INLDS Practicum 7

Exercises

Study the following two-parameter planar systems exhibiting local codim 2 bifurcations by combining analytical and numerical methods.

Ex.1 Takens normal form

$$\begin{cases} \dot{x} &=& \beta x + y + x^2, \\ \dot{y} &=& \alpha - 4x^2. \end{cases} \tag{1}$$

(a) Derive equations for the saddle-node and Andronov-Hopf bifurcations in the system.

(b) Prove that a Bogdanov-Takens (BT) bifurcation occurs in the system and find the corresponding parameter values.

(c) Compute the normal form coefficients a and b for the BT-bifurcation and verify that $ab \neq 0$.

(d) Use `pplane9` to produce all representative phase portraits of the system near the Bogdanov-Takens point. Sketch the bifurcation diagram of the system.

(e) For $\alpha = 0.25$ find numerically the value of β corresponding to the saddle homoclinic bifurcation.

Ex.2 A prey-predator model by Bazykin and Khibnik

$$\begin{cases} \dot{x} &=& \dfrac{x^2(1-x)}{n+x} - xy, \\ \dot{y} &=& -y(m-x), \end{cases} \tag{2}$$

where $x, y \geq 0$ and $0 < m < 1$.

(a) Using `pplane9`, produce several phase portraits of model (2) for different values of m and fixed $n = \frac{1}{4}$ and $n = \frac{1}{16}$. *Hint*: The most interesting phase portrait occurs at $(m, n) = (\frac{1}{5}, \frac{1}{16})$.

(b) For $m = \frac{1}{5}$ approximate by simulations the value of n corresponding to the collision and disappearance of two periodic orbits.

(c) Derive an equation for the Andronov-Hopf bifurcation in the model. *Hint*: Consider the orbitally-equivalent polynomial system

$$\begin{cases} \dot{x} &=& x^2(1-x) - xy(n+x), \\ \dot{y} &=& -y(m-x)(n+x). \end{cases} \tag{3}$$

(d) Compute the 1st Lyapunov coefficient l_1 along the Andronov-Hopf bifurcation line in (3) and prove that this coefficient vanishes at $(m, n) = (\frac{1}{4}, \frac{1}{8})$. Conclude that a Bautin bifurcation happens in the model.

(e) Sketch the bifurcation diagram of the model.

(f) **Challenge**: Prove that the Bautin bifurcation is nondegenerate by computing the corresponding 2nd Lyapunov coefficient l_2.

Homework

Hand-in is exercise 2.