INLDS Practicum 6

For each planar system below, construct its phase portrait for $\alpha = 0$ and for small $\alpha < 0$ and $\alpha > 0$ using the `MATLAB` tool `pplane9`. Identify the occuring bifurcation and try to support your conclusions by analytical arguments as outlined below.

Exercises

Ex.1 A system with a global bifurcation

$$\begin{cases} \dot{x} &= 1 - x^2 - \alpha xy, \\ \dot{y} &= xy + \alpha(1 - x^2). \end{cases} \tag{1}$$

Prove that for $\alpha = 0$ there exists a heteroclinic connection between two saddles.

(a) Determine the equilibria and classify them.

(b) Compute the heteroclinic solution explicitly and verify the limits $t \to \pm\infty$.

Ex.2 A system with a saddle homoclinic bifurcation

$$\begin{cases} \dot{x} &= -x + 2y + x^2, \\ \dot{y} &= (2 - \alpha)x - y - 3x^2 + \frac{3}{2}xy. \end{cases} \tag{2}$$

Recall that for $\alpha = 0$ the system has a homoclinic orbit with $\sigma < 0$.

(a) Predict the stability of the bifurcating cycle and the direction of its bifurcation.

(b) **Challenge:** Show that the splitting function $\beta(\alpha)$ has a regular zero at $\alpha = 0$, see Theorem 4.4 in *Applied Nonlinear Dynamics*.

Ex.3 A system with a fold bifurcation of a cycle

$$\begin{cases} \dot{x} &= \left(\alpha - \frac{1}{4}\right) x - y + x(x^2 + y^2) - x(x^2 + y^2)^2, \\ \dot{y} &= x + \left(\alpha - \frac{1}{4}\right) y + y(x^2 + y^2) - y(x^2 + y^2)^2. \end{cases} \tag{3}$$

(a) Introduce polar coordinates $x = r\cos\varphi,\ y = r\sin\varphi$.

(b) Analyze the number and stability of equilibria of the r-equation for various α. Plot the equilibria (vertically) versus α (horizontally).

(c) Show that $\alpha = 0$ the system (3) has a non-hyperbolic cycle, so that a fold bifurcation of cycles should occur in the system.

(d) **Challenge**: Compute the quadratic normal form coefficient for the fold bifurcation. *Hints*: Use separation of variables to define a suitable function g such that $g(r_0, r_1) = 0$, where $r_0 = r(0)$ and $r_1 = r(2\pi)$. Introduce the Poincaré map $P(r_0) = r_1$ and differentiate the relation $g(r_0, P(r_0)) = 0$ twice with respect to r_0 to get the second derivative of P at the critical fixed point $r_0 = \frac{1}{\sqrt{2}}$. Use limit values of the derivatives.

Homework

Hand-in is exercise 3.