# BIFURCATION PHENOMENA

## Lecture 3: Two-parameter bifurcations of planar ODEs

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## Literature


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2. L.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, and L.O. Chua *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II, World Scientific, Singapore, 2001

3. V.I. Arnol'd, V.S. Afraimovich, Yu.S. Il'yashenko, and L.P. Shil'nikov *Bifurcation theory*, In: V.I. Arnol'd (ed), Dynamical Systems V. Encyclopaedia of Mathematical Sciences, Springer-Verlag, New York, 1994

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## 1. LOCAL CODIM 2 BIFURCATIONS IN 2D

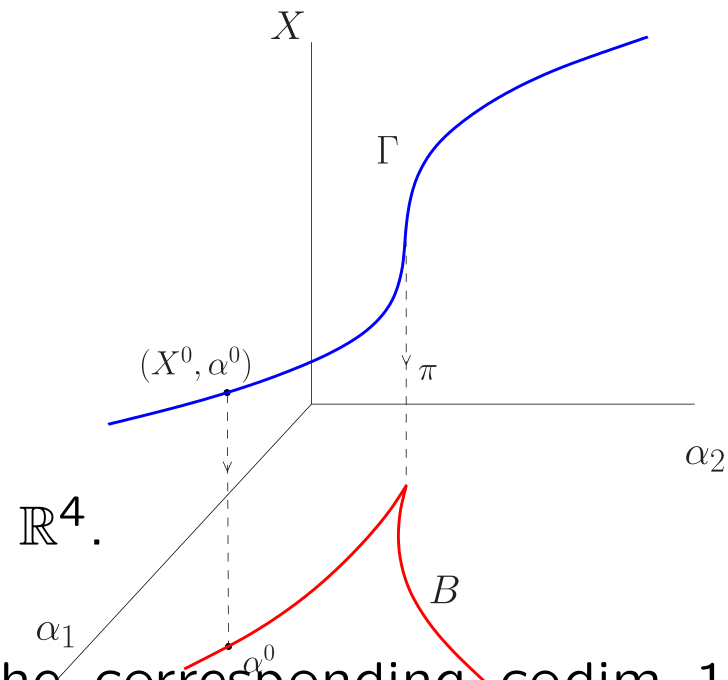
Consider a smooth 2D system depending on two parameters

$$\dot{X} = f(X, \alpha), \quad X \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^2.$$

**Curves of codim 1 bifurcations**:

$$\text{Fold}: \begin{cases} f(X,\alpha) &= 0, \\ \det\ f_X(X,\alpha) &= 0. \end{cases}$$

$$\text{Hopf}: \begin{cases} f(X,\alpha) &= 0, \\ \text{Sp}\ f_X(X,\alpha) &= 0. \end{cases}$$




In both cases, we have 3=2+1 equations in $\mathbb{R}^4$.

When we cross $B = \pi\Gamma$ in the $\alpha$-plane, the corresponding codim 1 bifurcation occurs.

One has to check that $\lambda_{1,2} = \pm i\omega$ along the Hopf curve.

## Local codim 2 cases in the plane:

1 $\lambda_1 = 0,\ a = 0$

$$\text{Fold} : \lambda_1 = 0 \quad \begin{cases} \dot{x} &= ax^2 + O(3) \\ \dot{y} &= \lambda_2 y + O(2) \end{cases}$$

2 $\lambda_1 = 0,\ \lambda_2 = 0$

$$\text{Hopf} : \lambda_{1,2} = \pm i\omega \quad \begin{cases} \dot{\rho} &= l_1 \rho^3 + O(4) \\ \dot{\varphi} &= \omega + O(1) \end{cases}$$

3 $\lambda_{1,2} = \pm i\omega,\ l_1 = 0$

To meet each case, we need to "tune" two parameters while following $\Gamma$ (or $B$) $\Rightarrow$ codim 2.

**Cusp bifurcation:** $\lambda_1 = 0,\ a = 0$

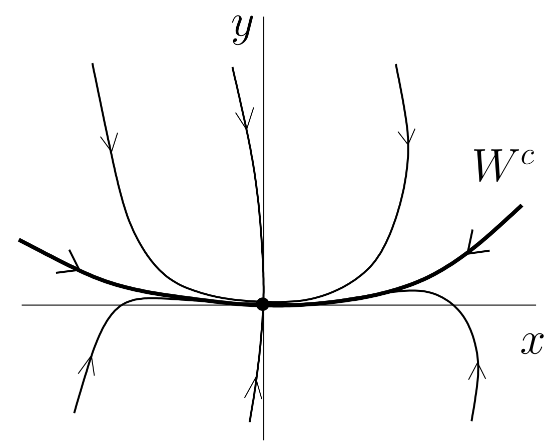
The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to

$$\begin{cases} \dot{x} &=& p_{11}xy + \frac{1}{2}p_{02}y^2 + \frac{1}{6}p_{30}x^3 + \ldots, \\ \dot{y} &=& \lambda_2 y + \frac{1}{2}q_{20}x^2 + q_{11}xy + \frac{1}{2}q_{02}y^2 + O(3). \end{cases}$$

It has an invariant 1D **center manifold** $W^c = \{(x, y) : y = W(x)\}$:

$$y = W(x) = \frac{1}{2}w_2 x^2 + O(3)$$

where $w_2 = -\dfrac{q_{20}}{\lambda_2}$.




Thus, the restriction of $\dot{X} = f(X, 0)$ to $W^c$ is

$$\dot{x} = cx^3 + O(4), \quad \text{where} \quad c = \frac{1}{6}\left(p_{30} - \frac{3}{\lambda_2}q_{20}p_{11}\right).$$

## Cusp normal form

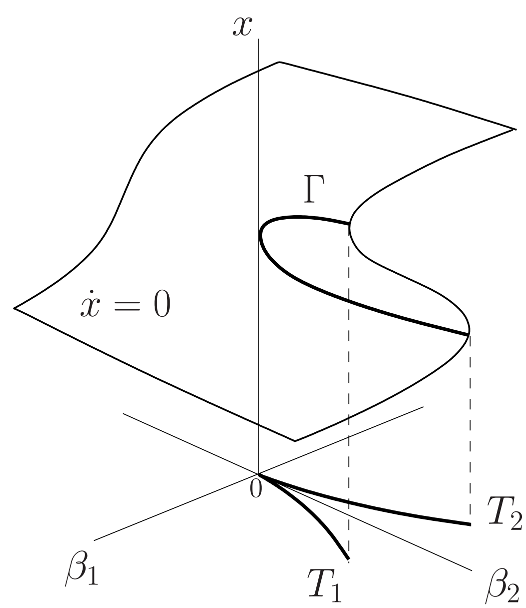
**Theorem 1** *If $c \neq 0$, then $\dot{X} = f(X, \alpha)$ is locally topologically equivalent near the cusp bifurcation to*

$$\begin{cases} \dot{x} &= \beta_1(\alpha) + \beta_2(\alpha)x + sx^3, \\ \dot{y} &= \lambda_2 y, \end{cases}$$

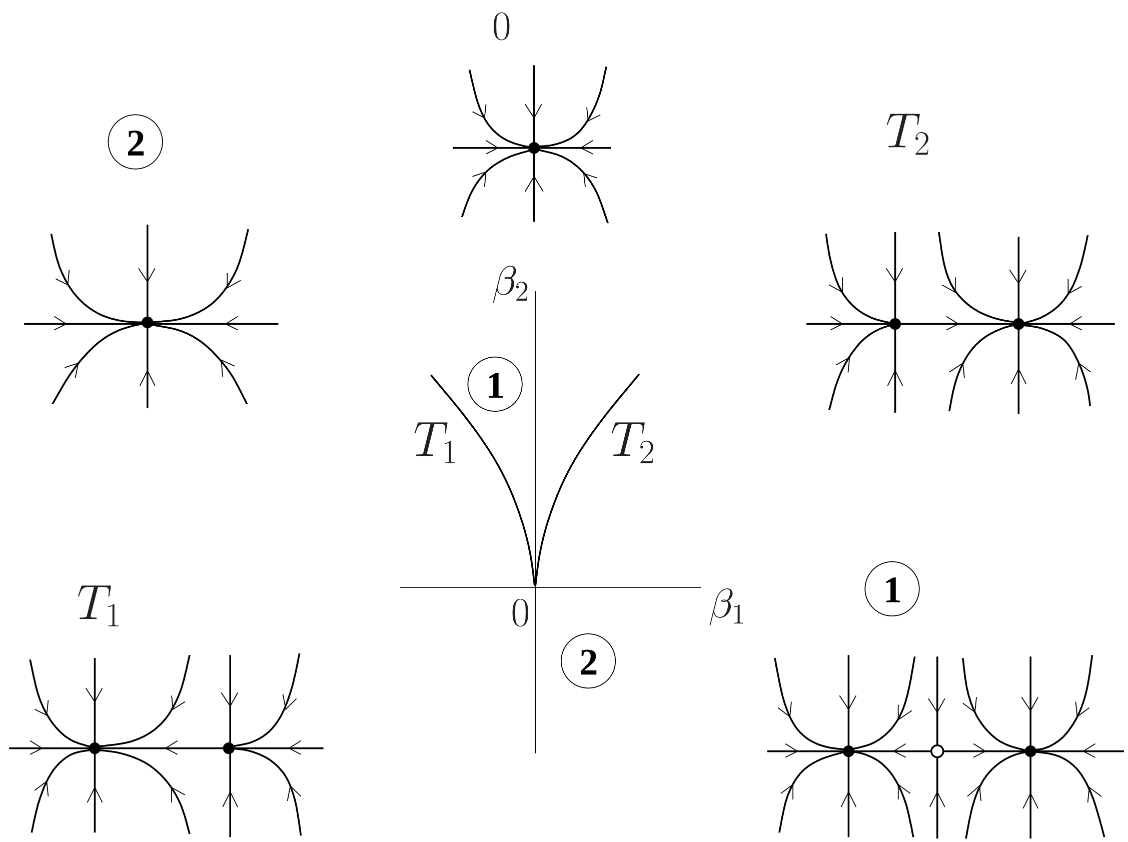
*where $\beta_1(0) = \beta_2(0) = 0$ and $s = \operatorname{sign}(c) = \pm 1$.*

**Fold curve(s)** $4\beta_2^3 + 27s\beta_1^2 = 0$

**Equilibrium manifold:**



## **Cusp bifurcation diagram** ($c < 0,\ \lambda_2 < 0$)




Three equilibria exist inside the wedge, pairwise colliding at its borders $T_{1,2}$ and leaving one equilibrium outside.

**Bogdanov-Takens bifurcation:** $\lambda_1 = \lambda_2 = 0 \quad (A = f_X(0,0) \neq 0)$

The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to

$$\begin{cases} \dot{x} &= y + \frac{1}{2}p_{20}x^2 + p_{11}xy + \frac{1}{2}p_{02}y^2 + O(3) \equiv P(x,y), \\ \dot{y} &= \frac{1}{2}q_{20}x^2 + q_{11}xy + \frac{1}{2}q_{02}y^2 + \frac{1}{6}q_{03}x^2 + O(3). \end{cases}$$

By a nonlinear local diffeomorphism (change of variables)

$$\begin{cases} \xi &= x, \\ \eta &= P(x,y), \end{cases}$$

this system can be reduced near the origin to

$$\begin{cases} \dot{\xi} &= \eta, \\ \dot{\eta} &= a\xi^2 + b\xi\eta + \ldots, \end{cases}$$

where

$$a = \frac{1}{2}q_{20}, \quad b = p_{20} + q_{11}.$$

**Bogdanov-Takens normal form**

**Theorem 2** *If $ab \neq 0$, then $\dot{X} = f(X, \alpha)$ is locally topologically equivalent near the BT-bifurcation to*

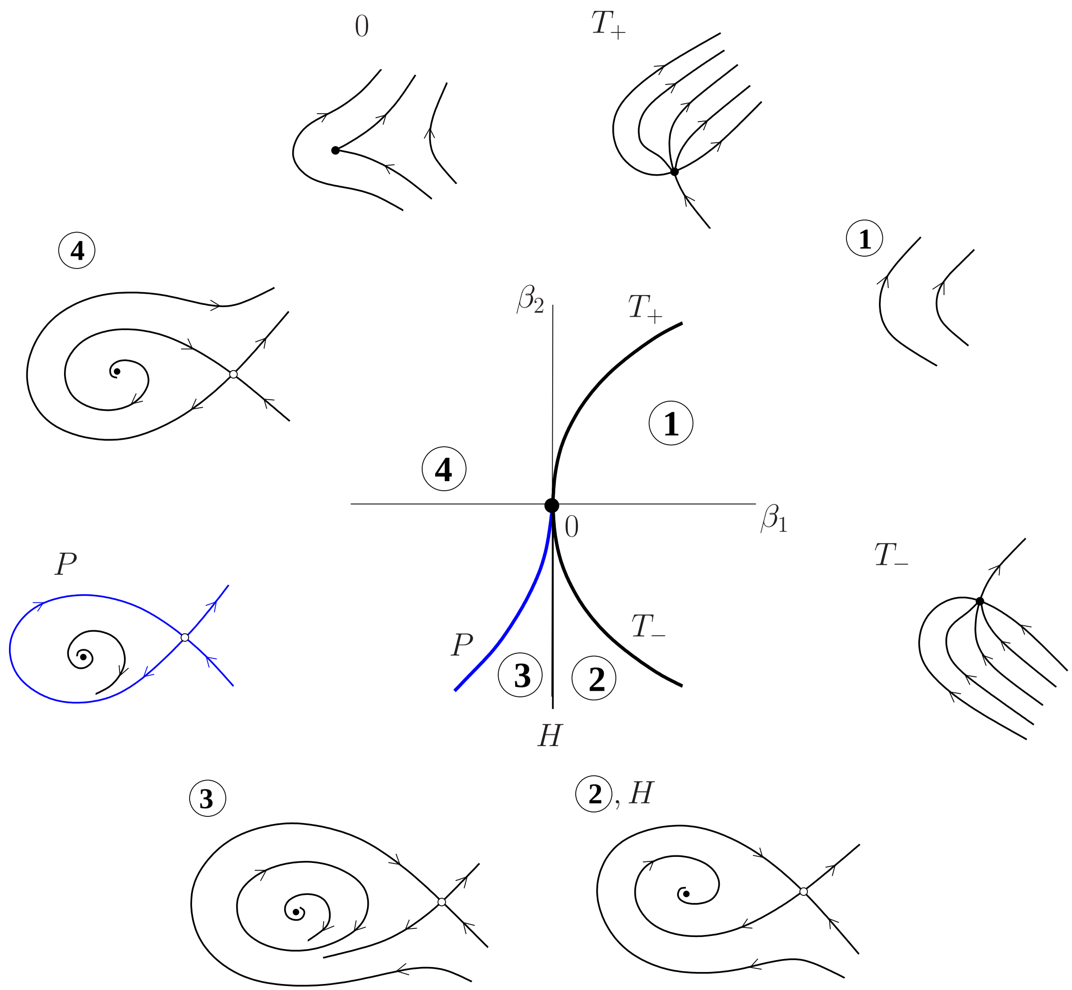
$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= \beta_1(\alpha) + \beta_2(\alpha)x + x^2 + sxy, \end{cases}$$

*where $\beta_1(0) = \beta_2(0) = 0$ and $s = \text{sign}(ab) = \pm 1$.*

**Bifurcation curves** $(ab < 0)$:

- **fold** $T : \beta_1 = \frac{1}{4}\beta_2^2$
- **Andronov-Hopf** $H : \beta_1 = 0,\ \beta_2 < 0$
- **saddle homoclinic** $P : \beta_1 = -\frac{6}{25}\beta_2^2 + O(3),\ \beta_2 < 0$ (**global bifurcation**)

## BT bifurcation diagram $(ab < 0)$




A unique limit cycle appears at Andronov-Hopf bifurcation curve $H$ and disappears via the saddle homoclinic orbit at the curve $P$.

**Bautin ("generalized Hopf") bifurcation:** $\lambda_{1,2} = \pm i\omega,\ l_1 = 0$

The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to the complex form

$$\dot{z} = i\omega z + \sum_{2 \le j+k \le 5} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(6),$$

which is locally smoothly equivalent to the **Poincaré normal form**

$$\dot{w} = i\omega w + c_1 w|w|^2 + c_2 w|w|^4 + O(6),$$

where the **Lyapunov coefficients**

$$l_j = \frac{1}{\omega}\Re(c_j)$$

satisfy

$$2l_1 = \frac{1}{\omega}\left(\Re(g_{21}) - \frac{1}{\omega}\Im(g_{20}g_{11})\right) \ \Rightarrow\ l_1 = \frac{1}{2\omega^2}\Re(ig_{20}g_{11} + \omega g_{21})$$

If $l_1 = 0$ then

$$
\begin{aligned}
12 l_2(0) \;=\;& \frac{1}{\omega}\Re(g_{32}) \\
+\;& \frac{1}{\omega^2}\Im\left[g_{20}\bar{g}_{31} - g_{11}(4g_{31} + 3\bar{g}_{22}) - \frac{1}{3}g_{02}(g_{40} + \bar{g}_{13}) - g_{30}g_{12}\right] \\
+\;& \frac{1}{\omega^3}\Big\{\Re\Big[g_{20}(\bar{g}_{11}(3g_{12} - \bar{g}_{30}) + g_{02}\left(\bar{g}_{12} - \frac{1}{3}g_{30}\right) + \frac{1}{3}\bar{g}_{02}g_{03}) \\
& \qquad + g_{11}(\bar{g}_{02}\left(\frac{5}{3}\bar{g}_{30} + 3g_{12}\right) + \frac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30})\Big] \\
& \qquad + 3\Im(g_{20}g_{11})\ \Im(g_{21})\Big\} \\
+\;& \frac{1}{\omega^4}\left\{\Im\left[g_{11}\bar{g}_{02}\left(\bar{g}_{20}^2 - 3\bar{g}_{20}g_{11} - 4g_{11}^2\right)\right]\right. \\
& \quad + \left.\Im(g_{20}g_{11})\left[3\Re(g_{20}g_{11}) - 2|g_{02}|^2\right]\right\}
\end{aligned}
$$

## Bautin normal form

**Theorem 3** *If* $l_2 \neq 0$ *and* $\omega \neq 0$*, then* $\dot{X} = f(X, \alpha)$ *is locally topologically equivalent near Bautin bifurcation to the normal form in the polar coordinates:*

$$\begin{cases} \dot{\rho} = \rho(\beta_1(\alpha) + \beta_2(\alpha)\rho^2 + s\rho^4), \\ \dot{\varphi} = 1, \end{cases}$$

*where* $\beta_1(0) = \beta_2(0) = 0$ *and* $s = \mathrm{sign}(l_2) = \pm 1$*.*

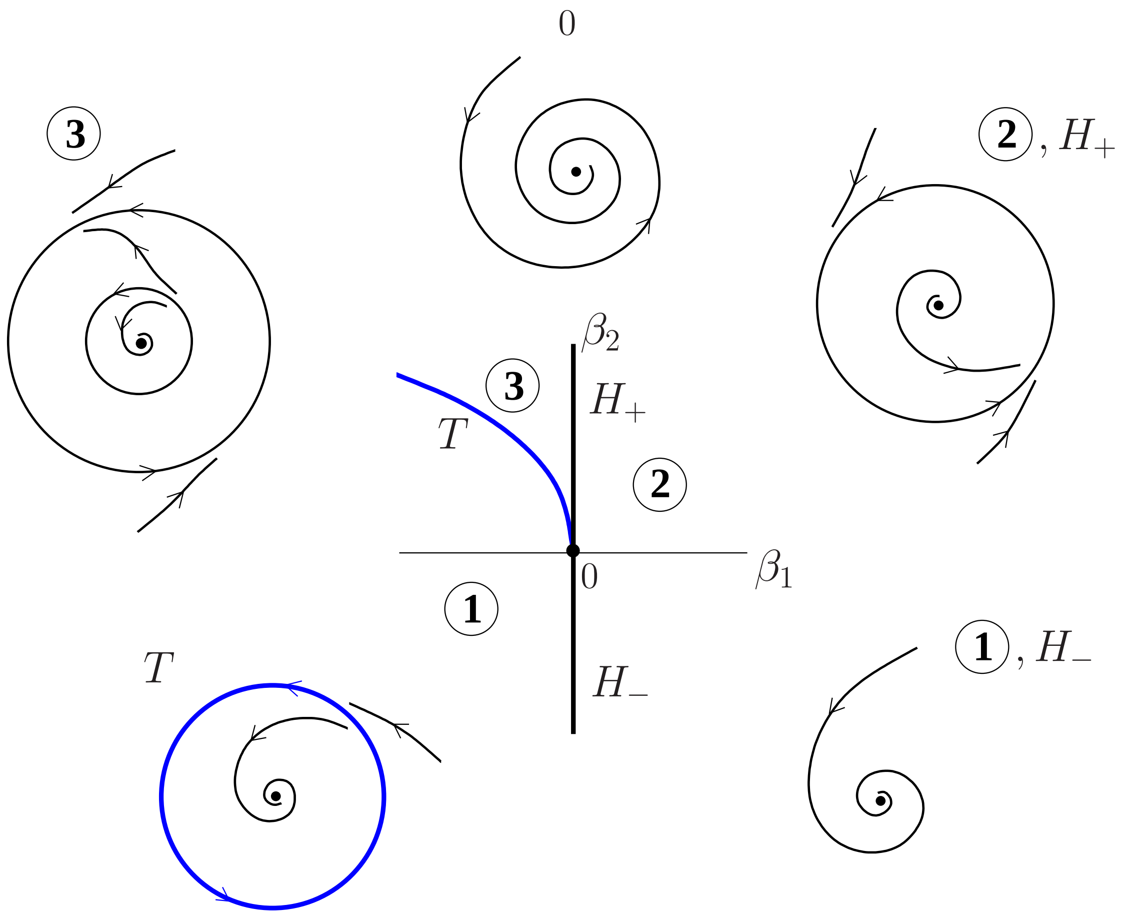
**Bifurcation curves** $(l_2 < 0)$:

- **superctitical Andronov-Hopf** $H^- : \beta_1 = 0,\ \beta_2 < 0$

- **subctitical Andronov-Hopf** $H^+ : \beta_1 = 0,\ \beta_2 > 0$

- **cyclic fold** $T_c : \beta_1 = -\frac{1}{4}\beta_2^2,\ \beta_2 > 0$ (**global bifurcation**)

## **Bautin bifurcation diagram** ($l_2 < 0$)




In the wedge between $H^+$ and $T_c$ there exist two limit cycles born via different Andronov-Hopf bifurcations, which merge and disappear at the cyclic fold curve $T_c$.

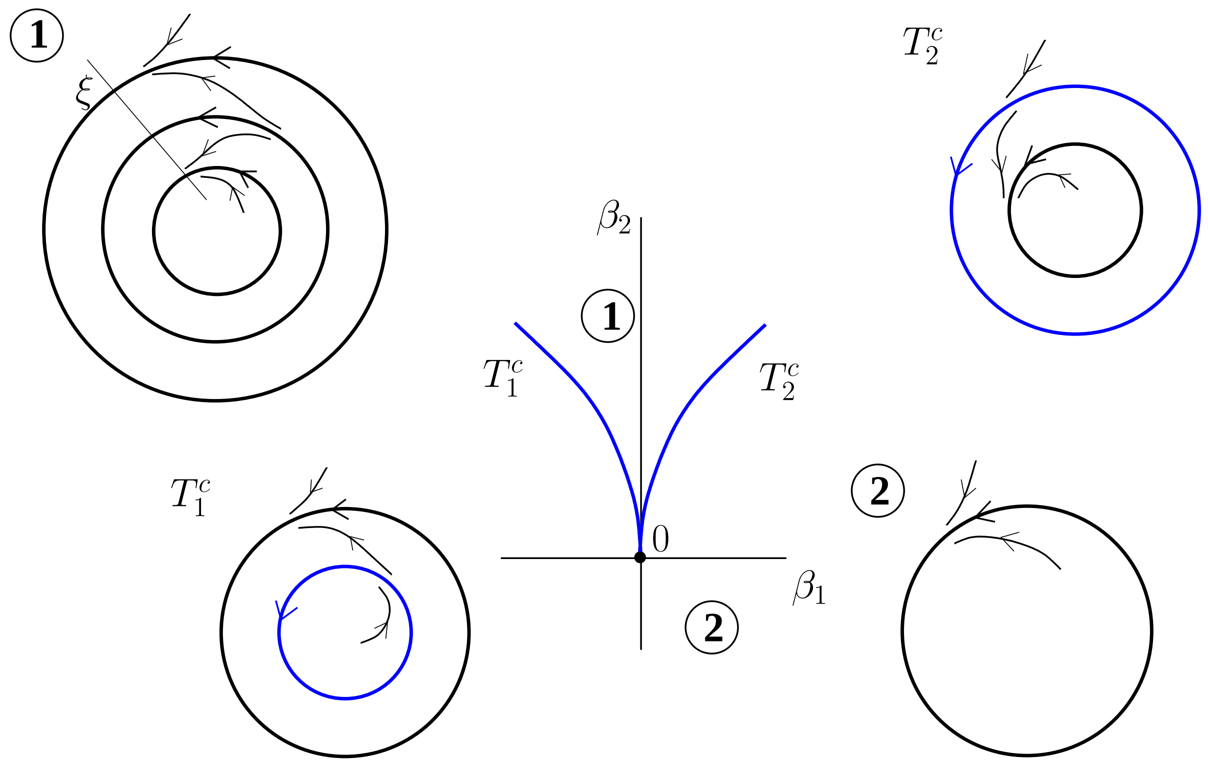
## 2. SOME GLOBAL CODIM 2 BIFURCATIONS IN 2D

• **Cyclic cusp** ($b = 0$): Critical Poincaré map $\xi \mapsto \xi + c\xi^3 + \ldots$.

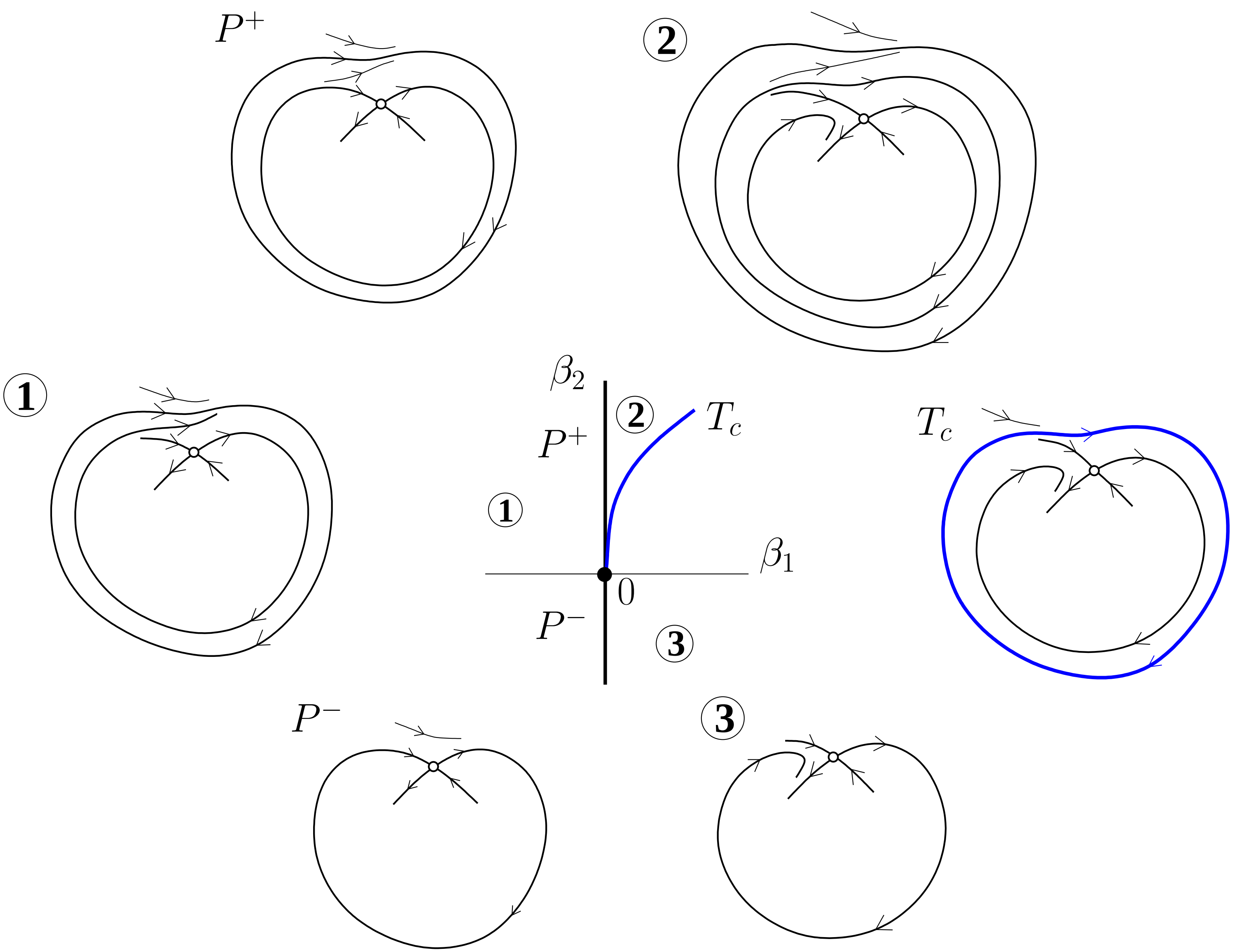
If $c \neq 0$ then the Poincaré map is locally topologically equivalent to

$$\xi \mapsto \beta_1(\alpha) + \beta_2(\alpha)\xi + \xi + s\xi^3,$$

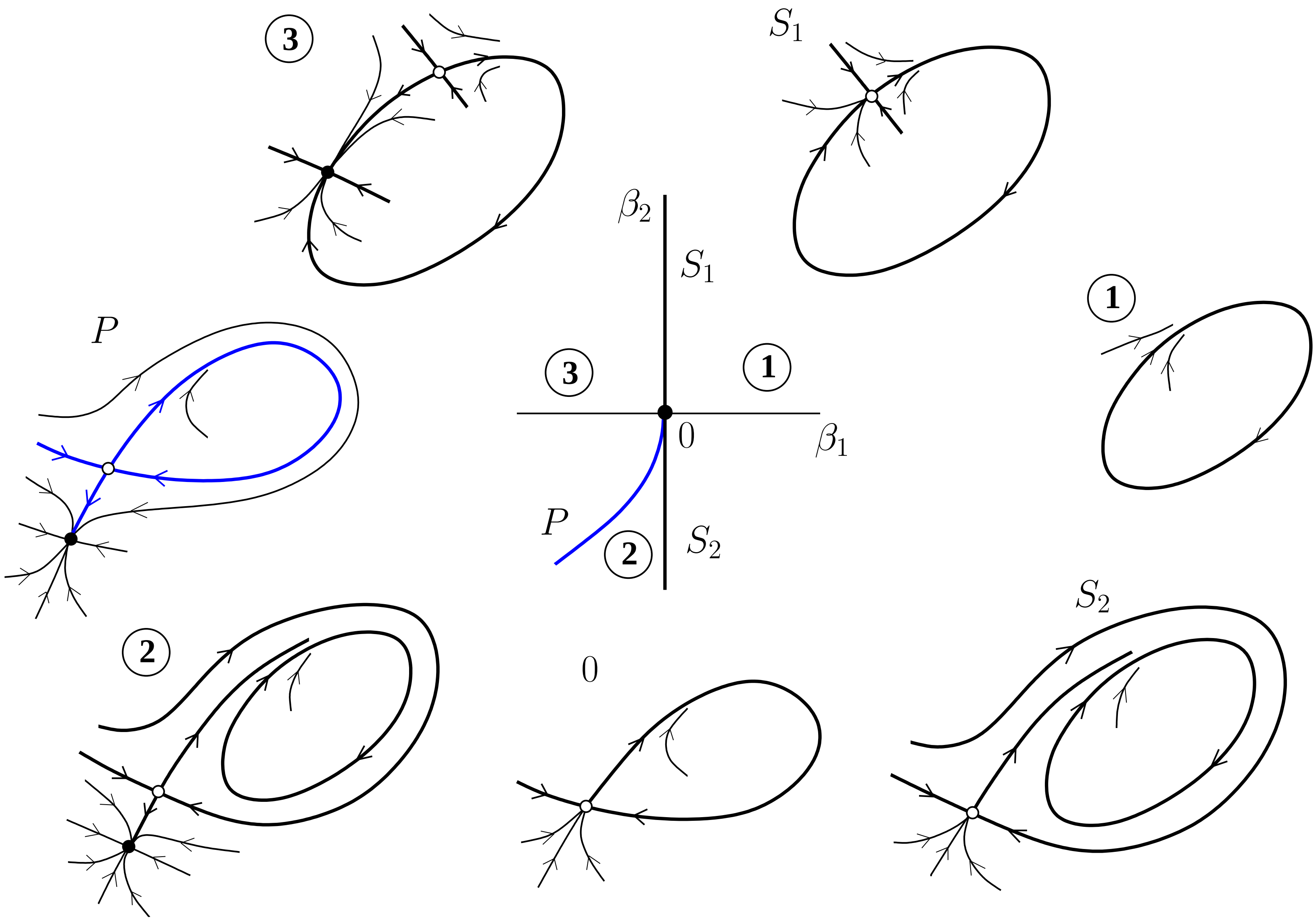
where $\beta_1(0) = \beta_2(0) = 0$ and $s = \mathrm{sign}(c) = \pm 1$.



- **Neutral saddle homoclinic orbit:** $\int_{-\infty}^{\infty} (\mathrm{div}\ f)(X^0(t))dt < 0$

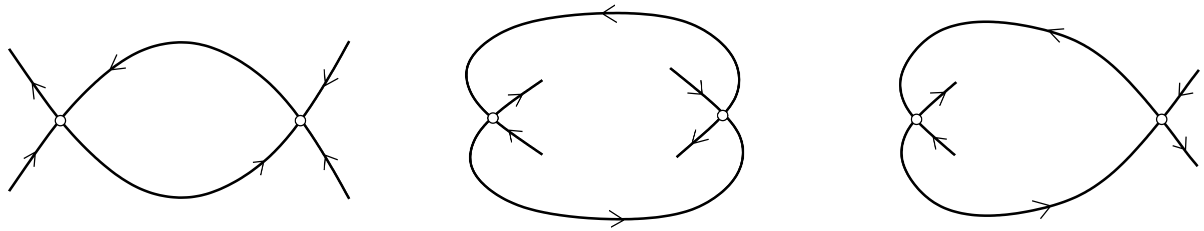


• **Non-central saddle-node homoclinic orbit**

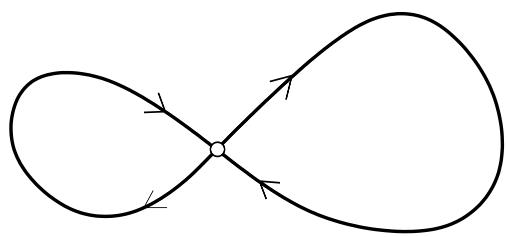


## Other global codim 2 cases:

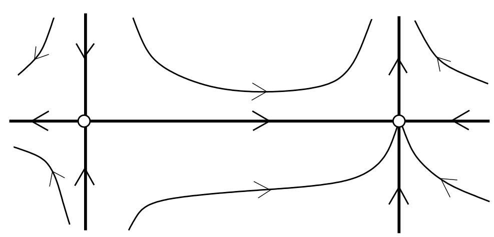
- **Heteroclinic cycles**



- **“Figure of 8”**

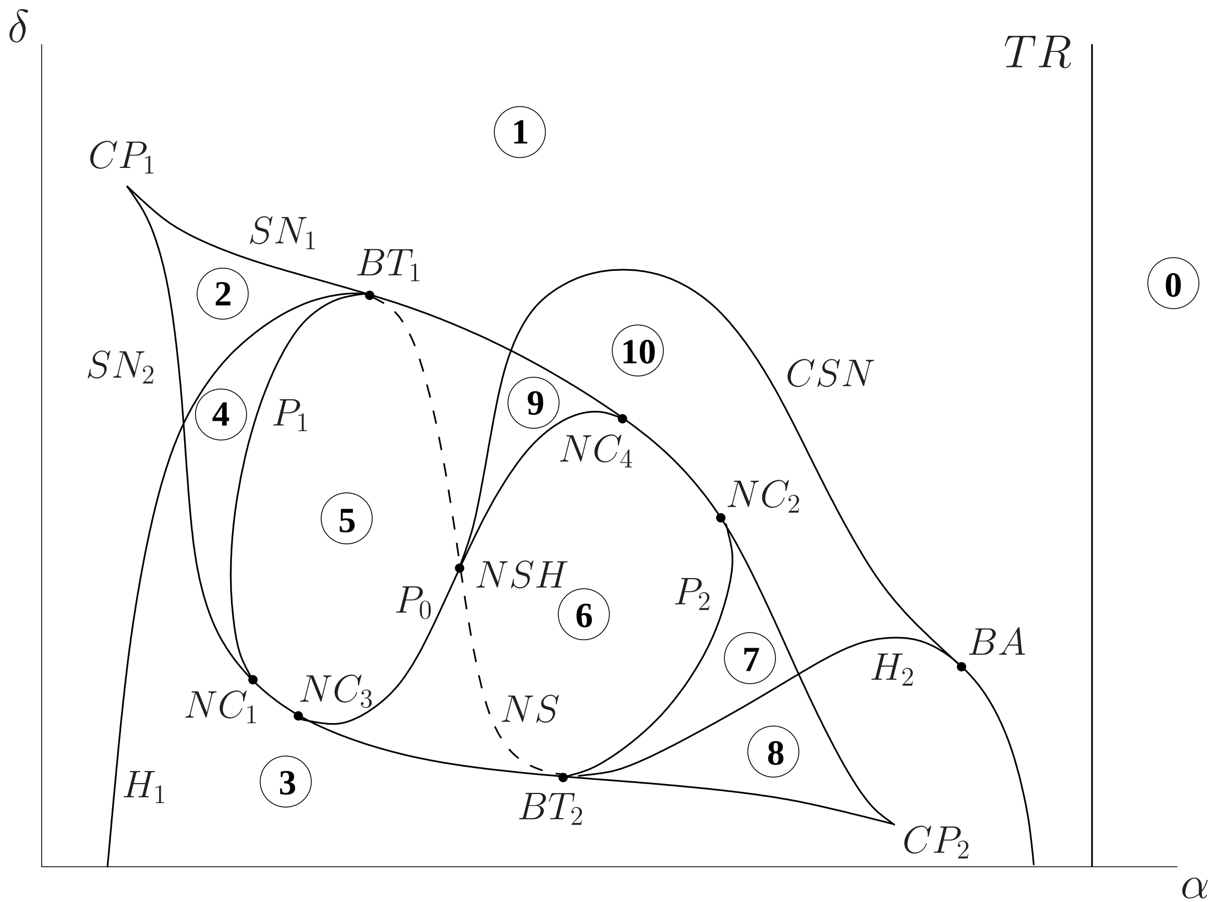


- **Saddle-to-saddle-node heteroclinic orbits**



## Example: Bazykin's prey-predator model

$$\begin{cases} \dot{x}_1 &= x_1 - \dfrac{x_1 x_2}{1+\alpha x_1} - \varepsilon x_1^2, \\ \dot{x}_2 &= -\gamma x_2 + \dfrac{x_1 x_2}{1+\alpha x_1} - \delta x_2^2 \ . \end{cases}$$



## Generic phase portraits: