious reasons the displacement function. In general, $\Delta(h, \varepsilon) \neq 0$. However, $\Delta(h, 0) = 0$, since $H$ is constant along orbits of the unperturbed system. Moreover, if $\Delta(h, \varepsilon) = 0$, the perturbed system (4.35) has a closed orbit for the corresponding value of $\varepsilon$ passing through the point in $\Sigma$ with the coordinate $h$. We will analyse the equation $\Delta(h, \varepsilon) = 0$ with the help of the Implicit Function Theorem.

Along the solutions of (4.35), one has

$$\Delta(h, \varepsilon) = \int_0^{T^{h,\varepsilon}} dH(x^{h,\varepsilon}(t)) = \int_0^{T^{h,\varepsilon}} [H_{x_1}(x^{h,\varepsilon}(t)) \dot{x}_1^{h,\varepsilon}(t) + H_{x_2}(x^{h,\varepsilon}(t)) \dot{x}_2^{h,\varepsilon}(t)] dt$$

$$= \varepsilon \int_0^{T^{h,\varepsilon}} [H_{x_1}(x^{h,\varepsilon}(t)) f_1(x^{h,\varepsilon}(t)) + H_{x_2}(x^{h,\varepsilon}(t)) f_2(x^{h,\varepsilon}(t))] dt$$

$$= \varepsilon \int_0^{T^{h,0}} [-\dot{x}_2^{h,0}(t) f_1(x^{h,0}(t)) + \dot{x}_1^{h,0}(t) f_2(x^{h,0}(t))] dt + O(\varepsilon^2),$$

where the last integral is computed along solutions of (4.35) with $\varepsilon = 0$. Since these solutions have clockwise orientation, we can write

$$\Delta(h, \varepsilon) = \varepsilon M(h) + O(\varepsilon^2), \quad (4.39)$$

where

$$M(h) = -\iint_{L_h} f_1 dx_2 - f_2 dx_1 = \int_{\Omega_h} \text{div} f(x) \, dx.$$ 

(The last equality, in which $\Omega_h$ is the domain inside the periodic orbit $L_h$ belonging to the level curve $H(x) = h$, is Green’s formula (4.9); note the change of sign to incorporate the clockwise orientation of $L_h$.)

By assumption,

$$M(0) = \iint_{H(x) = 0} f_1 dx_2 - f_2 dx_1 = \int_{\Omega_0} \text{div} f(x) \, dx = M_0 = 0. $$

Thus we have

$$M(h) = \int_0^h \left( \int_0^{T^{h,0}} \text{div} f(x(\tau, s)) |\det(J_1(\tau, s))| \, d\tau \right) \, ds,$$

where $J_1$ is the Jacobian matrix of the map $(\tau, h) \mapsto x(\tau, h) = x^{h,0}(\tau)$. Hence

$$M'(0) = \int_0^{T^{0,0}} \text{div} f(\varphi(\tau)) \, |\det(J_1(\tau, 0))| \, d\tau.$$
Differentiating the identity

\[ H(x_{h,0}(\tau)) = h \]

with respect to \( h \), we obtain

\[
H_{x_1}(x_{h,0}(\tau)) \frac{\partial x_{1,0}^{h,0}(\tau)}{\partial h} + H_{x_2}(x_{h,0}(\tau)) \frac{\partial x_{2,0}^{h,0}(\tau)}{\partial h} = 1. 
\]

Therefore,
\[
\det(J_1(\tau, h)) = \det \begin{pmatrix} \frac{\partial x_{1,0}^{h,0}(\tau)}{\partial \tau} & \frac{\partial x_{1,0}^{h,0}(\tau)}{\partial h} \\ \frac{\partial x_{2,0}^{h,0}(\tau)}{\partial \tau} & \frac{\partial x_{2,0}^{h,0}(\tau)}{\partial h} \end{pmatrix} = \det \begin{pmatrix} H_{x_2}(x_{h,0}(\tau)) & \frac{\partial x_{1,0}^{h,0}(\tau)}{\partial h} \\ -H_{x_2}(x_{h,0}(\tau)) & \frac{\partial x_{2,0}^{h,0}(\tau)}{\partial h} \end{pmatrix} = 1.
\]

Hence \( |\det(J_1(\tau, 0))| = 1 \) and

\[ M'(0) = \int_0^{T_0} \text{div } f(\varphi(t)) \, dt = M_1 \neq 0 \]

by the second assumption.

Thus, we have

\[ \Delta(h, \varepsilon) = \varepsilon(M_0 + hM_1 + O(h^2)) + O(\varepsilon^2) = \varepsilon F(h, \varepsilon) \]

for some smooth function \( F \). If \( \varepsilon = 0 \), \( \Delta = 0 \) for all small \( |h| \) and all orbits are closed (Hamiltonian case). If \( \varepsilon \neq 0 \), the equation

\[ F(h, \varepsilon) = 0 \]

is such that we can apply the Implicit Function Theorem. Indeed,

\[ F(0, 0) = M_0 = 0, \quad F_h(0, 0) = M_1 \neq 0, \]

by the assumptions we made. Thus, there is a unique smooth function \( h = h(\varepsilon), \, h(0) = 0 \), such that

\[ F(h(\varepsilon), \varepsilon) = 0 \]

for all small \( |\varepsilon| \). This implies that \( \Delta(h(\varepsilon), \varepsilon) = 0 \) for all \( \varepsilon \) with sufficiently small \( |\varepsilon| \neq 0 \). Therefore, there exists a unique cycle \( L_\varepsilon \) through \( h(\varepsilon) \).

(ii) Since the map \( h \mapsto h + \Delta(h, \varepsilon) \) has derivative

\[ 1 + \frac{\partial \Delta(h(\varepsilon), \varepsilon)}{\partial h} \]
in the fixed point \( h(\varepsilon) \) and the sign of
\[
\frac{\partial \Delta(h(\varepsilon), \varepsilon)}{\partial h}
\]
coincides with the sign of \( \varepsilon M_1 \), the stability assertions follow at once. □

Remarks:

(1) If \( L_0 \) has counter-clockwise orientation, the expression (4.39) will take the form
\[
\Delta(h, \varepsilon) = -\varepsilon M(h) + O(\varepsilon^2).
\]
The subsequent analysis can then be carried out with obvious modifications. It shows that statement (i) is still valid, but the cycle \( L_\varepsilon \) is stable when \( \varepsilon M_1 > 0 \) and unstable when \( \varepsilon M_1 < 0 \).

(2) When \( L_0 \) is oriented counter-clockwise,
\[
\int_{L_0} f_1(x)dx_2 - f_2(x)dx_1 = \int_0^{T_0} f(\varphi(t)) \wedge \dot{\varphi}(t) dt,
\]
where the wedge product of two vectors \( u, v \in \mathbb{R}^2 \) is defined by \( u \wedge v = u_1v_2 - u_2v_1 \).

Example 4.31 (Van der Pol equation)
The second-order equation,
\[
\ddot{x} + x = \varepsilon \dot{x}(1 - x^2),
\]
is called the van der Pol equation\(^3\). It can be rewritten as the equivalent planar system
\[
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = -x_1 + \varepsilon x_2(1 - x_1^2).
\end{cases}
\]
The system with \( \varepsilon = 0 \) is Hamiltonian (harmonic oscillator) with \( H(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) \), which has a family of 2\( \pi \)-periodic solutions
\[
\varphi(t) = \begin{pmatrix} r \sin t \\ r \cos t \end{pmatrix}, \quad r > 0.
\]
Since \( f_1 = 0, f_2 = x_2(1 - x_1^2) \), and the periodic orbits are oriented clockwise, we get
\[
M(r) = -\int_{x_1^2 + x_2^2 = r^2} f_1(x)dx_2 - f_2(x)dx_1 = \int_0^{2\pi} r^2 \cos^2 t(1-r^2 \sin^2 t) dt = \frac{\pi}{4} r^2(4-r^2).
\]
Therefore, \( M(r) = 0 \) for \( r = 2 \). Along this solution,
\[
M_1 = \int_0^{2\pi} \text{div} f(\varphi(t)) dt = \int_0^{2\pi} (1 - 4 \sin^2 t) dt = -2\pi < 0.
\]
\(^3\)van der Pol, B. 'Forced oscillations in a circuit with nonlinear resistance (receptance with reactive triode)', London, Edinburgh and Dublin Phil. Mag. 3 (1927), 65-80
4.5. PLANAR HAMILTONIAN AND RELATED SYSTEMS

\[ x' = y, \quad y' = -x + \varepsilon y (1 - x^2) \]

\( \varepsilon = 0.1 \)

\[ x \quad y \quad 1 \quad 2 \quad 3 \]
\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \]

\( x \quad y \quad 1 \quad 2 \quad 3 \]
\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \]

\( \varepsilon = 1 \)

\[ x \quad y \quad 1 \quad 2 \quad 3 \]
\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \]

\( \varepsilon = 0.01 \)

\[ x \quad y \quad 1 \quad 2 \quad 3 \]
\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \]

\( \varepsilon = 0 \)

\[ x \quad y \quad 1 \quad 2 \quad 3 \]
\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \]

Figure 4.21: Phase portraits of Van der Pol equation: (a) \( \varepsilon = 1 \); (b) \( \varepsilon = 0.1 \); (c) \( \varepsilon = 0.01 \); (d) \( \varepsilon = 0 \).

Thus, by Theorem 4.30, a unique and stable limit cycle bifurcates from the circle \( r = 2 \) for small \( \varepsilon > 0 \) (see Figure 4.21). One can prove that (4.41) has exactly one limit cycle for all \( \varepsilon > 0 \) (see Exercise 4.7.6).

Finally, let us consider perturbations of a saddle homoclinic orbit. Suppose that a Hamiltonian planar system

\[ \dot{x} = \begin{pmatrix} H_{x_2}(x) \\ -H_{x_1}(x) \end{pmatrix}, \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \quad (4.42) \]

has an orbit \( \Gamma_0 \) homoclinic to a simple saddle point \( x_0 \). Let us denote the corresponding solution by \( \gamma(t) \), so that

\[ \lim_{t \to \pm \infty} \gamma(t) = x_0. \]

Let \( H(x_0) = h_0 \), so that \( \Gamma_0 \subset \{ x \in \mathbb{R}^2 : H(x) = h_0 \} \).

Introduce now the following two-parameter perturbation of (4.42):

\[ \dot{x} = \begin{pmatrix} H_{x_2}(x) \\ -H_{x_1}(x) \end{pmatrix} + \varepsilon f(x, \mu), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \quad (4.43) \]
where $\varepsilon, \mu \in \mathbb{R}$ are parameters, and $f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ is a smooth function. The reason for introducing the second parameter will become clear later. Finally, suppose that $f(x_0, \mu) = 0$ for $\mu \in \mathbb{R}$. This assumption implies that $x_0$ is an equilibrium for all values of both parameters.

$H(x_{\mp}(0)) \quad H(x_{\mp}(0))$

\[\Sigma\]

$x_0$

$\Gamma_0$

Figure 4.22: A segment $\Sigma$ orthogonal to the saddle homoclinic orbit $\Gamma_0$, together with unperturbed and perturbed orbits asymptotic to the saddle.

Consider a segment $\Sigma$ orthogonal to $\dot{\gamma}(0)$ at $\gamma(0) \in \Gamma_0$. For small values of the parameter $\varepsilon$ and any $\mu \in \mathbb{R}$, there exist two solutions of (4.43), say $x_{\mp}(t)$, such that

\[x_{\pm}(0) \in \Sigma, \quad \lim_{t \to \pm \infty} x_{\pm}(t) = x_0.\]

Clearly, these solutions belong to the stable and unstable manifolds of the saddle, respectively. Similar to the limit cycle case, parametrize $\Sigma$ near $x_0$ by the value $h$ of the Hamiltonian function $H(x)$ and introduce the split function

\[\Psi(\mu, \varepsilon) = H(x_{+\varepsilon}(0)) - H(x_{-\varepsilon}(0))\]

(see Figure 4.22). This is a smooth function near the origin in the $(\mu, \varepsilon)$-plane. Obviously, $\Psi(\mu, 0) = 0$ for all values of $\mu$. Indeed, at $\varepsilon = 0$ the system (4.43) reduces to (4.42) and has the homoclinic orbit $\Gamma_0$, so that $x_{+0}(t) = x_{-0}(t) = \gamma(t)$ for all $t \in \mathbb{R}$. If $\Psi(\mu, \varepsilon) = 0$ for some $\varepsilon \neq 0$, then the perturbed system (4.43) has a homoclinic orbit $\Gamma$ to the saddle $x_0$ at these parameter values. This explains the appearance of the second parameter: Under a generic perturbation $\varepsilon f(x)$, the invariant manifolds split, so one needs another parameter, e.g., $\mu$, to tune to “compensate” for this in order to preserve the homoclinic orbit in the perturbed system.

We have

\[\Psi(\mu, \varepsilon) = \int_{-\infty}^{0} dH\left(x_{-\varepsilon}(t)\right) - \int_{0}^{\infty} dH\left(x_{+\varepsilon}(t)\right)\]

\[= \int_{-\infty}^{0} \langle H_x(x_{-\varepsilon}(t), \dot{x}_{-\varepsilon}(t)) dt + \int_{0}^{\infty} \langle H_x(x_{+\varepsilon}(t), \dot{x}_{+\varepsilon}(t)) dt\]

\[= \varepsilon M(\mu) + O(\varepsilon^2),\]

where

\[M(\mu) = \int_{-\infty}^{\infty} [-\dot{\gamma}(t) f_1(\gamma(t), \mu) + \dot{\gamma}(t) f_2(\gamma(t), \mu)] dt = \int_{\Gamma_0} f_2(x, \mu) dx_1 - f_1(x, \mu) dx_2.\]
This function is called the Melnikov homoclinic integral. The integral converges absolutely, since \( f(\gamma(t), \mu) \) and \( \dot{\gamma}(t) \) tend exponentially fast to zero as \( t \to \pm \infty \).

Fix some \( \mu_0 \in \mathbb{R} \) and consider \( M_0 = M(\mu_0) \) and

\[
M_1 = M'(\mu_0) = \int_{\Gamma_0} \frac{\partial f_2}{\partial \mu}(x, \mu_0)dx_1 - \frac{\partial f_1}{\partial \mu}(x, \mu_0)dx_2,
\]

then

\[
\Psi(\mu, \varepsilon) = \varepsilon(M_0 + M_1(\mu - \mu_0) + O((\mu - \mu_0)^2)) + O(\varepsilon^2) = \varepsilon \Phi(\mu, \varepsilon).
\]

If \( M_0 = 0 \) but \( M_1 \neq 0 \), then we can apply the Implicit Function Theorem to the equation \( \Phi(\mu, \varepsilon) = 0 \). Indeed,

\[
\Phi(\mu_0, 0) = M_0 = 0, \quad \Phi(\mu_0, 0) = M_1 \neq 0.
\]

Thus, there is a unique smooth function \( \mu = \mu(\varepsilon) \), \( \mu(0) = \mu_0 \), such that

\[
\Phi(\mu(\varepsilon), \varepsilon) = 0
\]

for all small \( |\varepsilon| \). This implies that \( \Psi(\mu(\varepsilon), \varepsilon) \equiv 0 \), so that there exists a unique homoclinic orbit \( \Gamma \) to the saddle \( x_0 \) in (4.43) for all sufficiently small \( |\varepsilon| \), provided that \( \mu = \mu(\varepsilon) \).

The considerations above can be summarized in the following theorem.

**Theorem 4.32 (Melnikov, 1963)** Let \( \Gamma_0 \) be an orbit homoclinic to a saddle equilibrium of (4.43) for \( \varepsilon = 0 \). Suppose that for some \( \mu = \mu_0 \) holds

\[
\int_{\Gamma_0} f_2(x, \mu_0)dx_1 - f_1(x, \mu_0)dx_2 = 0,
\]

while

\[
\int_{\Gamma_0} \frac{\partial f_2}{\partial \mu}(x, \mu_0)dx_1 - \frac{\partial f_1}{\partial \mu}(x, \mu_0)dx_2 \neq 0,
\]

then there exists a unique function \( \mu(\varepsilon) \) with \( \mu(0) = \mu_0 \), and an annulus around \( \Gamma_0 \) in which the system (4.43) has, for all \( \varepsilon \) with sufficiently small \( |\varepsilon| \) and \( \mu = \mu(\varepsilon) \), a homoclinic orbit \( \Gamma \to \Gamma_0 \) as \( \varepsilon \to 0 \).

**Example 4.33 (Singular normal form for Bogdanov-Takens bifurcation)**

Consider the following planar system

\[
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -1 + x_1^2 + \varepsilon(\mu x_2 + x_1 x_2),
\end{aligned}
\]

(4.45)

that appears in the analysis of a generic two-parameter system having an equilibrium with a double zero eigenvalue at some critical parameter values.

---

For $\varepsilon = 0$ the system (4.45) is Hamiltonian with

$$H(x, y) = \frac{x^2}{2} + x - \frac{x^3}{3}$$

and has a center $(-1, 0)$ and a saddle $(x_0, y_0) = (1, 0)$. (Verify!) The saddle is simple and has a homoclinic orbit corresponding with the following explicit solution

$$\gamma_1(t) = 1 - 3 \text{sech}^2 \left( \frac{t}{\sqrt{2}} \right),$$

$$\gamma_2(t) = 3\sqrt{2} \text{sech}^2 \left( \frac{t}{\sqrt{2}} \right) \tanh \left( \frac{t}{\sqrt{2}} \right),$$

where

$$\text{sech}(z) = \frac{2}{e^z + e^{-z}}, \quad \tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$ 

Notice that $(x_0, y_0)$ is an equilibrium point of (4.45) for all values of $(\varepsilon, \mu)$. The Melnikov homoclinic integral defined by (4.44) can then be computed explicitly:

$$M(\mu) = \int_{-\infty}^{\infty} \gamma_2(t)[\mu \gamma_2(t) + \gamma_1(t) \gamma_2(t)] \, dt = \frac{24\sqrt{2}}{35}(7\mu - 5).$$

Crealy, for

$$\mu_0 = \frac{5}{7},$$

we have $M(\mu_0) = 0$ and $M'(\mu_0) \neq 0$. Thus, Theorem 4.32 implies that, for sufficiently small $|\varepsilon|$ and

$$\mu = \mu(\varepsilon) = \frac{5}{7} + O(\varepsilon),$$

the perturbed system (4.45) has a homoclinic orbit to the saddle equilibrium $(1, 0)$.

\hfill $\Diamond$

### 4.6 References

The best references for the qualitative theory of autonomous planar ODEs and the theory of their bifurcations are still the classical books [Andronov, Leontovich, Gordon & Maier 1971, Andronov, Leontovich, Gordon & Maier 1973]. Pontryagin’s method to locate limit cycles by perturbing planar Hamiltonian systems is also presented in [Andronov et al. 1973]. Further results on nonlinear planar ODEs, e.g., the index theory and blow-up techniques to study degenerate equilibrium points, can be found in [Arnol’d 1973, Arnol’d 1983, Arnol’d & Il’yashenko 1988, Perko 2001] and, in particular, in [Dumortier, Llibre & Artés 2006].

The theory of Hamiltonian systems is a classical and highly developed topic, see [Verhulst 1996] for a brief introduction and [Arnol’d 1989, Marsden & Ratiu 1999] for advanced presentations.

Phase portraits of various prey-predator models and their bifurcations are studied in [Bazykin 1998].