TUTORIAL II:

One-parameter bifurcation analysis of equilibria with

MATCONT

Yu.A. Kuznetsov
Department of Mathematics
Utrecht University
Budapestlaan 6
3508 TA, Utrecht

N. Neirynck
Department of Applied Mathematics,
Computer Science and Statistics
Ghent University
Krijgslaan 281, S9
B-9000, Ghent

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This session was tested on MATCONT7.1 with MATLAB2019b. It is devoted to the numerical continuation of equilibria in systems of autonomous ODEs depending on one parameter

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R},$$

and detection of their bifurcations.

1 An ecological model with multiple equilibria and limit points

Consider the following system appearing in mathematical ecology:

$$\begin{cases} \dot{x} &= rx(1-x) - \dfrac{xy}{x+a}, \\ \dot{y} &= -cy + \dfrac{xy}{x+a} - \dfrac{dy^2}{y^2 + b^2}. \end{cases} \tag{1}$$

Fix

$$r = 2, \quad a = 0.6, \quad b = c = 0.25,$$

and consider d as a bifurcation parameter with initial value $d = 0.1$. The aim is to locate equilibria of (1) and study their dependence on d.

1.1 System specification

Start MATCONT, choose **Select|System|New**, and input a new ODE system – say `EcoMod` – into MATCONT as shown in Figure 1.

1.2 Locate an equilibrium by numerical integration

Select **Type|Initial point|Point** and check that the default **Type|Curve|Orbit** is selected.

In the appearing **Integrator** window, increase the integration `Interval` to 200 and set `RelTolerance` to `1e-7` and `AbsTolerance` to `1e-10` (to assure smaller integration steps).

Via the **Starter** window, input the initial point

```
x          1.2
y           1
```

and the initial values of the parameters, namely

```
r           2
a          0.6
b          0.25
c          0.25
d          0.1
```

Open a **Plot2D** window with **Window/Output|Graphic|2D plot** and a **Layout** window by clicking **MatCont|Layout** in the **Plot2D** window. Select `x` and `y` as variables along the corresponding axes and the following visibility limits:

```
Abscissa:        0        1.2
Ordinate:        0        2.0
```

as shown in Figure 2.

Start **Compute|Forward**. You will get an orbit converging to an equilibrium, see Figure 3

To see how the coordinates vary along the orbit, select **Window/Output|Numeric** which opens a **Numeric** window. Then input another initial point in the **Starter** window
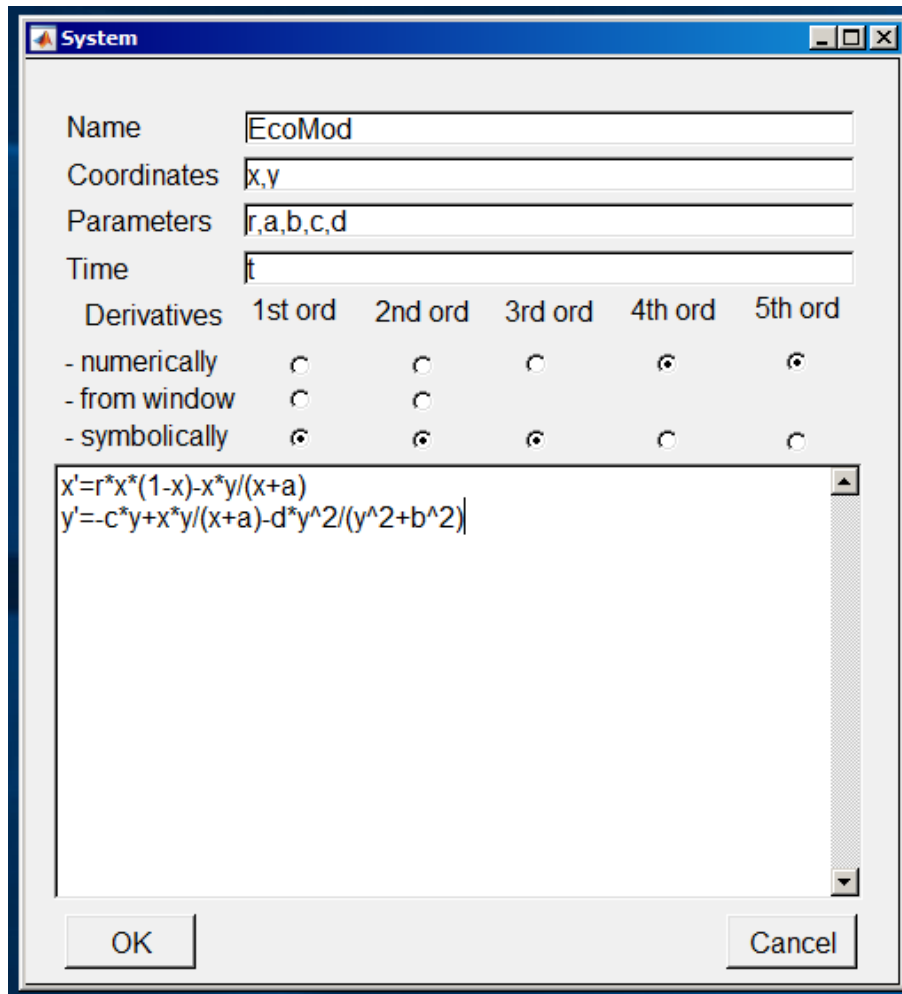
Figure 1: The ecological model.

```
x          1.0
y          0.001
```

and **Compute|Forward**. You get the second orbit from Figure 3 tending to the same equilibrium, whose (approximate) coordinates can be seen in the **Numeric** window shown in Figure 4. Close both the **Numeric** window and the **Control** window.

1.3 Equilibrium curve

We can continue the equilibrium found by integration with respect to the parameter d. Open the **Data Browser** window by pressing **Select|Initial Point** in the main MATCONT window. Then click on `Last Point` and **Select Point**, as shown in Figure 5.

To tell MATCONT to continue an equilibrium curve, we have to specify the initial point type and the curve type. Selecting **Type|Initial Point|Equilibrium** we set the point type `EP` and the same (default) curve type, so that we prepare to compute the `EP_EP` equilibrium curve, as indicated in the main MATCONT window. The **Starter** window is now modified and the **Integrator** window is replaced by a **Continuer** window.

The initial values of the parameters and the equilibrium coordinates are visible in the **Starter** window. Press the radio-button next to parameter `d`, indicating that it will be a bifurcation
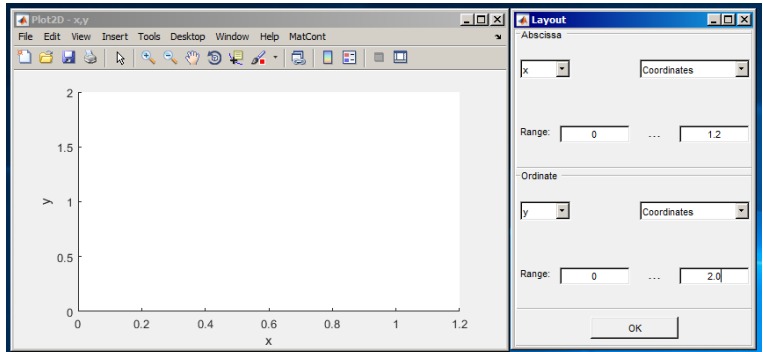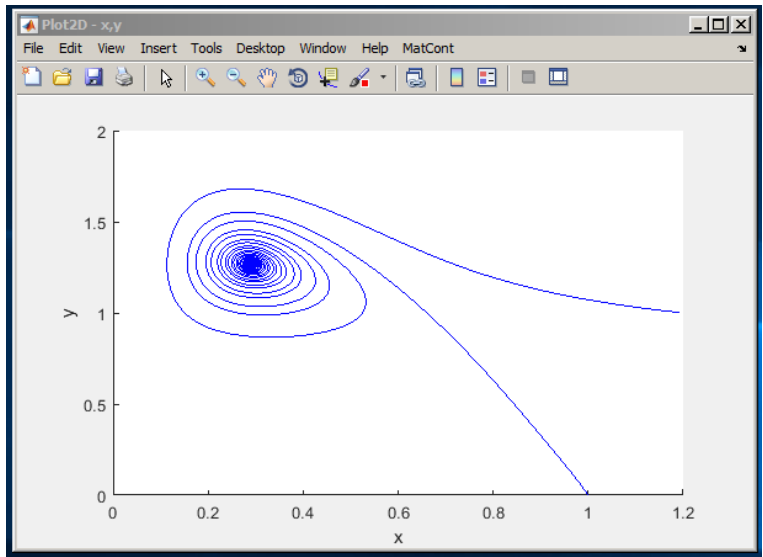
Figure 2: Opening a **Plot2D** window.

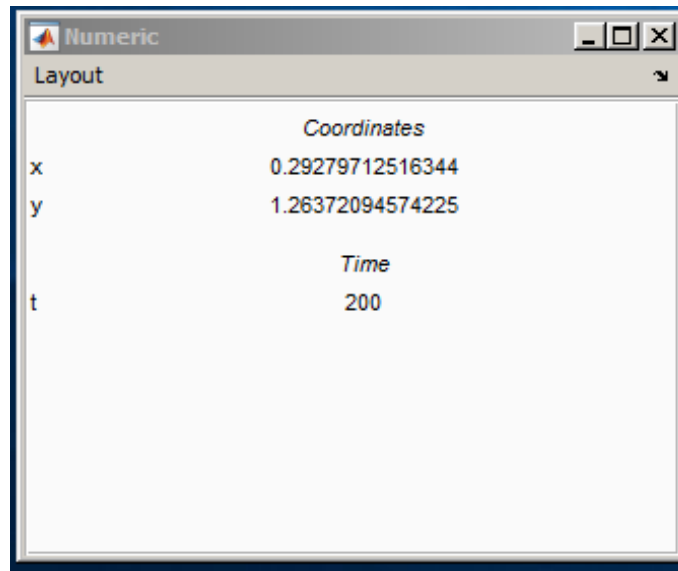Figure 3: Phase orbits approaching a stable equilibrium for $d = 0.1$.
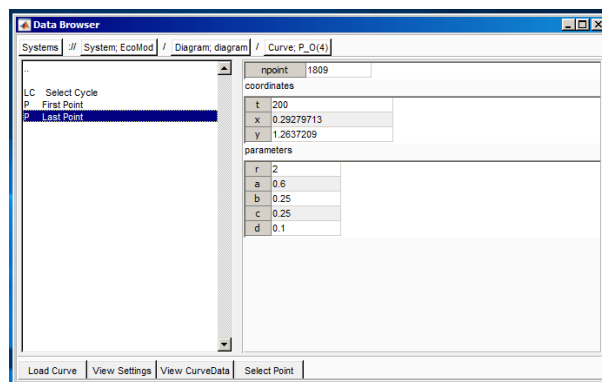
Figure 4: The coordinates of the stable equilibrium.

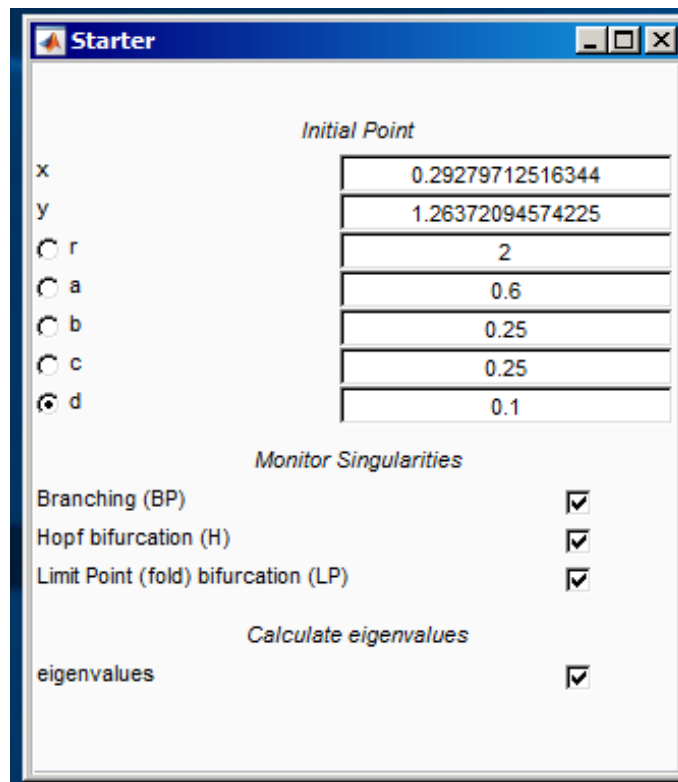Figure 5: **Data Browser** window with `Last Point` selected.

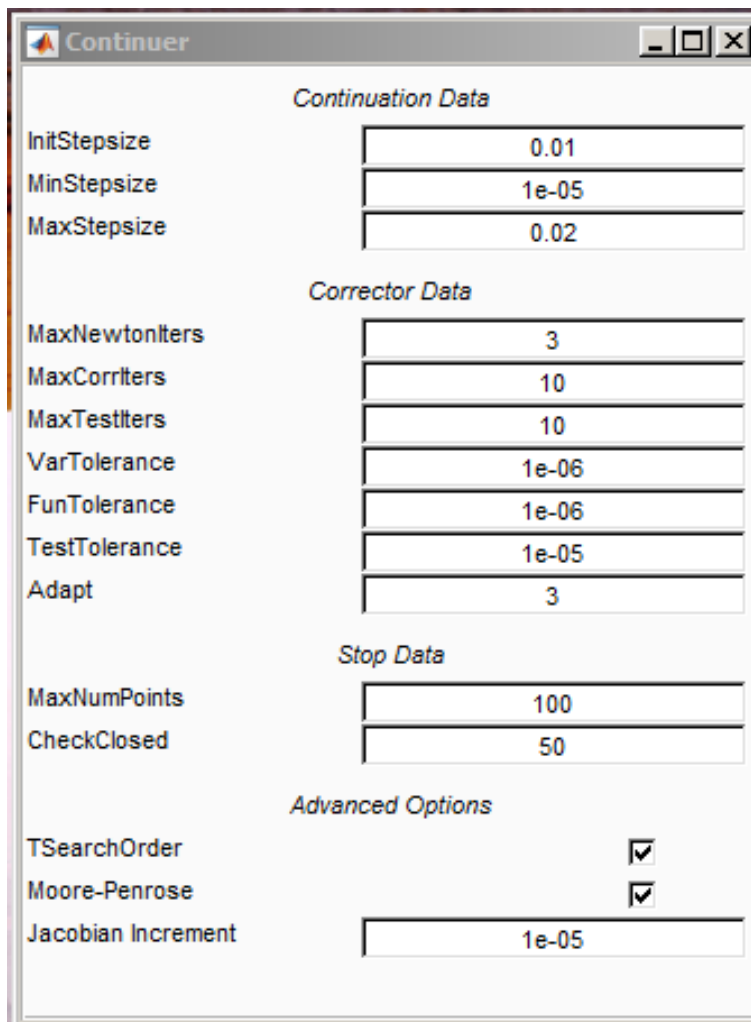Figure 6: **Starter** window for the equilibrium continuation: Parameter d is activated.

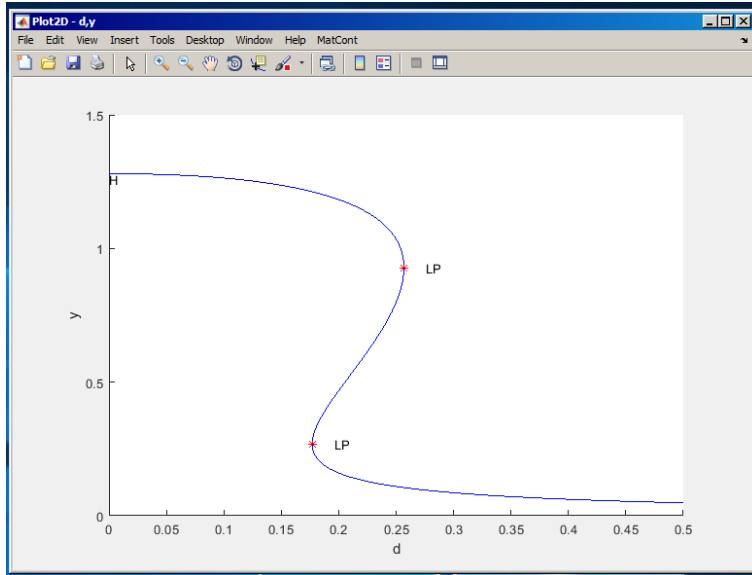Figure 7: Continuation parameters in the **Continuer** window.

Figure 8: Equilibrium manifold in the (d, y)-plane: LP's denote limit points, H is a Hopf point.

parameter (i.e. *activate* parameter d). The resulting **Starter** window is shown in Figure 6. There you can also see which singularities will be monitored (all), and whether the equilibrium eigenvalues will be computed (yes).

In the **Continuer** window, several default numerical parameters related to the continuation are listed. Change only

```
MaxStepsize        0.02
MaxNumPoints       100
```

(see Figure 7).

Use **MatCont|Layout** menu in the **Plot2D** window to select the parameter d as abscissa and the coordinate y as ordinate with the visibility limits:

```
Abscissa:          0              0.5
Ordinate:          0              1.5
```

and clear the window by pressing **MatCont|Clear** in the **Plot2D** window.

Now click **Compute|Forward** to get part of a sigmoidal curve as in Figure 8. During the computations a **Control** window is opened and the continuation is stopped each time when a singularity is found, see Figure 9. Then press **Resume** in the **Control** window to advance the continuation.

Along the forward branch, MATCONT stoped at two *limit points* labeled LP. After finishing this continuation press the **View Result** button in the **Control** window. This opens a **Data Browser** window. Press the **View CurveData** button at the bottom of the **Data Browser** window. This opens a spreadsheet view of all computational values that will be stored as data of the curve. Figure 10 shows part of this spreadsheet; the **Curve Data** correspond to the x,v,s,h,f output of the continuer as described in the MATCONT manual.

Now click **Compute|Backward** to complete the sigmoid. Along the backward branch, MATCONT stops at a *Hopf point* labeled H. Terminate the computation by pressing **Stop** there.

To determine stability of the equilibria and read the bifurcation parameter values, open **Window/Output|Numeric** and select **Layout** in the **Numeric** window.

The **Layout** menu offers the choice to visualise during the continuation the coordinates (i.e. state variables), active parameters, testfunctions, eigenvalues, current stepsize, userfunc-
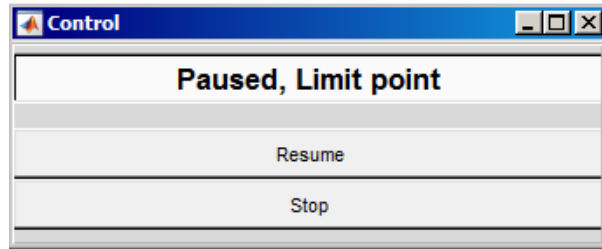
8

Figure 9: The **Control** window when paused at an LP point.

Figure 10: Spreadsheet view of the **Curve Data**.

tions (when userfunctions are present), and current number of computed points. Check only the `Coordinates`, `Parameters` and `Eigenvalues`.

This makes the state variables, active parameters and eigenvalues of the equilibrium visible in the **Numeric** window (upon resizing it).
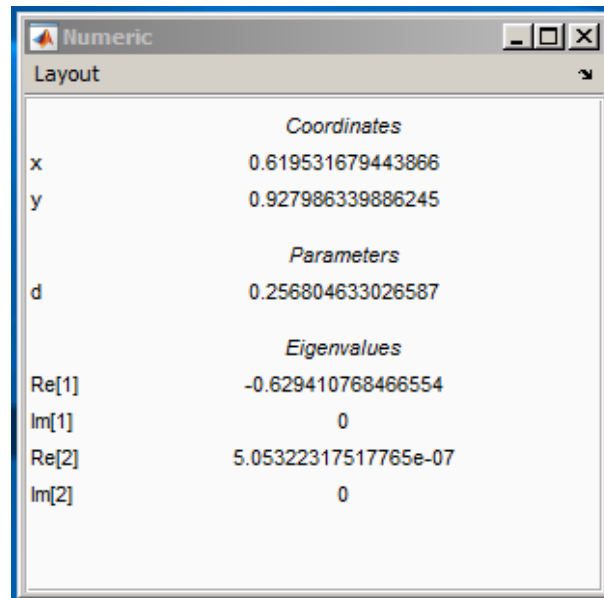

Figure 11: **Numeric** window at the first limit point.

Clear the **Plot2D** window with **MatCont|Clear** and repeat **Compute|Forward**. In the **Numeric** window you can read the LP parameters:

$$d_2 = 0.256805\ldots, \quad d_1 = 0.176927\ldots$$

(see Figures 11 and 12; your actual numerical values might slightly differ). One eigenvalue (close to) zero is also present in each case. The existence of three equilibria for $d \in (d_1, d_2)$ is evident from the figure.

In the MATLAB Command Window, the value of the *fold normal form* coefficient a is shown at each limit point:

```
label = LP, x = ( 0.619532 0.927986 0.256805 )
a=-5.311546e-01
label = LP, x = ( 0.911266 0.268200 0.176927 )
a=5.681504e-01
```

Finally, clear the **Plot2D** window once more and recompute the equilibrium curve forward with **Options|Suspend Computation** mode `At Each Point` selected as shown in Figure 13. Resume computations after each computed point and monitor the eigenvalues in the **Numeric** window. You should see that, indeed, the upper and lower branches correspond to linearly stable equilibria (with all Re $\lambda_i < 0$), while the middle branch correspond to a linear saddle (with $\lambda_1 < 0 < \lambda_2$). Since $a \neq 0$ within numerical accuracy, the equilibrium curve is approximately a (scaled) parabola near each of the limit points.

Restore the original pause option (`At Special Points`) via **Options|Suspend Computation** and close the **Numeric** window.
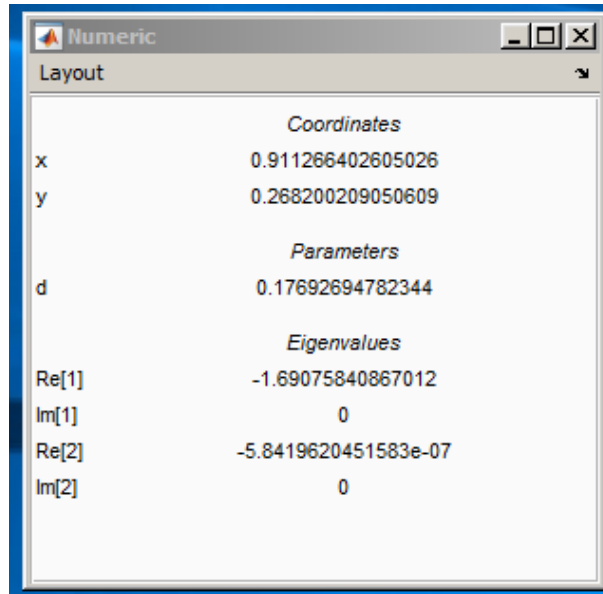
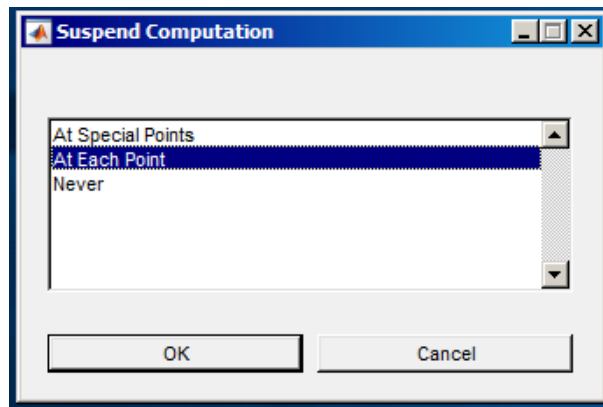Figure 12: **Numeric** window at the second limit point.

Figure 13: Pause at each computed point.

1.4 Phase portrait

To verify our conclusions, compute several orbits to get a phase portrait of the system at $d = 0.2$ as in Figure 14. To open a **Plot2D** window with previous visibility layout, use **Window/Output| Graphic|Previous|Plot2D**. Start always at $x = 1.2$ but with different values for y.
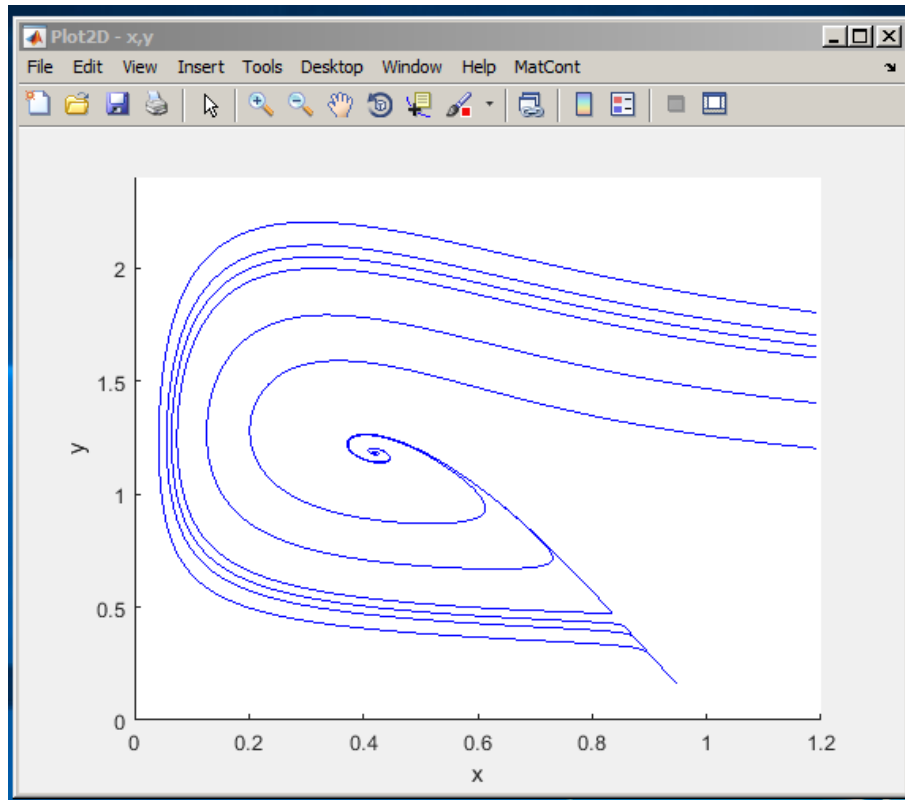
Figure 14: Two stable equilibria (a focus and a node) and a saddle in the ecological model at $d = 0.2$. The stable manifold of the saddle separates the domain of attraction of the focus from that of the node.

2 Limit and branching points in a discretization of Bratu-Gelfand PDE

Consider the following evolution problem for $u = u(x, t)$ with $x \in [0, 1], t \geq 0$:

$$u_t = u_{xx} + \alpha e^u, \quad u(0, t) = u(1, t) = 0 \text{ for all } t, \tag{2}$$

where α is a real parameter. It is called the *Bratu-Gelfand* problem. In this section we study the behaviour of a stationary solution of (2) as a function of α using a finite-difference approximation over an equidistant mesh. One such time-independent solution is obvious: $u \equiv 0$ for $\lambda = 0$.

2.1 Discretization

Characterize a solution to (2) at time t by its values $u_k(t) = u(x_k, t)$ at the uniformly distributed mesh points

$$x_k = kh, \quad h = \frac{1}{N + 1}, \quad k = 0, 1, \ldots, N, N + 1,$$

for some $N > 1$. Then approximate the spatial derivative in (2) by finite differences at the inner points

$$\dot{u}_k(t) = \frac{u_{k-1}(t) - 2u_k(t) + u_{k+1}(t)}{h^2} + \alpha \exp(u_k(t)) = 0, \quad k = 1, 2, \ldots, N, \tag{3}$$

and add the boundary conditions in the form

$$\begin{cases} u_0(t) & = & 0, \\ u_{N+1}(t) & = & 0. \end{cases} \tag{4}$$

After elimination of $u_0(t)$ and $u_{N+1}(t)$ and introduction of a rescaled time τ such that $t = h^2 \tau$, the equations (3) and (4) take the form of the ODE-system

$$\dot{U} = F(U, \lambda), \tag{5}$$

where $U = (u_1, u_2, \ldots, u_N)$ and $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is given by (3) (with $u_0 = u_n = 0$) and

$$\lambda := h^2 \alpha = \frac{\alpha}{(N+1)^2} \ .$$

An equilibrium of this system approximates a stationary solution to (2) with $O(h^2)$-accuracy.

In this section, we will study a crude spatial discretization of (2) with only two internal points (i.e. $N = 2$ and $h = \frac{1}{3}$), so that (5) becomes

$$\begin{cases} \dot{u}_1 & = & -2u_1 + u_2 + \lambda e^{u_1}, \\ \dot{u}_2 & = & u_1 - 2u_2 + \lambda e^{u_2}, \end{cases} \tag{6}$$

where $\lambda = h^2 \alpha = \frac{1}{9}\alpha$. The system (6) has $\mathbb{Z}_2$-*symmetry*: It is invariant under the involution $(u_1, u_2) \mapsto (u_2, u_1)$.

We will compute the equilibrium manifold of (6) in the (u_1, u_2, λ)-space starting from point $(0, 0, 0)$ corresponding to the trivial stationary solution of (2), namely $u(x, t) \equiv 0$.

2.2 System specification

Input a new ODE system into MATCONT, namely

```
U1'=-2*U1+U2+LAMBDA*exp(U1)
U2'=U1-2*U2+LAMBDA*exp(U2)
```

and choose to generate the derivatives of order 1 and 2 symbolically.

2.3 Symmetric equilibrium branch

Select **Type|Initial Point|Equilibrium**. Two windows will appear: **Starter** and **Continuer** windows corresponding to the equilibrium continuation (curve type EP_EP). In the **Starter** window, activate the parameter LAMBDA. Since (6) has an equilibrium $u_1 = u_2 = 0$ for $\lambda = 0$, no further changes to the **Starter** parameters are required.

Open a **Plot2D** window and select the parameter LAMBDA and the coordinate U1 as abscissa and ordinate, respectively, with the visibility limits

```
Abscissa:           0          0.5
Ordinate:           0          8.0
```

To monitor the stability of the equilibrium, open a **Numeric** window and activate the eigenvalues to be visible via the **Numeric|Layout** window.

Click **Compute|Forward**. The first bifurcation point, namely a *limit point* LP, will be located at $\lambda = 0.367879\ldots$. In the MATLAB Command Window, you can read the following message:

```
label = LP, x = ( 1.000001 1.000001 0.367879 )
a=3.535537e-01
```

that gives the critical equilibrium coordinates and the critical parameter value, as well as the *quadratic normal form coefficient* $a = 0.35355\ldots$ at LP. Thus, the limit point is nondegenerate and the equilibrium manifold near LP looks like a parabola.

Further computation detects a *neutral saddle* labeled H at $\lambda = 0.270671...$, where the equilibrium has eigenvalues $\mu_{1,2} = \pm 1$, so that their sum is zero. In the MATLAB Command Window, you can read the following message:

```
label = H , x = ( 2.000000 2.000000 0.270671 )
Neutral saddle
```

This is not a bifurcation point for the equilibrium, since it is a hyperbolic saddle.

Resume computations to get the second bifurcation point at $\lambda = 0.14936122...$, namely a *branching point* labelled BP, with the following message in the MATLAB Command Window

```
label = BP, x = ( 3.000000 3.000000 0.149361 )
```

Extend the equilibrium curve further until you leave the window. You should get Figure 15.
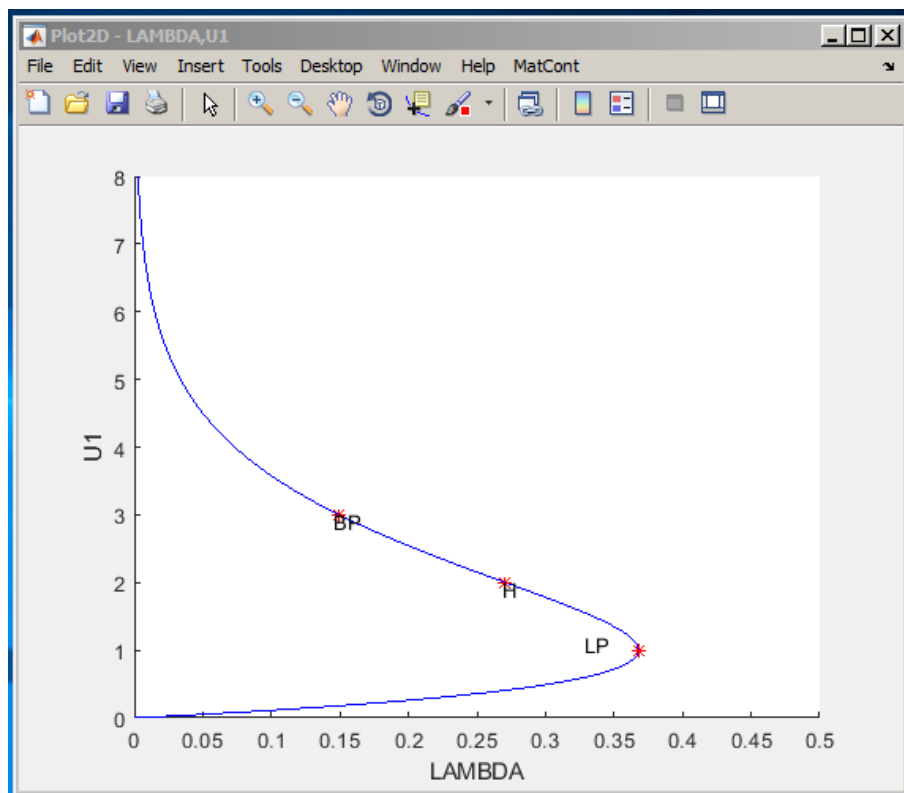

Figure 15: The curve of symmetric equilibria of (6) with limit point LP, neutral saddle H, and branching point BP.

To get more information on the BP point, single-click at the label and the message

```
*** EP_EP(1): Branch point (BP)
        npoint:   53
```

will appear in the MATLAB command window. It tells us that the BP point is a `Branch point` that was detected along the `EP_EP(1)` curve as its 53−th point.

To get more information on the H point, single-click at the label and the message

```
*** EP_EP(1): Neutral Saddle Equilibrium (H)
        npoint:        37
        1st Lyapunov:  Neutral saddle
```

will appear in the MATLAB command window. It tells us the H point is a `Neutral Saddle` that was detected along the `EP_EP(1)` curve as its 37−th point. Since it is not a Hopf point, it does not have a 1st Lyapunov coefficient.

To get more information on the LP point, single-click at the label and the message

```
*** EP_EP(1): Limit point (LP)
        npoint:  22
        a:       3.5355366e-01
```

will appear in the MATLAB command window. It tells us the LP point is a `Limit point` that was detected along the `EP_EP(1)` curve as its 22−th point. Is normal form coefficient a is nonzero, so it is nondegenerate.

Notice that along the whole computed branch we have $u_1 = u_2$, i.e. it is composed of *symmetric equilibria*. Check by looking at the eigenvalues in the **Numeric** window that only the lower part of the branch is stable, while it is a saddle between LP and BP, and a repeller above it.

2.4   Symmetry breaking

Let us compute the second branch passing through the branching point BP. Double-click with the computer mouse at the BP point in the **Plot2D** window presented in Figure 15. This opens a small **Select BP as a new initial point** window, see Figure 16. Click **Yes** and the (u_1, u_2, λ)-values corresponding to BP will be loaded and by default the initializer `init_BP_EP` to the new equilibrium curve is loaded as well.
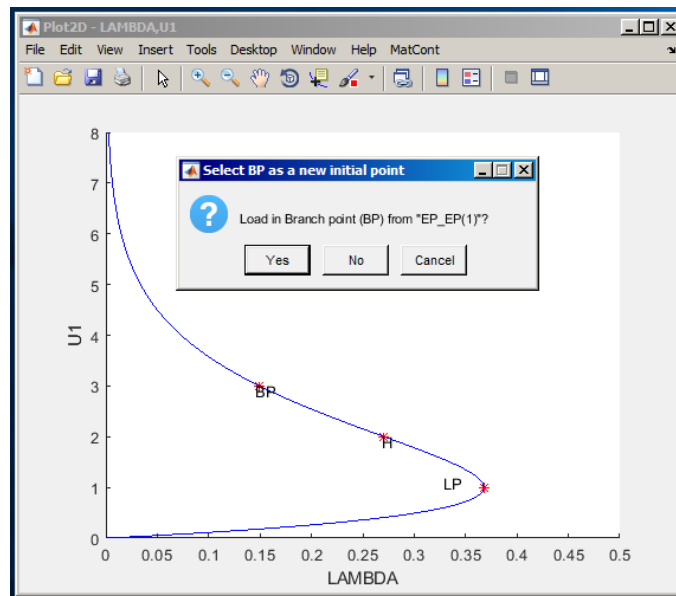

Figure 16: BP selection.

In the appearing **Starter** window, untick monitoring all singularities as in Figure 17.
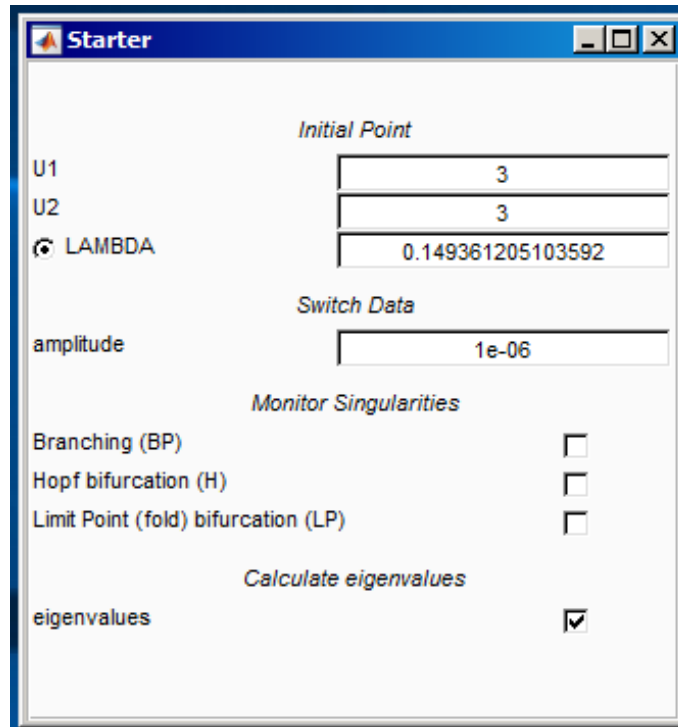
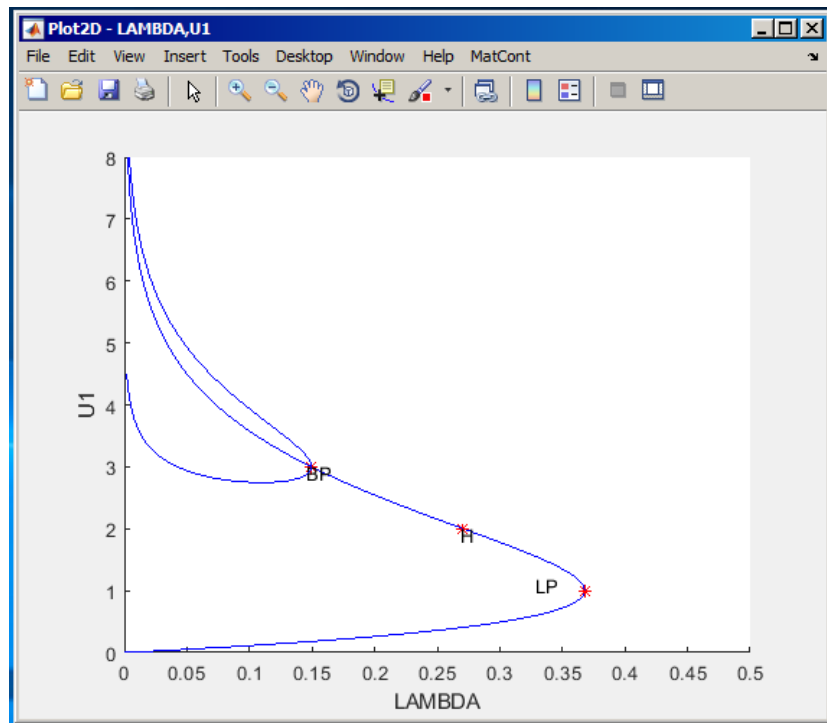Figure 17: **Starter** window to switch branches at BP.

Figure 18: Two equilibrium branches passing through a branch point BP.

Leave the **Continuer** window as it is and click **Compute|Forward** and
Compute|Backward. You will get Figure 18. Thus, the branching point BP corresponds, actually, to a *pitchfork bifurcation*: Two more branches of equilibria bifurcate vertically from this point. Moreover, each branch is composed of equilibria which are not symmetric, $u_1 \neq u_2$. However, one branch is mapped into the other by the involution. This phenomenon is called the *symmetry breaking*: We got non-symmetric equilibrium solutions in a symmetric system.

Warning: We have shown numerically that a *discretization* of the Bratu-Gelfand problem (2) has multiple stationary solutions and limit and branching points. It does not imply immediately that the original PDE problem has the same properties. To verify this, one has to repeat the continuation with smaller h (i.e. bigger N) and eventually make some error estimates. It should be noted that the appearance of the pitchfork is due to the equidistant mesh with a small number of points. If more accurate discretizations of (2) are used, the pitchfork and associated branches disappear. The limit point remains and corresponds to a limit point on the branch of the stationary solutions of (2).

2.5   Stopping at a zero of a user function

Sometimes we want to stop computation at a prescribed parameter value. This can be done in MATCONT by specifying a *user function*. For example, let us compare two non-symmetric equilibria at $\lambda = 0.1$. You can see the corresponding equilibrium coordinates and eigenvalues in
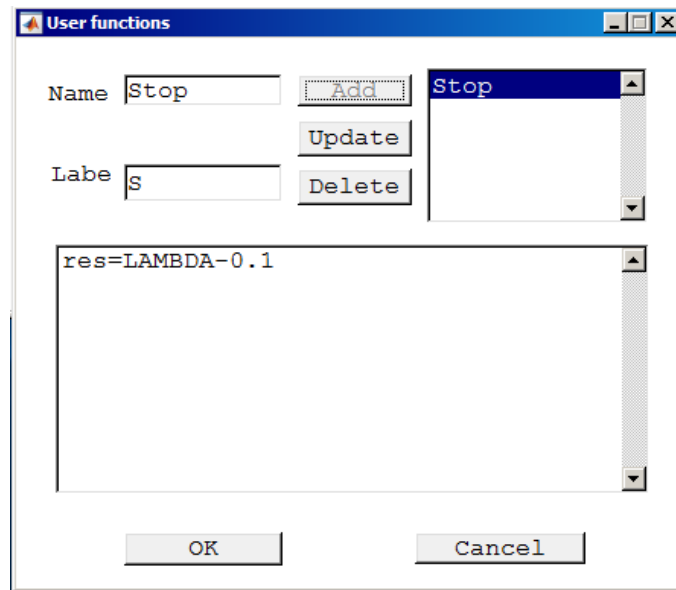
Figure 19: User function input.

the **Numeric** window.

User functions can be introduced in two different ways. One of them is by pressing **Select|System|Manage Userfunctions**. The other way is to right-click in the `Current System` field of the MATCONT main window and then selecting **Manage Userfunctions**. Use the second method. This opens a **User functions** window. Define a user function `Stop` with the associated label `S` by typing `res=LAMBDA-0.1` in the edit field of the **User functions** window (see Figure 19). Next press **Add** and then **OK** buttons. The **Starter** window will change and show a user function control. Click to activate it. If one wants to monitor the user function, the `User Functions` must also be activated in the layout of the **Numeric** window.
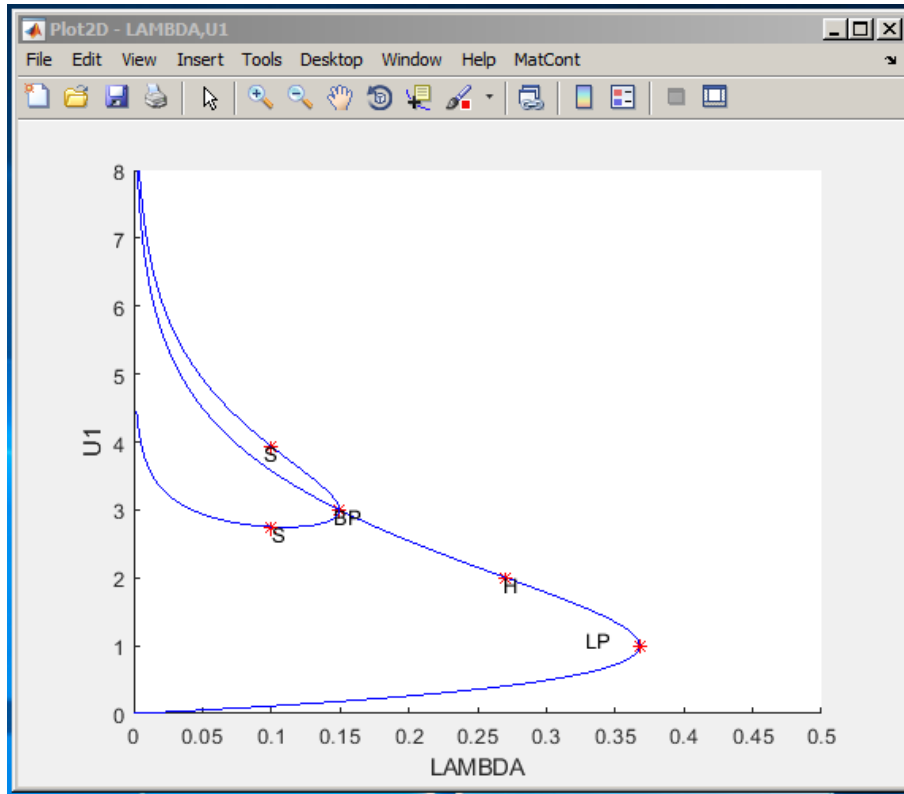
Figure 20: Zeros of a userfunction are labeled S.

Recomputing the non-symmetric branches with **Compute|Forward** and **Compute|Backward** will stop at two points labeled S, where the user-defined function Stop vanishes, see Figure 20.

3   Additional Problems

A. Compute the equilibrium manifold of the scalar ODE

$$\dot{y} = x^2 + y^2 - 1, \quad y \in \mathbb{R}, \tag{7}$$

where $x \in \mathbb{R}$ is a parameter.

B. Consider the following chemical model

$$\begin{cases} \dot{x} &= 2q_1 z^2 - 2q_5 x^2 - q_3 xy, \\ \dot{y} &= q_2 z - q_6 y - q_3 xy, \\ \dot{s} &= q_4 z - K q_4 s, \end{cases}$$

where $z = 1 - x - y - s$ and

$$q_1 = 2.5, \quad q_2 = 1.55, \quad q_3 = 10, \quad q_4 = 0.0675, \quad q_5 = 1.0, \quad q_6 = 0.1,$$

and

$$K = 2.0.$$

Given an equilibrium $(x, y, s) = (0.0032\dots, 0.8838\dots, 0.0376\dots)$, find two more positive equilibria of the system.

Hint: Compute a curve in the (x, y, s)-space defined by the first two equations:

$$\begin{cases} 2q_1 z^2 - 2q_5 x^2 - q_3 xy &= 0, \\ q_2 z - q_6 y - q_3 xy &= 0, \end{cases}$$

and detect zeros of the user-defined function $F = q_4 z - K q_4 s$ along this curve[1]

C. Find a real eigenvalue and the corresponding eigenvector of the matrix

$$B = \begin{pmatrix} -5 & 2 & -3 \\ -5 & 0 & -4 \\ 8 & -2 & 6 \end{pmatrix}$$

by continuation of the eigenvector $v_1 = (0, 0, 1)^T$ of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

corresponding to its eigenvalues $\lambda_1 = -2$.

1. Consider a one-parameter family of matrices

$$C(a) = aB + (1 - a)A, \quad C(0) = A, \ C(1) = B,$$

and the following continuation problem in the (v, λ, a)-space:

$$\begin{cases} C(a)v - \lambda v &= 0, \\ v^T v - 1 &= 0. \end{cases}$$

Explain, how it can be used to solve the formulated problem.

2. Setup an auxiliary 4D-system of ODEs

$$\begin{cases} \dot{v} &= C(a)v - \lambda v, \\ \dot{\lambda} &= v^T v - 1 \end{cases}$$

and continue its equilibrium manifold in the (v, λ, a)-space. Use $(v_1, \lambda_1, 0)$ as the initial point. Stop when $a = 1$ using a user-defined function.

3. Continue other two eigenvectors of A and produce a plot tracing all three real eigenvalues in the (a, λ)-plane. What happens in the limit points of these curves?

4. Can you setup a continuation problem for (real and imaginary) parts of complex eigenvalues and eigenvectors of $C(a)$?

D. Study equilibria of the following *complex* ODE:

$$\dot{z} = \alpha + z^2, \quad z \in \mathbb{C}^1, \tag{8}$$

where the parameter α and the time are real. *Hint*: Write $z = x + iy$, where (x, y) are real. Then (8) takes the form

$$\begin{cases} \dot{x} &= \alpha + x^2 - y^2, \\ \dot{y} &= 2xy. \end{cases} \tag{9}$$

At $\alpha = -1$ this system has equilibrium $(x_0, y_0) = (1, 0)$, which can be continued.

Plot also the phase portraits in the (x, y)-plane of (9) for $\alpha = -0.25$, $\alpha = 0$, and $\alpha = 0.25$

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[1]As described in Section 2.5, click **Select|System|Manage Userfunction** in the MATCONT window and type the expression for F after `res=` in the edit field of the appearing **User functions** window. Fill in the **Name** and **Label** fields, and press **Add** and **OK**.

E. Study discretizations of the Bratu-Gelfand problem in more detail:

1. For $N = 2$, find *analytically* the parameter value λ_{BP} corresponding to the branching point in system (6) and compare it with the numerical result in Section 2.

2. Analyse numerically the effect of increasing the number of equidistant mesh points to $N = 3$, $N = 4$, and and $N = 10$ on the α-values corresponding to the limit and branching points in the finite-difference approximation (3)-(4) of the Bratu-Gelfand problem (2).

   Hint: Do not forget that $\alpha = (N + 1)^2 \lambda$.