INLDS Practicum 1

For each planar system below, construct its phase portrait numerically using the `MATLAB` tool `pplane9` and then try to prove its essential features analytically.

Exercises

Ex.1 Two systems without cycles

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x - y + x^2, \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} &= y, \\ \dot{y} &= -x - y + x^2 + y^2. \end{cases}$$

Ex.2 A system with infinite number of cycles

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= x + xy - x^3. \end{cases}$$

Hint: Consider the transformation $(x, y, t) \to (-x, y, -t)$.

Ex.3 A system with one stable cycle

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + y(1 - x^2 - y^2). \end{cases}$$

Hint: Introduce polar coordinates $x = r\cos\varphi,\ y = r\sin\varphi$.

Ex.4 A system with an unstable cycle

$$\begin{cases} \dot{x} &= x(x^2 + y^2 - 2x - 3) - y, \\ \dot{y} &= y(x^2 + y^2 - 2x - 3) + x. \end{cases}$$

Ex.5 Van der Pol oscillator

$$\begin{cases} \dot{x} &= y - x^3 + x, \\ \dot{y} &= -\varepsilon x, \end{cases}$$

with $\varepsilon = 1.0, 0.1$, and 0.01.

Homework

Hand-in exercise is number 4. A solution should contain a phase plane with a periodic orbit obtained with `pplane9`. Also prove rigorously the existence of at least one periodic orbit using the Poincaré-Bendixson Theorem. *Hint*: Introduce polar coordinates $x = r\cos\varphi,\ y = r\sin\varphi$, and consider an annulus $r_1 < r < r_2$ with suitable $r_{1,2} > 0$.

Challenge: Prove that the cycle is unique. *Hint*: Consider the curve $\operatorname{div} F(x, y) = 0$ as the inner border of another annulus.