# BIFURCATION PHENOMENA

## Lecture 4: Bifurcations in $n$-dimensional ODEs

Yuri A. Kuznetsov

August, 2010

# Contents

## Literature


1. L.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, and L.O. Chua *Methods of Qualitative Theory in Nonlinear Dynamics*, Part I, World Scientific, Singapore, 1998

2. L.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, and L.O. Chua *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II, World Scientific, Singapore, 2001

3. V.I. Arnol'd, V.S. Afraimovich, Yu.S. Il'yashenko, and L.P. Shil'nikov *Bifurcation theory*, In: V.I. Arnol'd (ed), Dynamical Systems V. Encyclopaedia of Mathematical Sciences, Springer-Verlag, New York, 1994

4. Yu.A. Kuznetsov *Elements of Applied Bifurcation Theory*, 3rd ed. Applied Mathematical Sciences 112, Springer-Verlag, New York, 2004

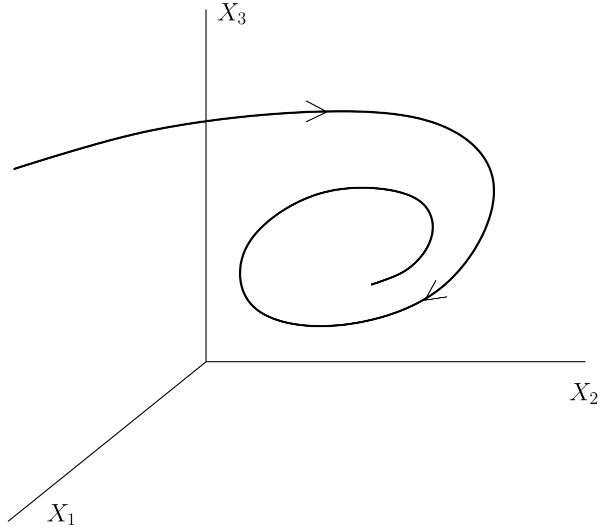
5. Yu.A. Kuznetsov, O. De Feo, and S. Rinaldi (2001), Belyakov homoclinic bifurcations in a tritrophic food chain model, *SIAM J. Appl. Math.* **62**, 462–487

# 1. SOLUTIONS AND ORBITS

Consider a smooth system

$$\dot{X} = f(X), \quad X \in \mathbb{R}^n.$$

**Orbits**, **phase portraits**, and **topological equivalence** are defined as in the case $n = 2$



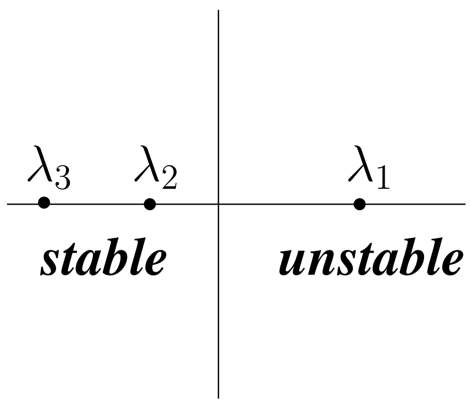

- **Equilibria**: $f(X_0) = 0$

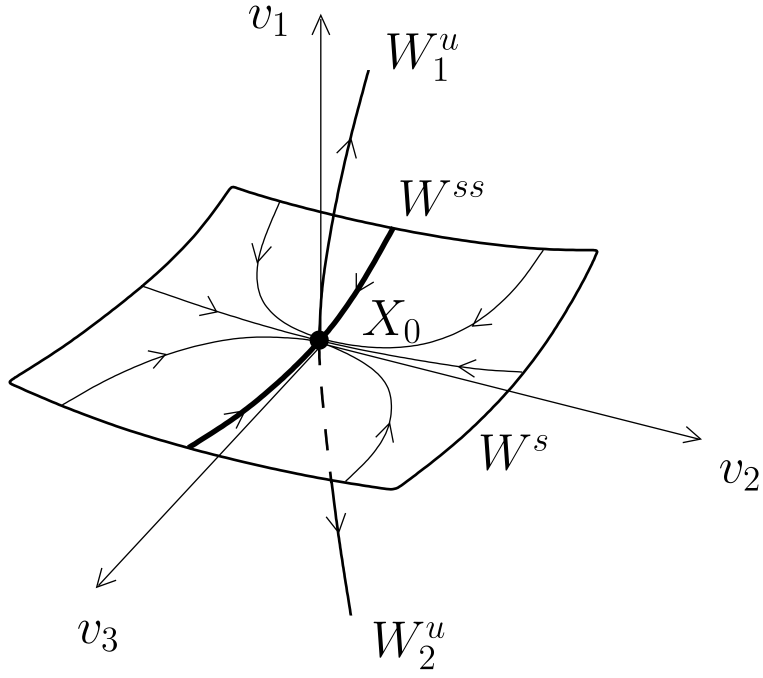
**Definition 1** *An equilibrium is called* **hyperbolic** *if* $\Re(\lambda) \neq 0$ *for all eigenvalues of its Jacobian matrix* $A = f_X(X_0)$.

**Theorem 1 (Grobman-Hartman)** *If equilibrium* $X_0 = 0$ *is hyperbolic,* $\dot{X} = f(X)$ *is locally topologically equivalent near the origin to* $\dot{Y} = AY$.

## Stable and unstable invariant manifolds of equilibria:

If a hyperbolic equilibrium $X_0$ has $n_s$ eigenvalues with $\Re(\lambda) < 0$ and $n_u$ eigenvalues with $\Re(\lambda) > 0$, it has the $n_s$-dimensional smooth invariant manifold $W^s$ composed of all orbits approaching $X_0$ as $t \to \infty$, and the $n_u$-dimensional smooth invariant manifold $W^u$ composed of all orbits approaching $X_0$ as $t \to -\infty$



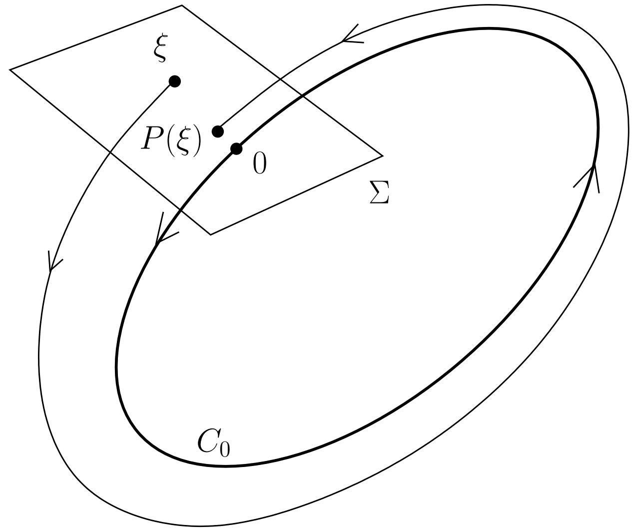




$$Av_j = \lambda_j v_j$$

- **Periodic orbits (cycles)**

The **Poincaré map** $\xi \mapsto \tilde{\xi} = P(\xi)$ is defined on a smooth $(n-1)$-dimensional crossection:

$$P : \Sigma \to \Sigma.$$




If $C_0$ coresponds to $\xi = 0$ then $P(0) = 0$ and $P(\xi) = M\xi + O(2)$

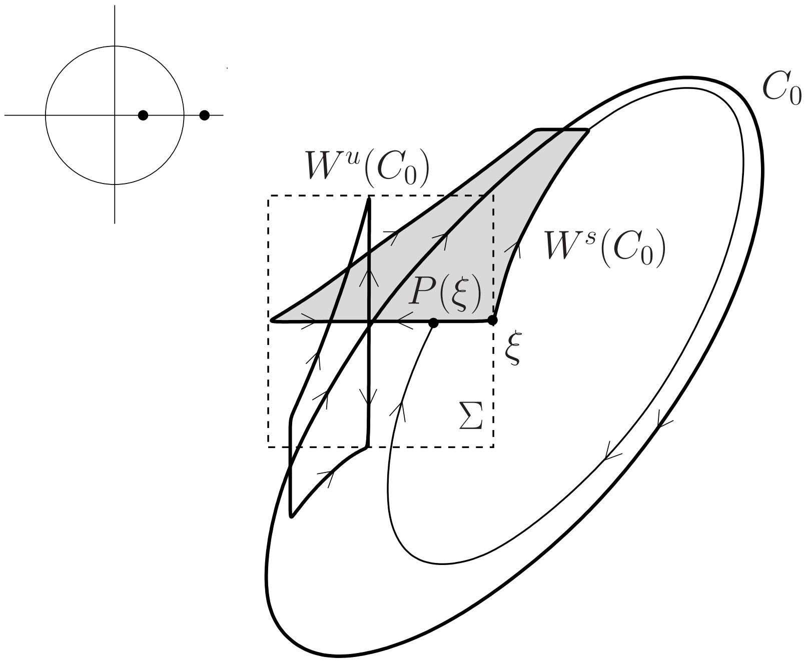
$$\mu_1\mu_2\cdots\mu_{n-1} = \exp\left(\int_0^{T_0} (\text{div } f)(X^0(t))dt\right) > 0$$

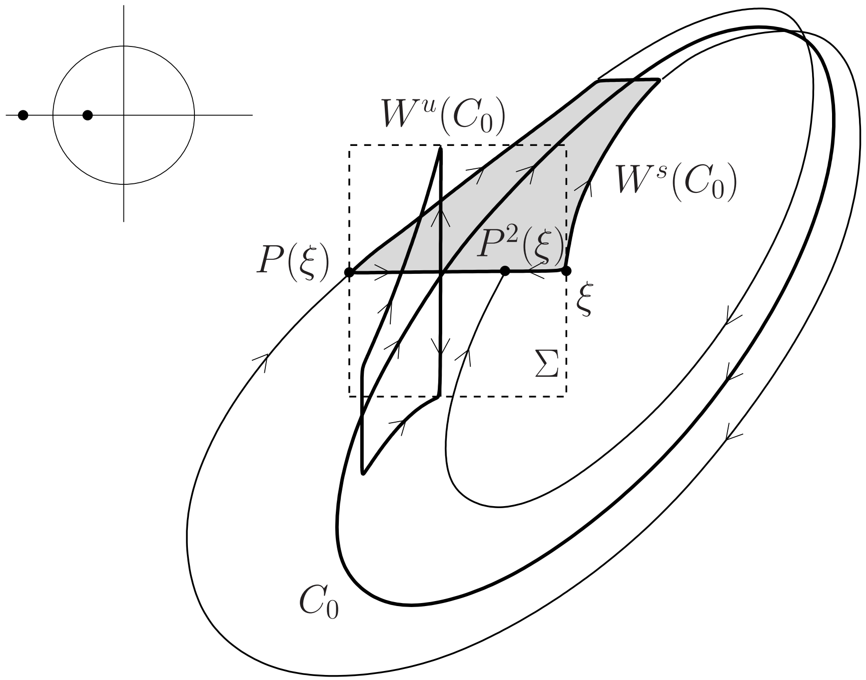
**Definition 2** *A cycle is called* **hyperbolic** *if* $|\mu| \neq 1$ *for all eigenvalues (***multipliers***) of the matrix* $M = P_\xi(0)$*.*

**Theorem 2 (Grobman-Hartman for maps)** *The Poincaré map* $\xi \mapsto P(\xi)$ *of a hyperbolic cycle is locally topologically equivalent near the origin to* $\xi \mapsto M\xi$*.*

## Stable and unstable invariant manifolds of cycles:

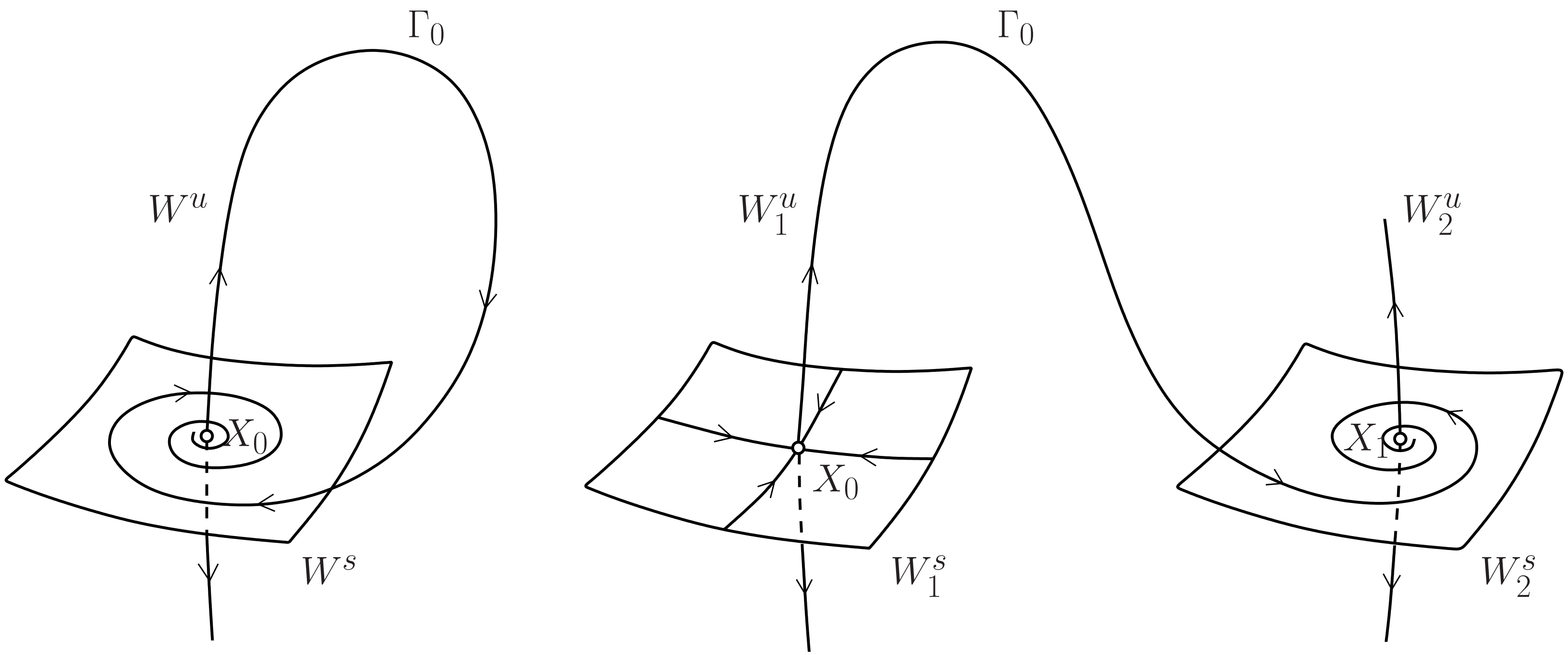
If a hyperbolic cycle $C_0$ has $m_s$ multipliers with $|\mu| < 1$ and $m_u$ multipliers with $|\mu| > 1$, it has the $(m_s + 1)$-dimensional smooth invariant manifold $W^s$ composed of all orbits approaching $C_0$ as $t \to \infty$, and the $(m_u + 1)$-dimensional smooth invariant manifold $W^u$ composed of all orbits approaching $C_0$ as $t \to -\infty$



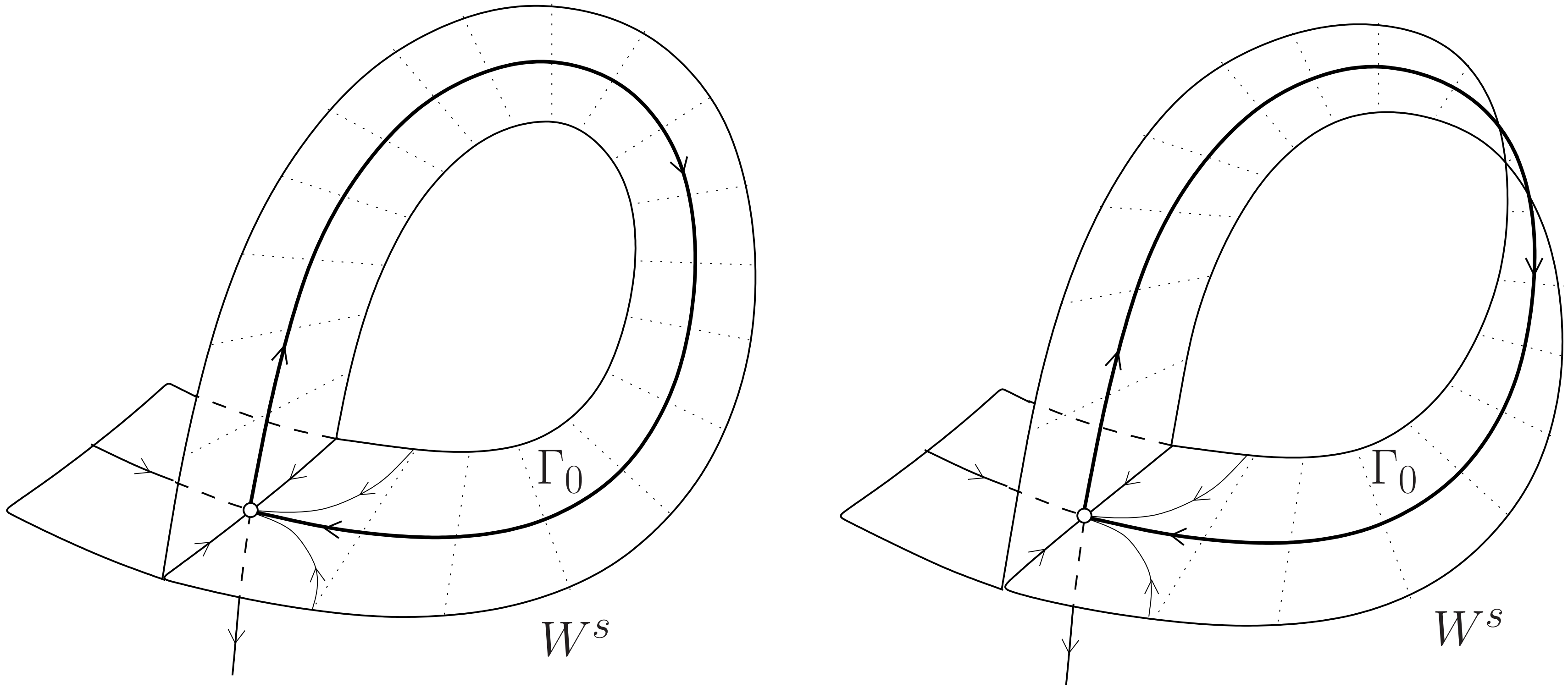



- **Connecting orbits**

**Homoclinic orbits** are intersections of $W^u$ and $W^s$ of an equilibrium/cycle.
**Heteroclinic orbits** are intersections of $W^u$ and $W^s$ of two different equilibria/cycles.



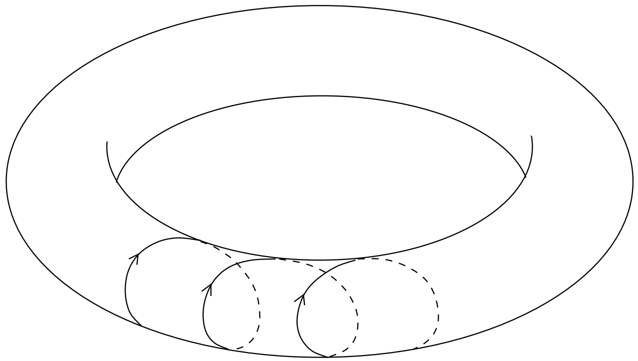
Generically, the closure of the 2D invariant manifold near a homoclinic orbit $\Gamma_0$ to an equilibriun with real eigenvalues (**saddle**) in $\mathbb{R}^3$ is either **simple (orientable)** or **twisted (non-orientable)**:



- **Compact invariant manifolds**

1. tori

   **Example**: 2D-torus $\mathbb{T}^2$ with periodic or quasi-periodic orbits

2. spheres

3. Klein bottles

- **Strange (chaotic) invariant sets**

  - have **fractal** structure (not a manifold);

  - contain **infinite** number of hyperbolic cycles;

  - demonstrate **sensitive dependence** of solutions on initial conditions;

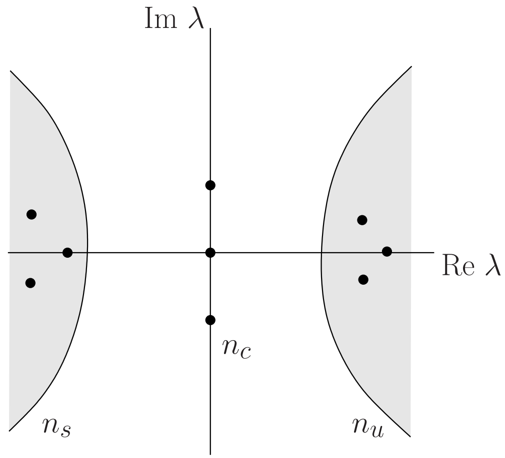
  - can be attracting (**strange attractors**);

  - orbits can be coded by sequences of symbols (**symbolic dynamics**).

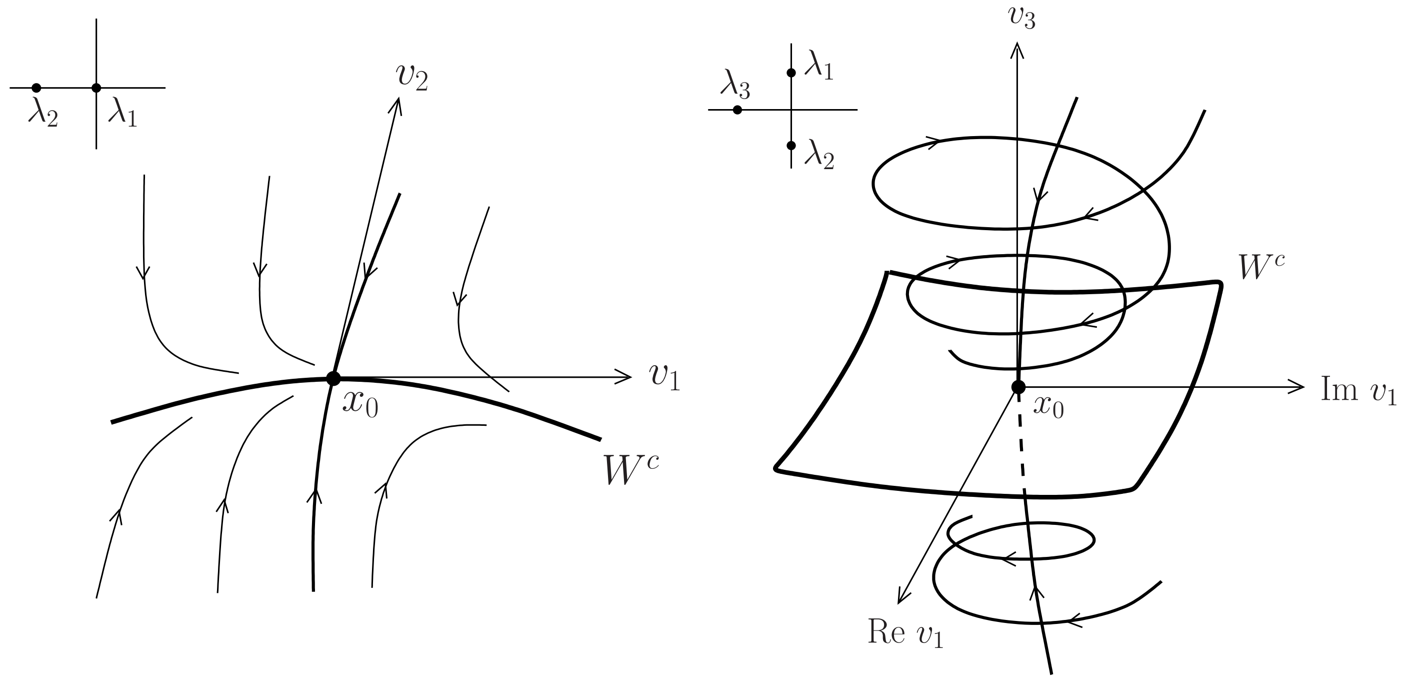
## 2. BIFURCATIONS OF N-DIMENSIONAL ODES $\dot{X} = f(X, \alpha)$

- **Local (equilibrium) bifurcations**

  **Center manifold reduction**: Let $X_0 = 0$ be non-hyperbolic with stable, usntable, and critical eigenvalues:




**Theorem 3** *For all sufficiently small $\|\alpha\|$, there exists a local invariant* **center manifold** *$W^c(\alpha)$ of dimension $n_c$ that is locally attracting if $n_u = 0$, repelling if $n_s = 0$, and of saddle type if $n_s n_u > 0$. Moreover $W^c(0)$ is tangent to the critical eigenspace of $A = f_X(0, 0)$.*




**Remark**: $W^c(0)$ is **not unique**; however, all $W^c(0)$ have the same Taylor expansion.

**Theorem 4** *If $\dot{\xi} = f(\xi, \alpha)$ is the restriction of $\dot{X} = f(X, \alpha)$ to $W^c(\alpha)$, then this system is locally topologically equivalent to*

$$\begin{cases} \dot{\xi} &= f(\xi, \alpha), \quad \xi \in \mathbb{R}^{n_c}, \alpha \in \mathbb{R}^m, \\ \dot{x} &= -x, \quad x \in \mathbb{R}^{n_s}, \\ \dot{y} &= y, \quad y \in \mathbb{R}^{n_u}. \end{cases}$$

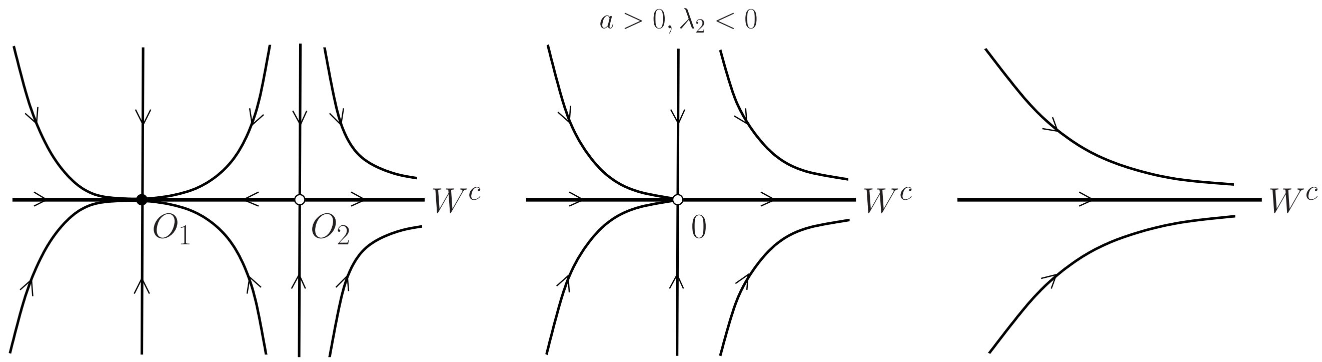
**Codim 1 equilibrium bifurcations:** $\alpha \in \mathbb{R}$

$$f(X,0) = AX + \frac{1}{2}B(X,X) + \frac{1}{6}C(X,X,X) + O(4)$$

- **Fold (saddle-node)**: $\lambda_1 = 0$ ($n_c = 1$)

  Let $a = \frac{1}{2}\langle q, B(q,q)\rangle$ where $Aq = A^{\mathsf{T}}p = 0$ with $\langle p,q\rangle = \langle q,q\rangle = 1$.

  If $a \neq 0$ then the restriction of $\dot{X} = f(X,\alpha)$ to its $W^c(\alpha)$ is locally topologically equivalent to $\dot{\xi} = \beta(\alpha) + a\xi^2$.




$\beta < 0$ $\beta = 0$ $\beta > 0$

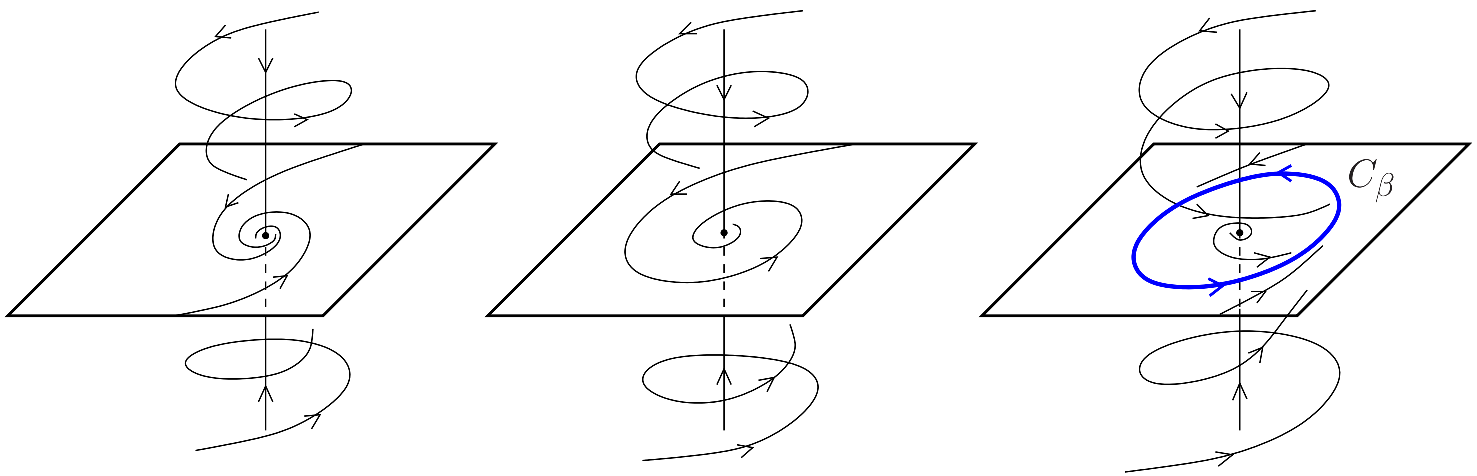
- **Andronov-Hopf:** $\lambda_{1,2} = \pm i\omega, \omega > 0 \quad (n_c = 2)$

$$l_1 = \frac{1}{2\omega}\Re\Big[\langle p, C(q,q,\bar{q})\rangle - 2\langle p, B(q, A^{-1}B(q,\bar{q}))\rangle + \langle p, B(\bar{q}, (2i\omega E_n - A)^{-1}B(q,q))\rangle\Big],$$

where $Aq = i\omega q,\ A^{\mathsf{T}}p = -i\omega p, \langle p, q\rangle = \langle q, q\rangle = 1.$

If $l_1 \neq 0$ then the restriction of $\dot{X} = f(X,\alpha)$ to its $W^c(\alpha)$ is locally topologically equivalent to $\begin{cases} \dot{\rho} = \rho(\beta(\alpha) + l_1\rho^2), \\ \dot{\varphi} = 1. \end{cases}$

$l_1 < 0,\ \lambda_3 < 0$




$\beta < 0$ $\beta = 0$ $\beta > 0$

**Codim 2 equilibrium bifurcations:** $\alpha \in \mathbb{R}^2$

1. **Cusp**: $\lambda_1 = 0, a = 0$ ($n_c = 1$)

   If $c \neq 0$, then the restriction of $\dot{X} = f(X, \alpha)$ to $W^c(\alpha)$ is locally topologically equivalent to $\dot{\xi} = \beta_1(\alpha) + \beta_2(\alpha)\xi + s\xi^3$, where $s = \text{sign}(c) = \pm 1$.

2. **Bogdanov-Takens**: $\lambda_1 = \lambda_2 = 0$ ($n_c = 2$)

   If $ab \neq 0$, then the restriction of $\dot{X} = f(X, \alpha)$ to $W^c(\alpha)$ is locally topologically equivalent to $\dot{x} = y$, $\dot{y} = \beta_1(\alpha) + \beta_2(\alpha)x + x^2 + sxy$, where $s = \text{sign}(ab) = \pm 1$.

3. **Bautin**: $\lambda_{1,2} = \pm i\omega, \omega > 0$ ($n_c = 2$)

   If $l_2 \neq 0$, then the restriction of $\dot{X} = f(X, \alpha)$ to $W^c(\alpha)$ is locally topologically equivalent to $\dot{\rho} = \rho(\beta_1(\alpha) + \beta_2(\alpha)\rho^2 + s\rho^4)$, $\dot{\varphi} = 1$, where $s = \text{sign}(l_2) = \pm 1$.

4. **Fold-Hopf**: $\lambda_1 = 0,\ \lambda_{2,3} = \pm i\omega, \omega > 0\ (n_c = 3)$

Generically, the restriction of $\dot{X} = f(X, \alpha)$ to $W^c(\alpha)$ is smoothly orbitally equivalent to

$$\begin{cases} \dot{\xi} &= \beta_1(\alpha) + \xi^2 + s\rho^2 + P(\xi, \rho, \varphi, \alpha), \\ \dot{\rho} &= \rho(\beta_2(\alpha) + \theta(\alpha)\xi + \xi^2) + Q(\xi, \rho, \varphi, \alpha), \\ \dot{\varphi} &= \omega_1(\alpha) + \theta_1(\alpha)\xi + R(\xi, \rho, \varphi, \alpha), \end{cases}$$

where $s = \pm 1,\ \theta(0) \neq 0, \omega_1(0) > 0, P, Q, R = \mathcal{O}(\|(\xi, \rho)\|^4)$.

The bifurcation diagrams **depend on** $O(4)$-terms. “Big picture” is determined by the ‘truncated normal form’ without the $O(4)$-terms.

There exist **invariant tori** and **homoclinic orbits** near the fold-Hopf bifurcation.

5. **Hopf-Hopf**: $\lambda_{1,2} = \pm \omega_1,\ \lambda_{3,4} = \pm i\omega_2, \omega_j > 0$ ($n_c = 4$)

Generically, the restriction of $\dot{X} = f(X, \alpha)$ to $W^c(\alpha)$ is smoothly orbitally equivalent to

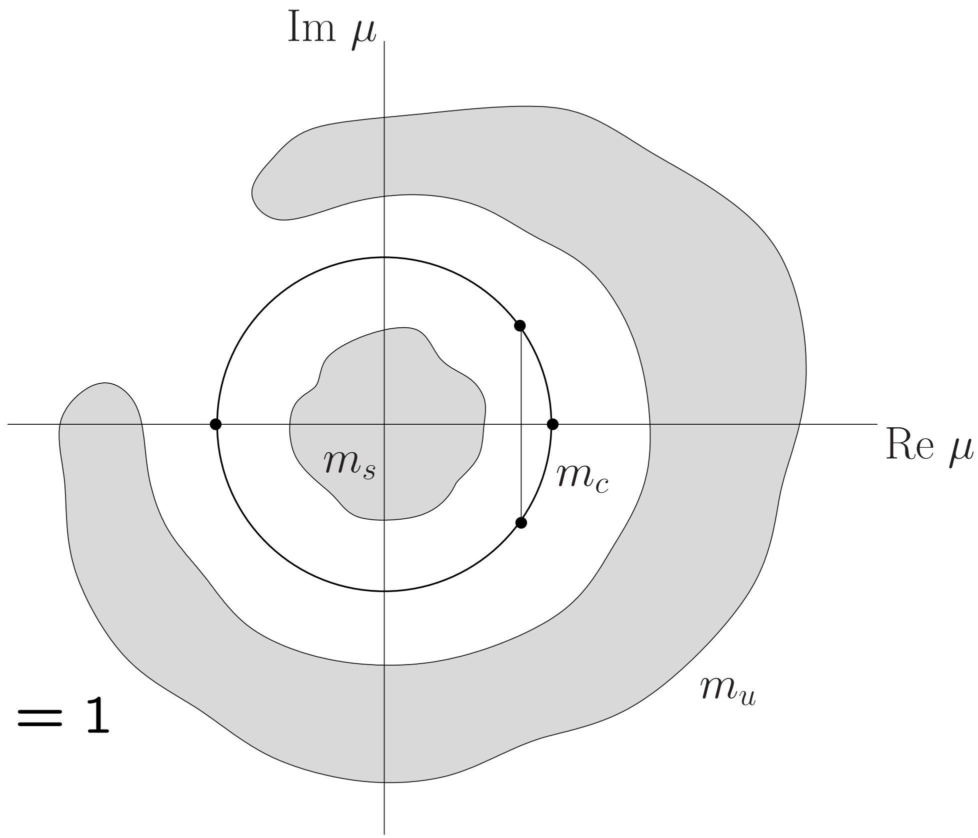
$$\begin{cases} \dot{r}_1 &= r_1(\beta_1(\alpha) + p_{11}(\alpha) r_1^2 + p_{12}(\alpha) r_2^2 + s_1(\alpha) r_2^4) + \Phi_1(r, \varphi, \alpha), \\ \dot{r}_2 &= r_2(\beta_2(\alpha) + p_{21}(\alpha) r_1^2 + p_{22}(\alpha) r_2^2 + s_2(\alpha) r_1^4) + \Phi_2(r, \varphi, \alpha), \\ \dot{\varphi}_1 &= \omega_1(\alpha) + \Psi_1(r, \varphi, \alpha), \\ \dot{\varphi}_2 &= \omega_2(\alpha) + \Psi_2(r, \varphi, \alpha) \end{cases}$$

where $\Phi_j = \mathcal{O}(\|r\|^6),\ \Psi_j = \mathcal{O}(\|r\|)$.

The bifurcation diagrams **depend on** $\Phi_j$- and $\Psi_j$-terms. “Big picture” is determined by the ‘truncated normal form’ without these terms.

There exist **invariant tori** and **homoclinic orbits** near the Hopf-Hopf bifurcation.

## Local bifurcations of cycles

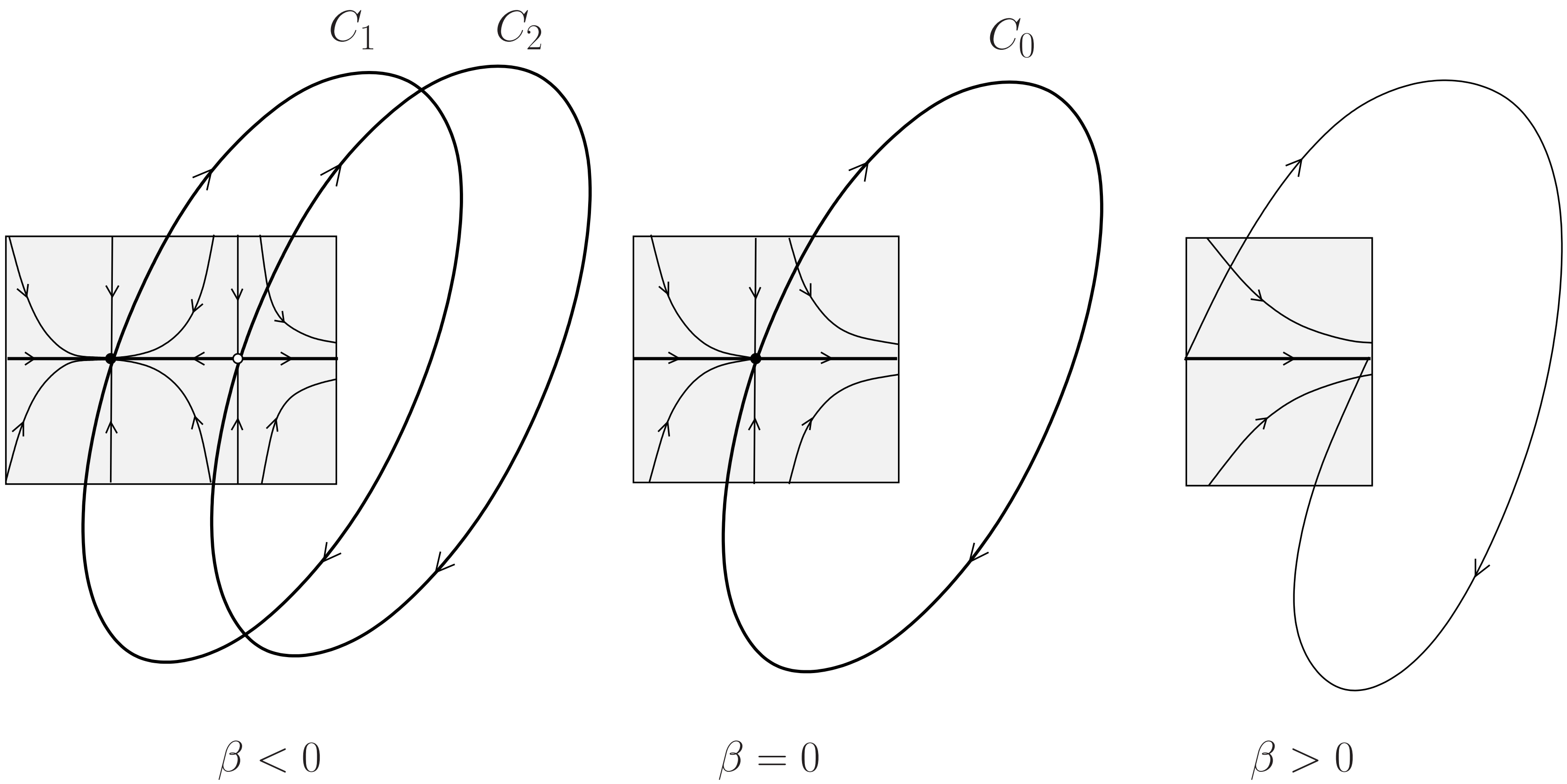



## Critical cases of codim 1:

- **cyclic fold (saddle-node)**: $\mu_1 = 1$
- **period-doubling**: $\mu_1 = -1$
- **Neimark-Sacker (torus):** $\mu_{1,2} = e^{\pm i\theta},\ 0 < \theta < \pi$

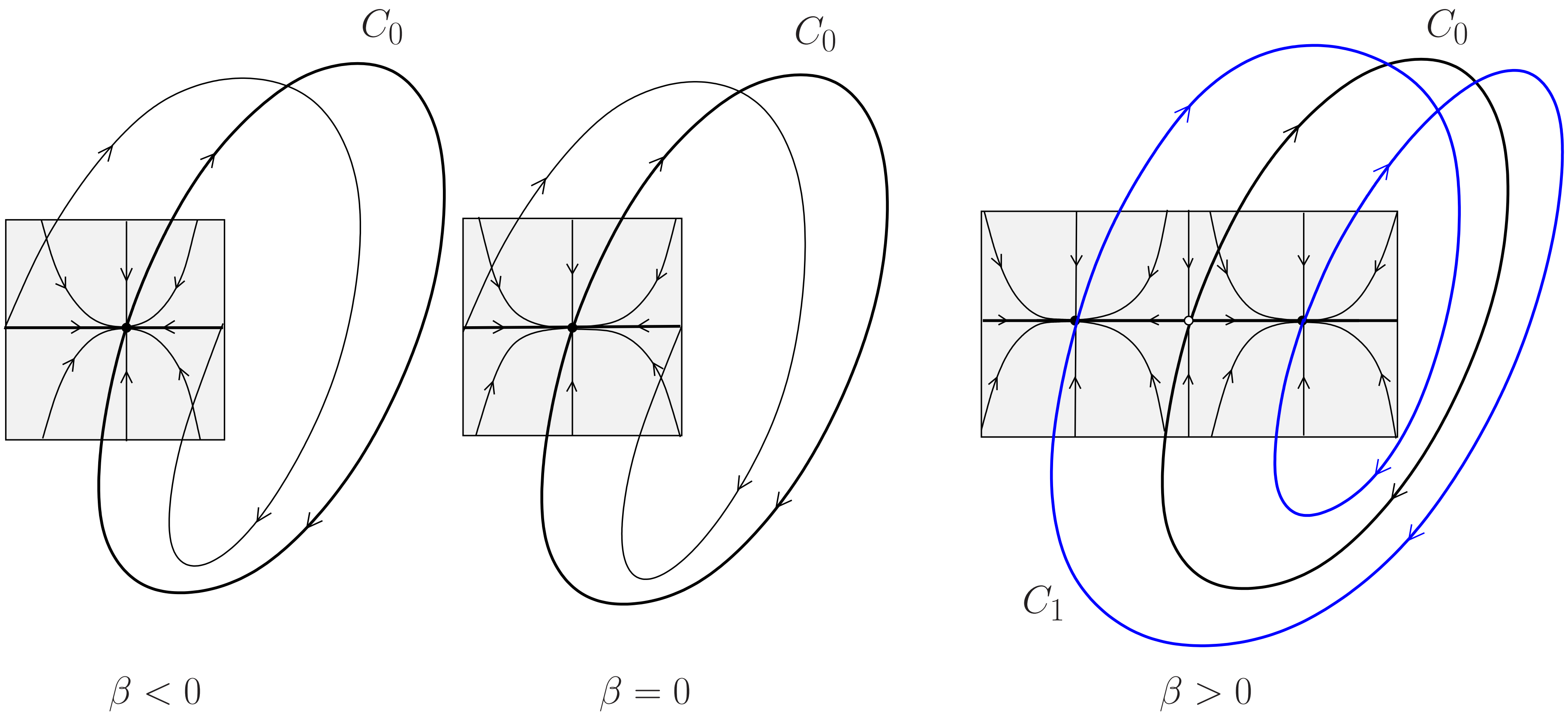
• **Fold bifurcation of cycles:** $\mu_1 = 1$ $(m_c = 1)$

If $b \neq 0$ then the restriction of the Poincaré map to its $W^c(\alpha)$ is locally topologically equivalent to $\xi \mapsto \tilde{\xi} = \xi + \beta(\alpha) + a\xi^2$.



- **Period-doubling:** $\mu_1 = -1$ $(m_c = 1)$

If $c \neq 0$ then the restriction of the Poincaré map to its $W^c(\alpha)$ is locally topologically equivalent to $\xi \mapsto \tilde{\xi} = -(1 + \beta(\alpha))\xi + c\xi^3$.

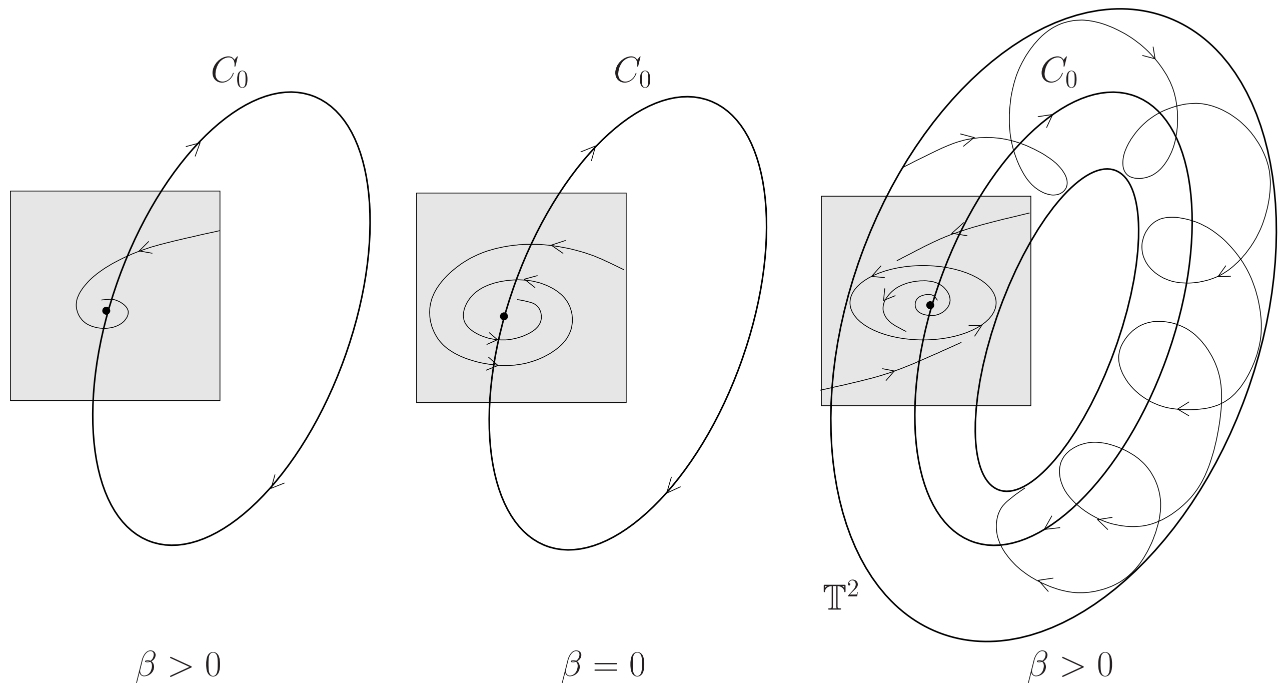


- **Torus:** $\mu_1 = -1$ $(m_c = 1)$

If $d(0) \neq 0$ and $e^{ik\theta} \neq 1$ for $k = 1, 2, 3, 4$, then the restriction of the Poincaré map to its $W^c(\alpha)$ is locally smoothly equivalent to

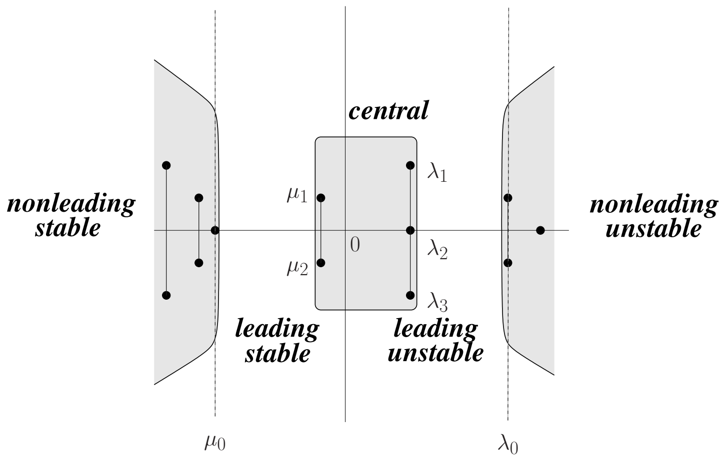
$$\begin{pmatrix} \rho \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} \rho(1 + \beta(\alpha) + d(\alpha)\rho^2) \\ \varphi + \theta(\alpha) \end{pmatrix} + \begin{pmatrix} R(\rho, \varphi, \alpha) \\ S(\rho, \varphi, \alpha) \end{pmatrix},$$

where $R = O(\rho^4)$, $S = O(\rho^2)$



## Codim1 bifurcations of homoclinic orbits to equilibria

- **Homoclinic orbit to a hyperbolic equilibrium**:

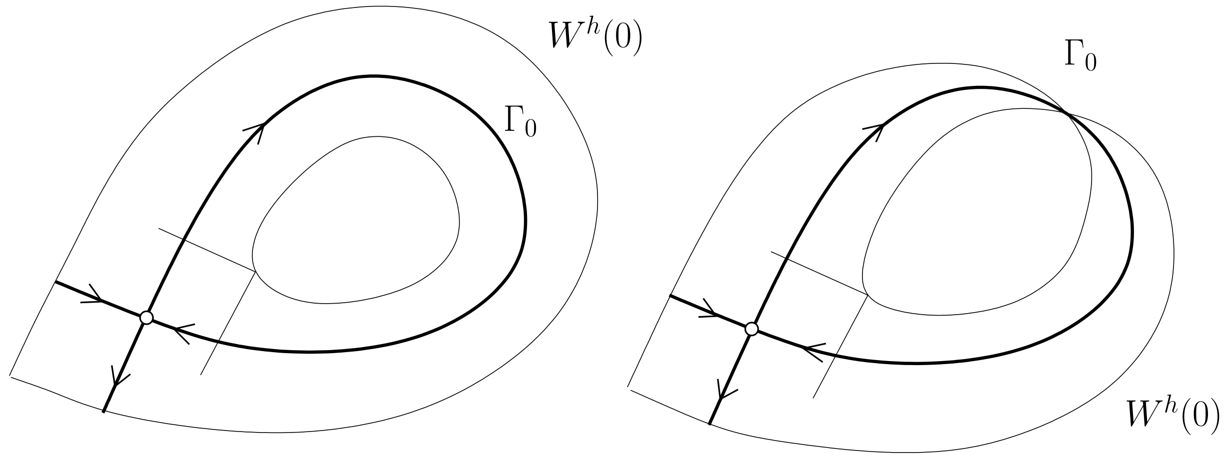



**Definition 3 Saddle quantity** $\sigma = \Re(\mu_1) + \Re(\lambda_1)$.

**Theorem 5 (Homoclinic Center Manifold)** *Generically, there exists an invariant finitely-smooth manifold* $W^h(\alpha)$ *that is tangent to the central eigenspace at the homoclinic bifurcation.*

**Saddle homoclinic orbit:** $\sigma = \mu_1 + \lambda_1$

Assume that $\Gamma_0$ approaches $X_0$ along the leading eigenvectors.



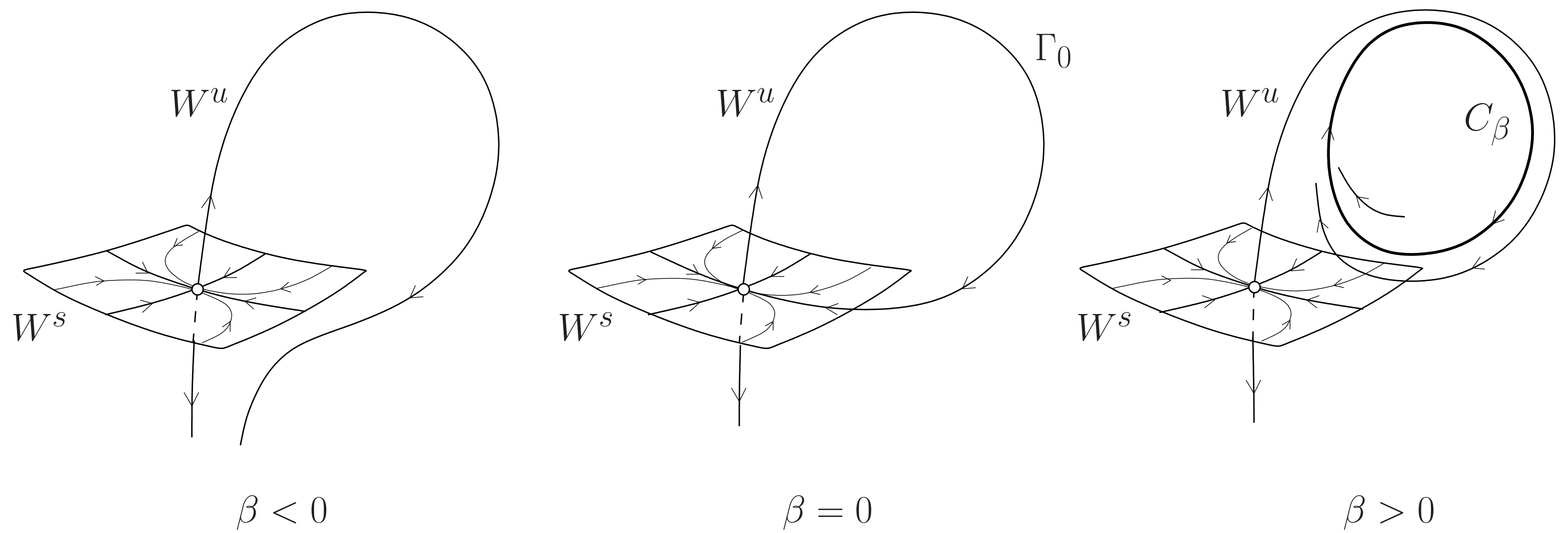

The Poincaré map near $\Gamma_0$:

$$\xi \mapsto \tilde{\xi} = \beta + A\xi^{-\frac{\mu_1}{\lambda_1}} + \dots$$

where generically $A \neq 0$, so that a unique hyperbolic cycle bifurcates from $\Gamma_0$ (stable in $W^h$ if $\sigma < 0$ and unstable in $W^h$ if $\sigma > 0$).

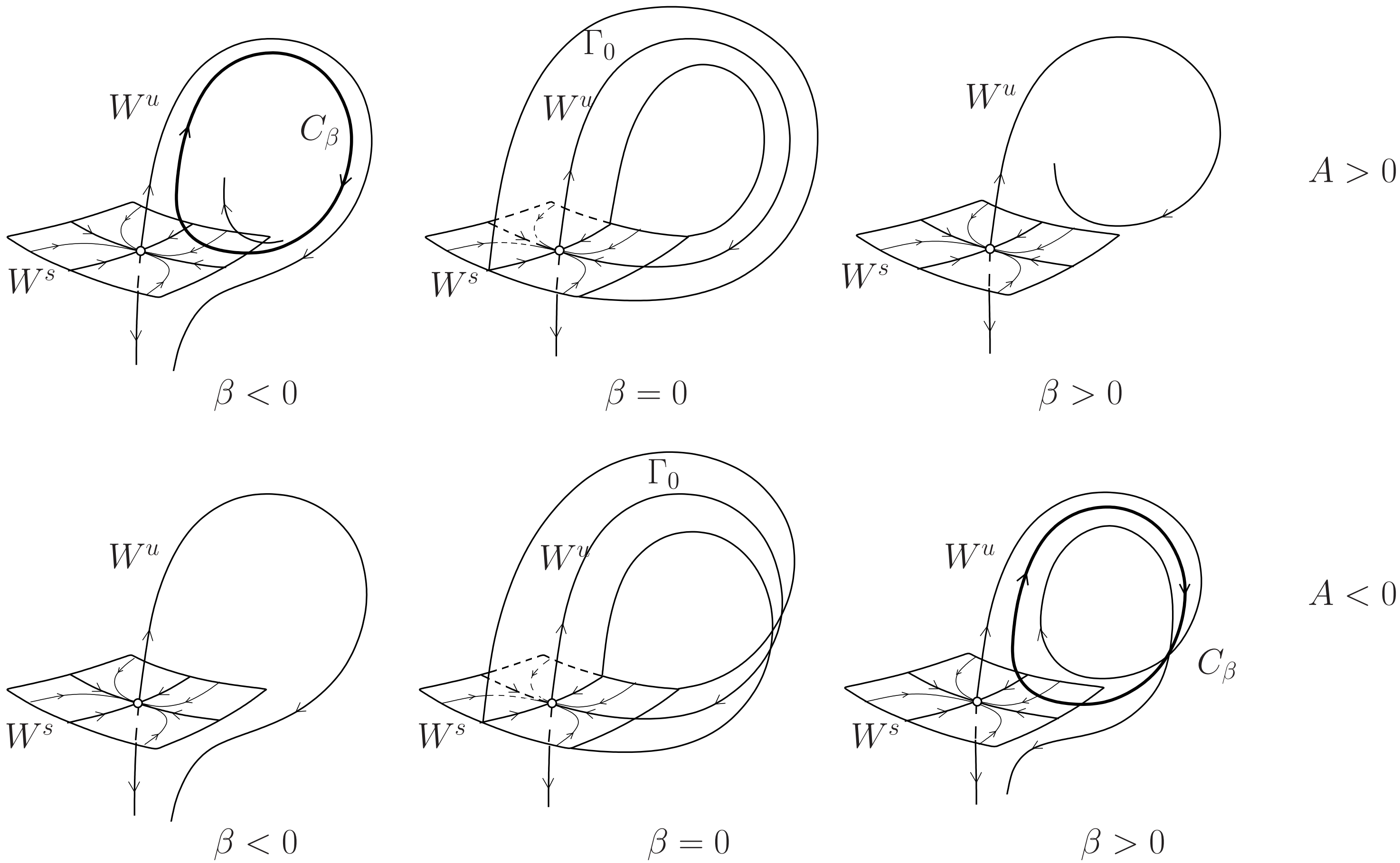
**3D saddle homoclinic bifurcation with $\sigma < 0$:**

Assume that $\mu_2 < \mu_1 < 0 < \lambda_1$ (otherwise reverse time: $t \mapsto -t$).



**3D saddle homoclinic bifurcation with $\sigma > 0$:**

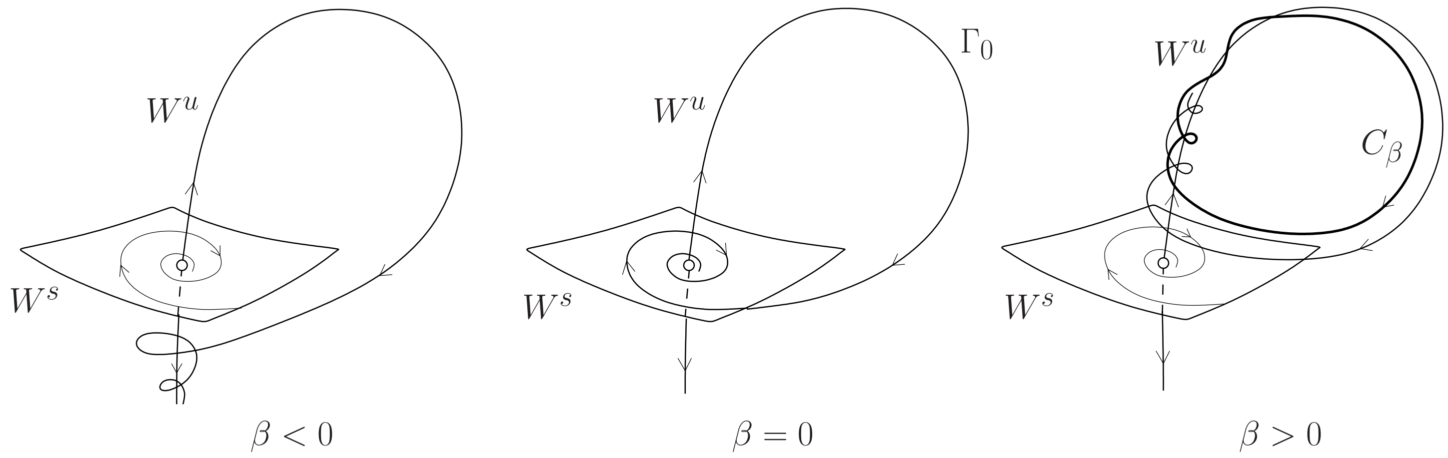
Assume that $\mu_2 < \mu_1 < 0 < \lambda_1$ (otherwise reverse time: $t \mapsto -t$).



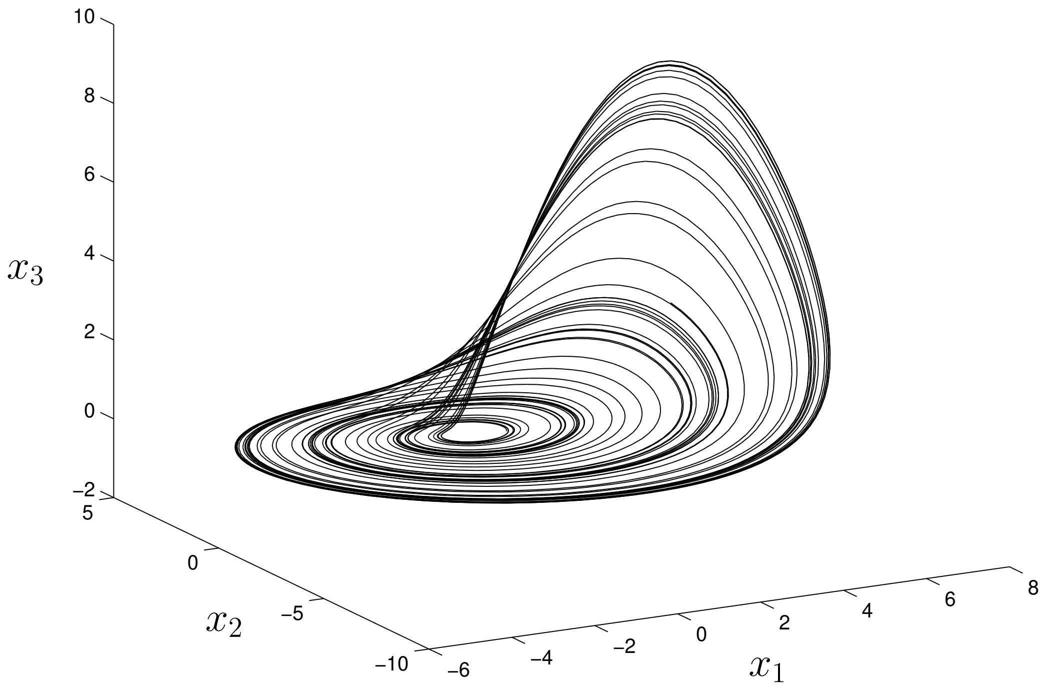
**Saddle-focus homoclinic orbit:** $\sigma = \Re(\mu_1) + \lambda_1$

**3D saddle-focus homoclinic bifurcation with $\sigma < 0$:**

Assume that $\Re(\mu_2) = \Re(\mu_1) < 0 < \lambda_1$ (otherwise reverse time: $t \mapsto -t$).



**3D saddle-focus homoclinic bifurcation with $\sigma > 0$:**

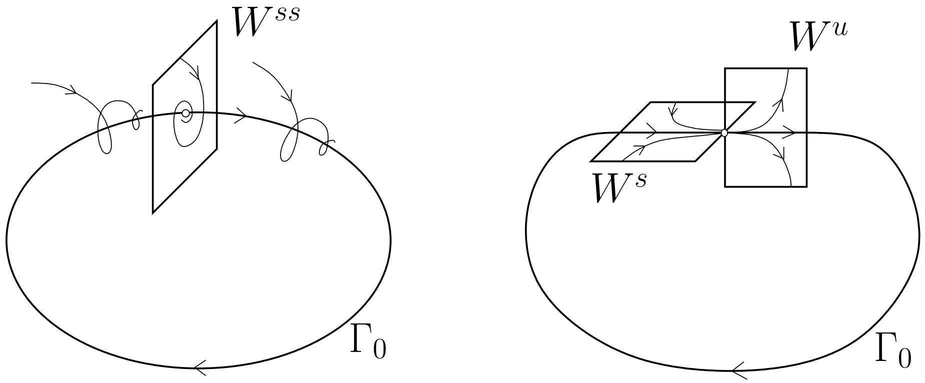



**CHAOTIC INVARIANT SET**

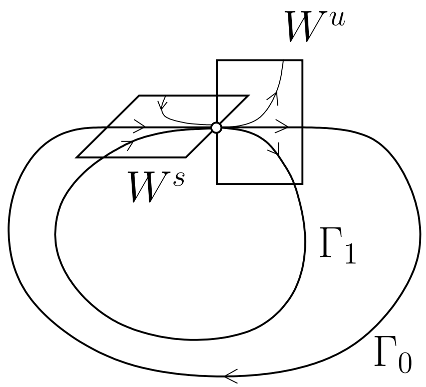
**Focus-focus homoclinic orbit**: $\sigma = \Re(\mu_1) + \Re(\lambda_1)$

**CHAOTIC INVARIANT SET**

- **Homoclinic orbit(s) to a non-hyperbolic equilibrium**

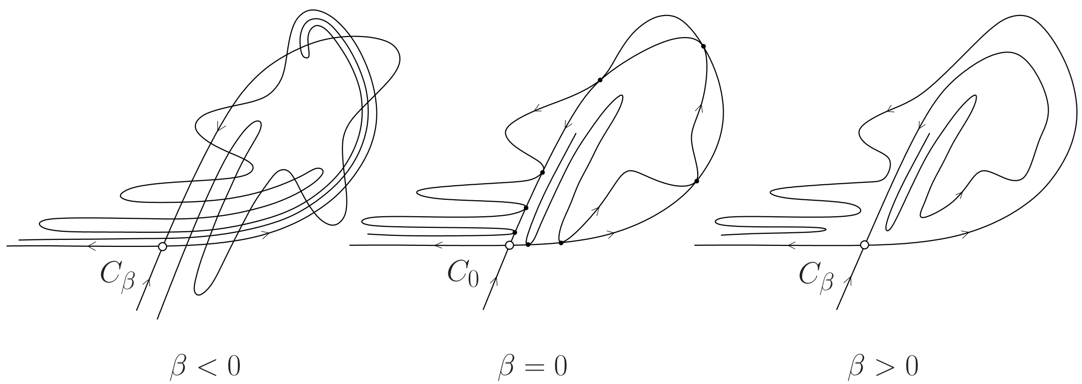



**One homoclinic orbit**: $\Rightarrow$ a unique hyperbolic **cycle**

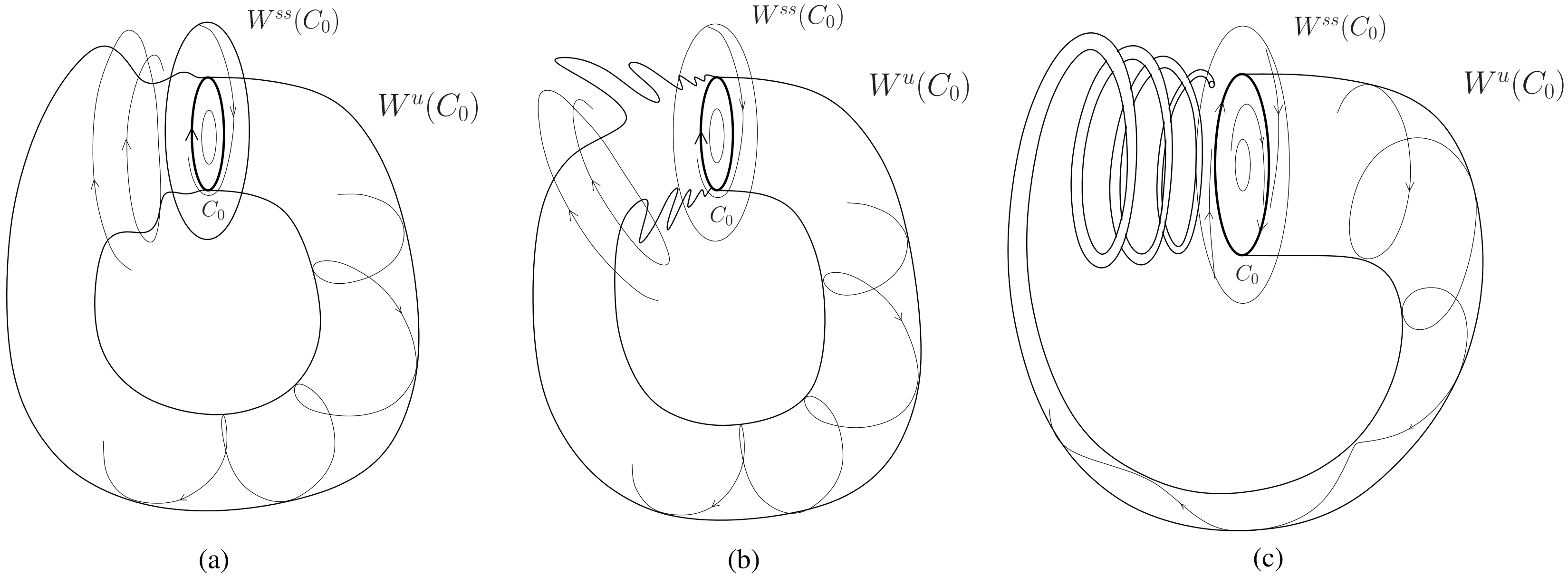



**Several homoclinic orbits**: $\Rightarrow$ **CHAOTIC INVARIANT SET**

- **Some other cases**




**Homoclinic tangency of a hyperbolic cycle: $\Rightarrow$ CHAOS**




**Homoclinics to nonhyperbolic cycle: $\Rightarrow$ torus/CHAOS/cycle**

## Example: Bifurcations in a food chain model

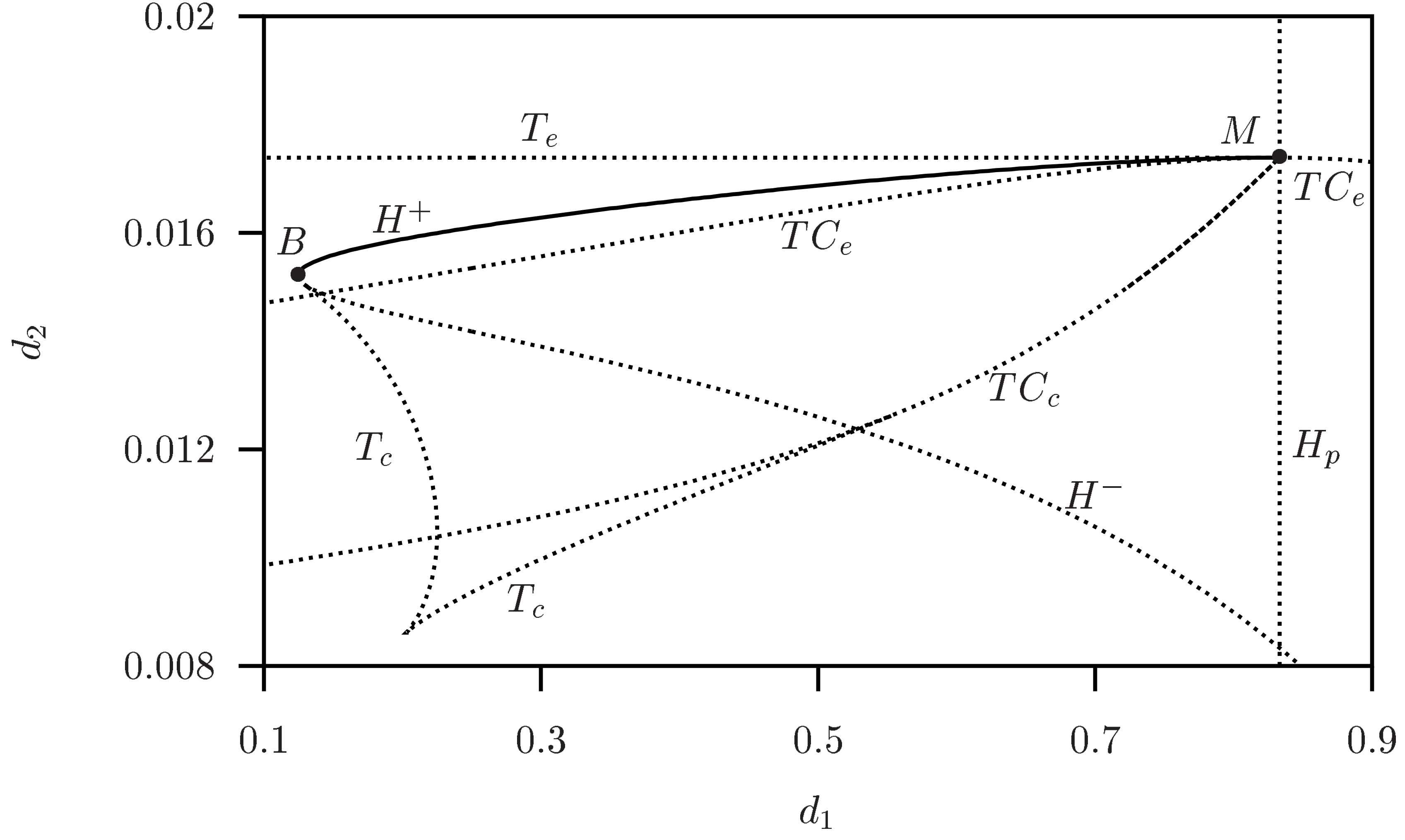
- The tri-trophic food chain model by Hogeweg & Hesper (1978):

$$\begin{cases} \dot{x}_1 = rx_1\left(1 - \frac{x_1}{K}\right) - \frac{a_1 x_1 x_2}{1 + b_1 x_1}, \\ \dot{x}_2 = e_1 \frac{a_1 x_1 x_2}{1 + b_1 x_1} - \frac{a_2 x_2 x_3}{1 + b_2 x_2} - d_1 x_2, \\ \dot{x}_3 = e_2 \frac{a_2 x_2 x_3}{1 + b_2 x_2} - d_2 x_3, \end{cases}$$

where

| | |
|---|---|
| $x_1$ | prey biomass |
| $x_2$ | predator biomass |
| $x_3$ | super−predator biomass |

## Local bifurcations



**Local and key global bifurcations**