Degenerate Bogdanov-Takens bifurcations in two and more dimensions

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Contents

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REFERENCES

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DEGENERATE BT BIFURCATIONS IN PLANAR ODES

- Classification of codim 3 BT points
- Bifurcations of a triple equilibrium with elliptic sector
- Example: A basic two-stage population model

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Classification of codim 3 BT points

- Consider a generic smooth family of planar autonomous ODEs

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \alpha \in \mathbb{R}^m.$$

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This indicates a *Bogdanov-Takens* (BT) *bifurcation*.

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  This indicates a *Bogdanov-Takens* (BT) *bifurcation*.

- The ODE at the BT-bifurcation is formally smoothly equivalent to

$$\begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \sum_{k \geq 2} \left(a_k w_0^k + b_k w_0^{k-1} w_1 \right). \end{cases}$$

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Classical codim 2 BT bifurcation

- Versal unfolding when $a_2 b_2 \neq 0$ (Bogdanov[1975], Takens[1974]):

$$\begin{cases} \dot{\xi}_0 &= \xi_1, \\ \dot{\xi}_1 &= \beta_1 + \beta_2 \xi_0 + a_2 \xi_0^2 + b_2 \xi_0 \xi_1. \end{cases}$$

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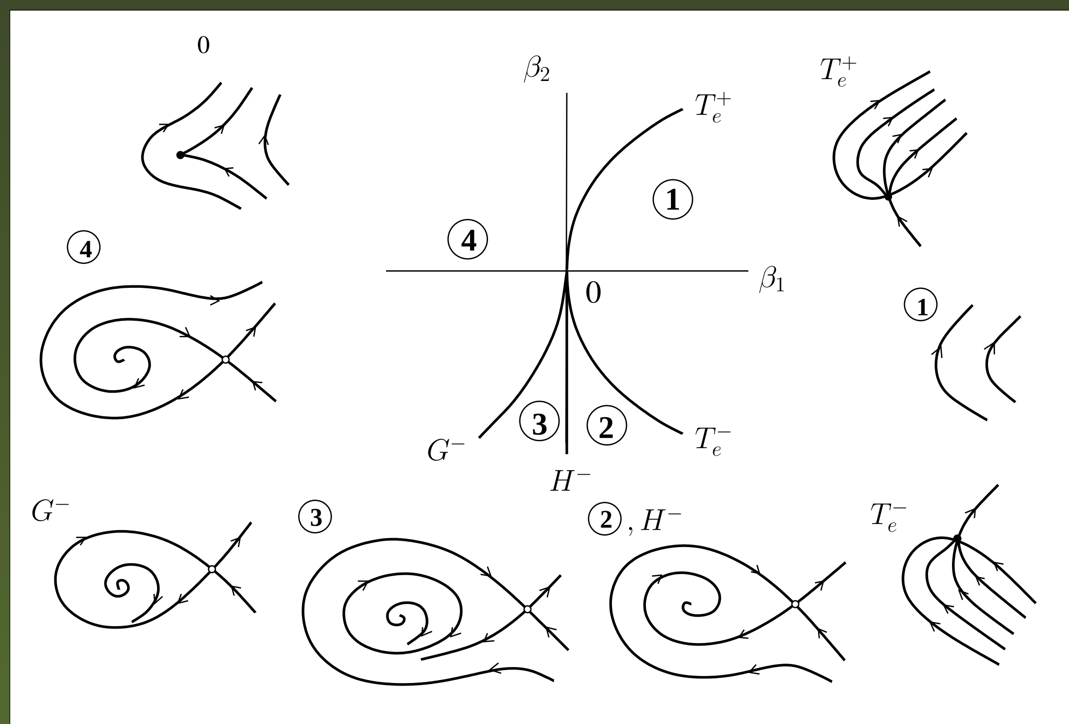
- The bifurcation diagram:



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Codim 3 BT bifurcation with double equilibrium

- If $b_2 = 0$ but $a_2 \neq 0$, the critical ODE is smoothly orbitally equivalent to

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= a_2 w_0^2 + b_4 w_0^3 w_1 + O(\|(w_0, w_1)\|^5). \end{cases}$$

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- Versal unfolding when $b_2 = 0$ but $a_2 b_4 \neq 0$ (Berezovskaya & Khibnik [1985], Dumortier, Roussarie & Sotomayor [1987]):

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- The bifurcation diagram includes a neutral saddle homoclinic and a degenerate Andronov-Hopf (Bautin) bifurcation curves.

Codim 3 BT bifurcation with triple equilibrium $(b_2 > 0)$

- If $a_2 = 0$ but $b_2 a_3 \neq 0$, the critical ODE is smoothly orbitally equivalent with $b_3' = b_3 - \dfrac{3b_2 a_4}{5a_3}$ to

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= a_3 w_0^3 + b_2 w_0 w_1 + b_3' w_0^2 w_1 + O(\|(w_0, w_1)\|^5). \end{cases}$$

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- If $a_3 > 0$ the origin is a topological *saddle*. If $a_3 < 0, b_2^2 + 8a_3 < 0$ and $b_3' \neq 0$, the origin is a topological *focus*. If $a_3 < 0$ and $b_2^2 + 8a_3 > 0$, the origin has one *elliptic sector*.

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- "Versal" unfolding in all cases (Dumortier, Roussarie, Sotomayor & Żołądek [1991]):

$$\begin{cases} \dot{\xi}_0 &= \xi_1, \\ \dot{\xi}_1 &= \beta_1 + \beta_2 \xi_0 + \beta_3 \xi_1 + a_3 \xi_0^3 + b_2 \xi_0 \xi_1 + b_3' \xi_0^2 \xi_1. \end{cases}$$

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Normal forms with $\mathbb{Z}_2$-symmetry

- In symmeric systems, degenerate BT bifurcations have smaller codimensions.

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Normal forms with $\mathbb{Z}_2$-symmetry

- In symmeric systems, degenerate BT bifurcations have smaller codimensions.
- The $\mathbb{Z}_2$-symmetry implies that certain coefficients in the critical normal form vanish, i.e.

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= a_3 w_0^3 + b_3 w_0^2 w_1 + O(\|(w_0, w_1)\|^5), \end{cases}$$

which leads to unfoldings like

$$\begin{cases} \dot{\xi}_0 &= \xi_1, \\ \dot{\xi}_1 &= \beta_1 \xi_0 + \beta_2 \xi_1 + a_3 \xi_0^3 + b_3 \xi_0^2 \xi_1, \end{cases}$$

provided $a_3 b_3 \neq 0$ (Carr [1981])

Bifurcations of a triple equilibrium with elliptic sector

- Truncated and scaled critical normal form:

$$\begin{cases} \dot{\xi} &= \eta, \\ \dot{\eta} &= \beta\xi\eta + \epsilon_1\xi^3 + \epsilon_2\xi^2\eta, \end{cases}$$

where $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$, and $\beta > 0$.

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- Saddle case: $\epsilon_1 = 1$, any ϵ_2 and β;
  Focus case: $\epsilon_1 = -1$ and $0 < \beta < 2\sqrt{2}$;
  Elliptic case: $\epsilon_1 = -1$ and $2\sqrt{2} < \beta$.

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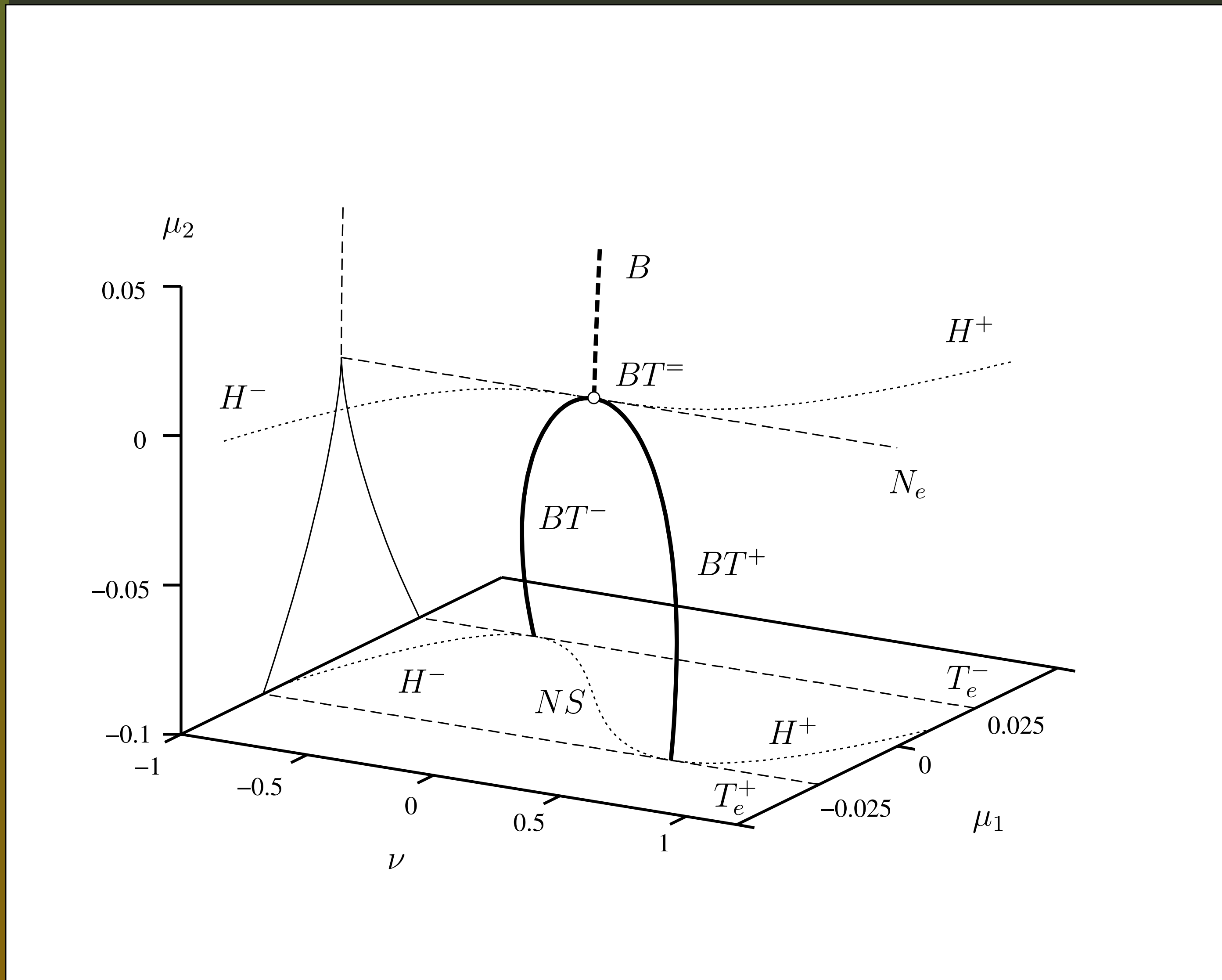
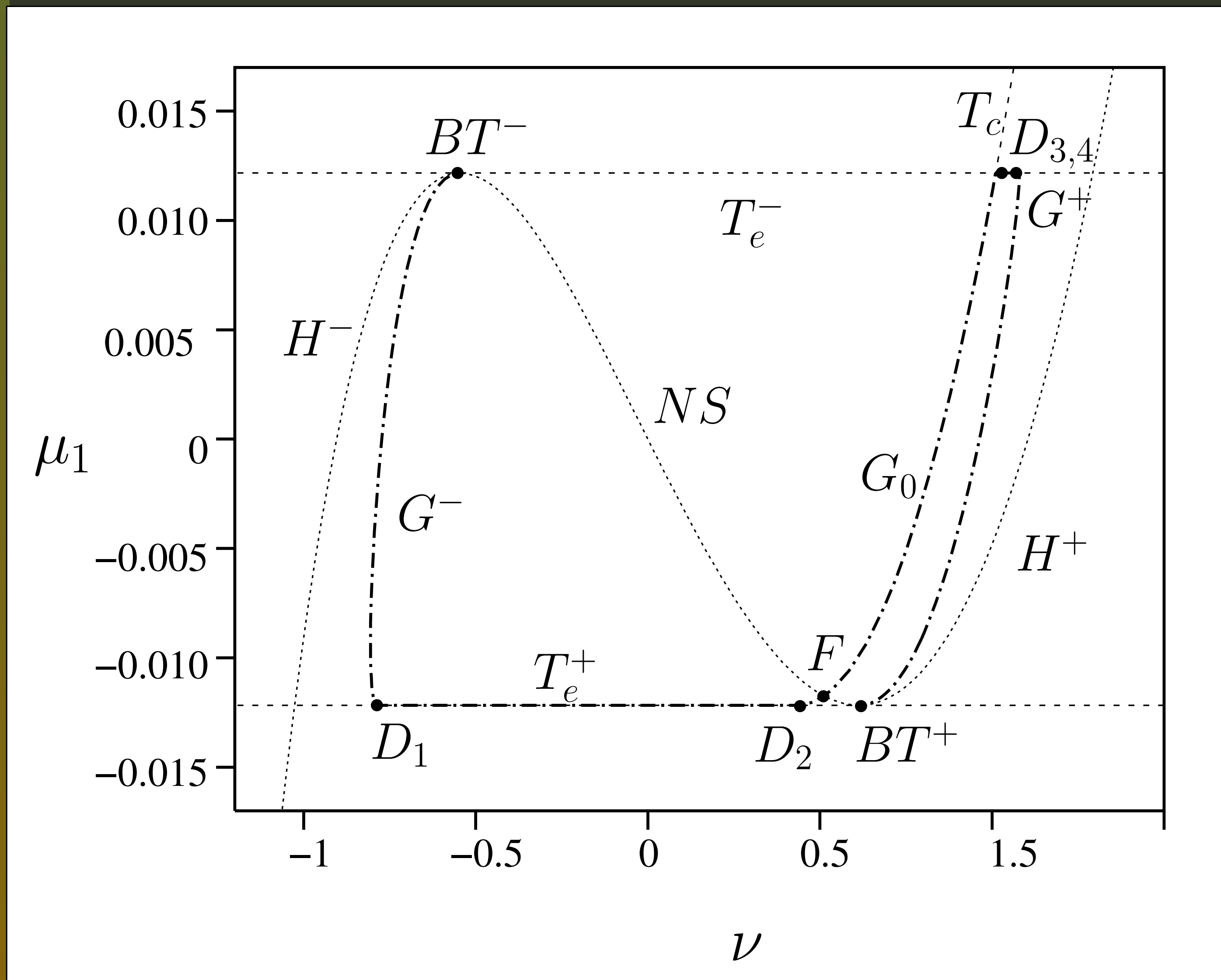
- Unfolding:

$$\begin{cases} \dot{\xi} &= \eta, \\ \dot{\eta} &= -\mu_1 - \mu_2\xi + \nu\eta + \beta\xi\eta - \xi^3 - \xi^2\eta. \end{cases}$$

Local bifurcations: $\beta = 3.175849820$



Local and global bifurcations: $\mu_2 = 0.1, \beta = 3.175849820$



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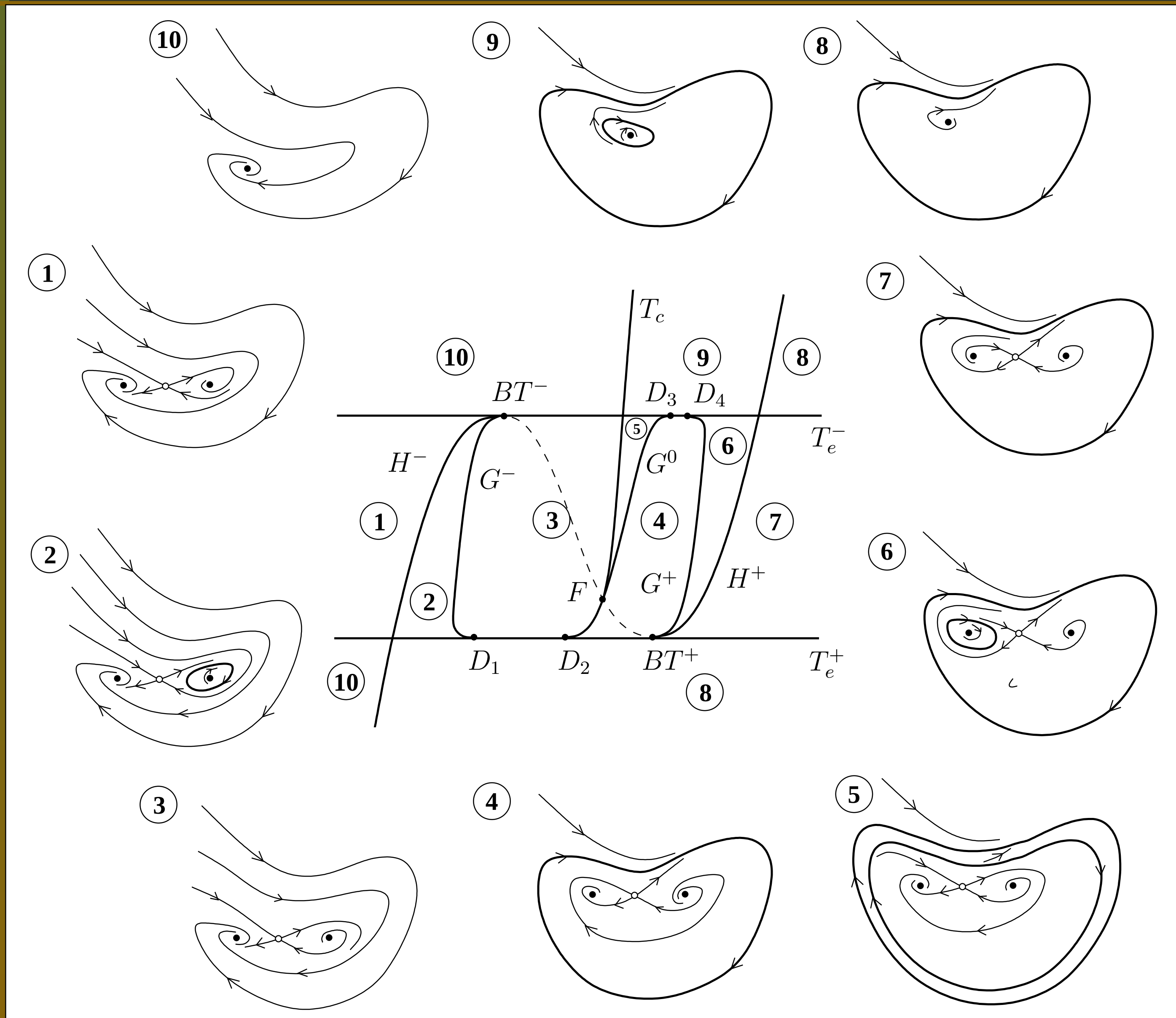
Schematic bifurcation diagram in the elliptic case



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Elliptic versus focus case

- The schematic bifurcation diagram differs drastically from the theoretical bifurcation diagram for the elliptic case given by Dumortier et al. [1991] who studied phase portraits in a *fixed* small neighborhood of the origin.

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- It turns out that generic two-parameter slices in the elliptic case are topologically equivalent to those in the focus case.
- However, the inner limit cycle demonstrates rapid amplitude changes ("canard-like" behavior) near the bifurcation curve T_c.
- The "big" homoclinic orbit to the neutral saddle (point F) shrinks not to the origin of the phase plane, but to the boundary of the elliptic sector that has a finite size in the unfolding.

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A basic two-stage population model

- The juvenile-adult model (Kostova, Li & Friedman [1999]):

$$\begin{cases} \dfrac{dL}{dt} = \dfrac{\mu}{m}\left(g(y)y - mL - f(L)L\right), \\ \dfrac{dy}{dt} = f(L)L - y, \end{cases}$$

where $f(L) = e^{-L}$, $g(y) = e^{(1/b)(a-y)}$.

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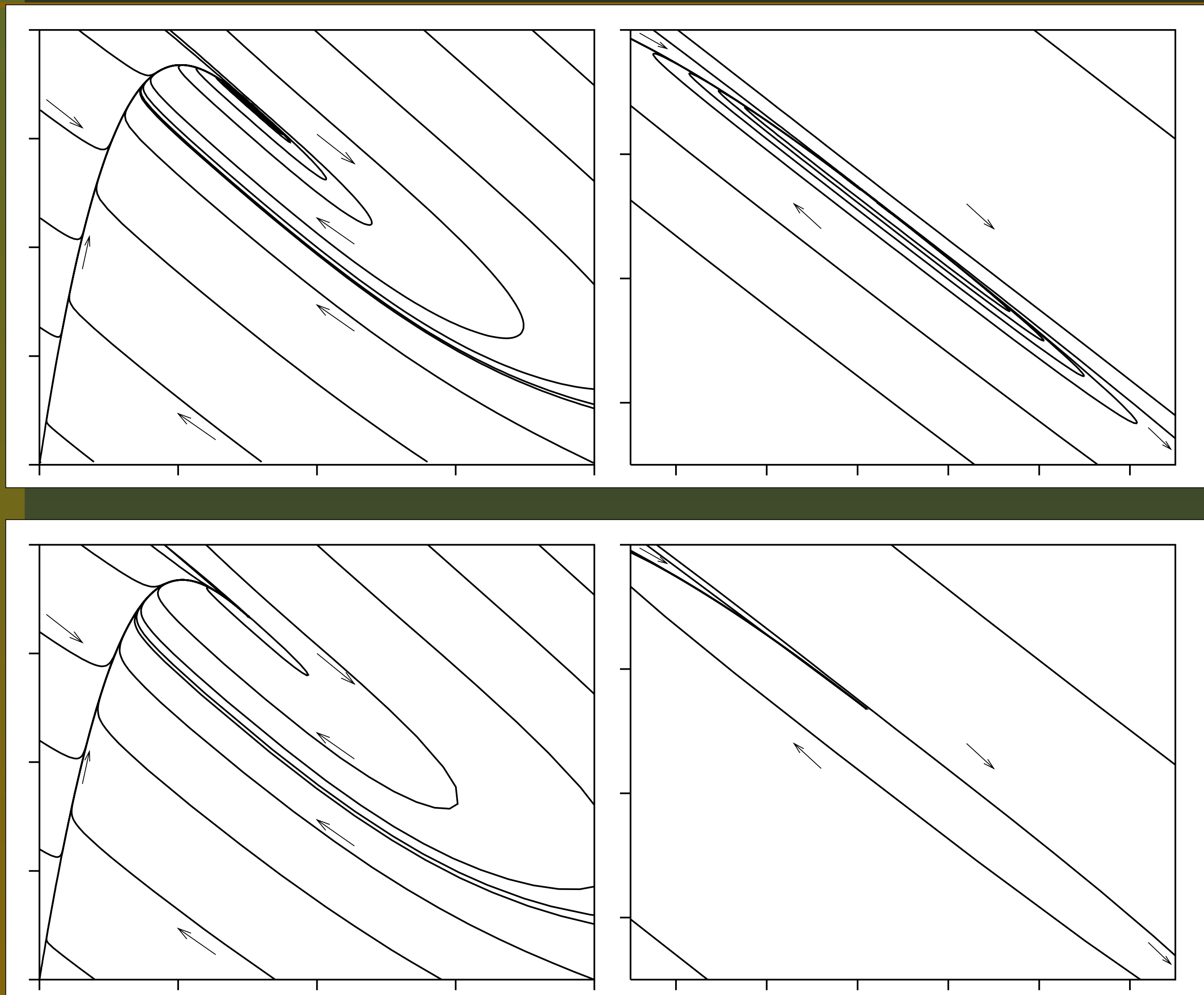
- For $b = 2.2$, we have
  $\mu^\sharp = 0.01179614,\ m^\sharp = 0.01192386945,\ a^\sharp = 0.4492276697$ and
  $L^\sharp = 1.513180178,\ y^\sharp = 0.33321523.$

Codim 4: $\beta = 2\sqrt{2}$ at $b = b^\natural = 1.7300228$



$b = 2.2$

$b = 1.5$

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NORMAL FORMS ON CENTER MANIFOLDS IN n-DIMENSIONAL ODES

- Combined reduction/normalization technique
- Explicit normal form coefficients
- Example: 6D-model of two coupled Faraday disk homopolar dynamos

Combined reduction/normalization technique

- Critical ODE: $\dot{x} = F(x), \quad x \in \mathbb{R}^n,$

  with Taylor expansion

$$F(x) = Ax + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + \frac{1}{24}D(x,x,x,x) + O(\|x\|^5).$$

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- Eigenvectors: $q_{0,1}, p_{0,1} \in \mathbb{R}^n$,

$$Aq_0 = 0, Aq_1 = q_0, A^T p_1 = 0, A^T p_0 = p_1$$

  with $\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 1, \langle p_0, q_1 \rangle = \langle p_1, q_0 \rangle = 0.$

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- Critical center manifold:

$$x = H(w_0, w_1) = w_0 q_0 + w_1 q_1 + \sum_{2 \le j+k \le 4} \frac{1}{j!k!} h_{jk} w_0^j w_1^k + O(\|(w_0, w_1)\|^5)$$

  where $(w_0, w_1) \in \mathbb{R}^2, h_{jk} \in \mathbb{R}^n$.

- Critical normal form:

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= a_2 w_0^2 + b_2 w_0 w_1 + a_3 w_0^3 + b_3 w_0^2 w_1 + a_4 w_0^4 + b_4 w_0^3 w_1 \\ & \quad + \; O(\|(w_0, w_1)\|^5). \end{cases}$$

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- Homological equation: $H_{w_0}\dot{w}_0 + H_{w_1}\dot{w}_1 = F(H(w_0, w_1)).$

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- Homological equation: $H_{w_0}\dot{w}_0 + H_{w_1}\dot{w}_1 = F(H(w_0, w_1))$.
- Collecting the $w_0^j w_1^k$-terms give singular linear systems for h_{jk}. Since these systems must be solvable, their right-hand sides should be orthogonal to p_1. Some of these Fredholm conditions will define the normal form coefficients, others can be satisfied using a freedom in selecting solutions of singular linear systems appearing at lower-order terms.

Explicit normal form coefficients: Quadratic terms

- The w_0^2-terms give

$$Ah_{20} = 2a_2q_1 - B(q_0, q_0).$$

The Fredholm solvability condition for this system implies

$$a_2 = \frac{1}{2}\langle p_1, B(q_0, q_0)\rangle.$$

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- The w_0w_1-terms give

$$Ah_{11} = b_2q_1 + h_{20} - B(q_0, q_1).$$

Its solvability leads to the expression

$$b_2 = \langle p_1, B(q_0, q_1)\rangle - \langle p_1, h_{20}\rangle.$$

- The w_1^2-terms give

$$Ah_{02} = 2h_{11} - B(q_1, q_1).$$

Since

$$\langle p_1, h_{11} \rangle = \langle p_0, h_{20} \rangle - \langle p_0, B(q_0, q_1) \rangle,$$

we get

$$\langle p_1, 2h_{11} - B(q_1, q_1) \rangle = 2\langle p_0, h_{20} \rangle - 2\langle p_0, B(q_0, q_1) \rangle - \langle p_1, B(q_1, q_1) \rangle.$$

The substitution $h_{20} \mapsto h_{20} + \delta_0 q_0$ with a properly selected δ_0 makes the right-hand side of this equation equal to zero. This does not affect the coefficient b_2, because $\langle p_1, q_0 \rangle = 0$.

Cubic terms

- The w_0^3-terms give

$$Ah_{30} = 6q_1 a_3 + 6h_{11} a_2 - 3B(h_{20}, q_0) - C(q_0, q_0, q_0).$$

Its solvability implies

$$a_3 = \frac{1}{6}\langle p_1, C(q_0, q_0, q_0)\rangle + \frac{1}{2}\langle p_1, B(h_{20}, q_0)\rangle - a_2\langle p_1, h_{11}\rangle.$$

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- The $w_0^2 w_1$-terms give

$$Ah_{21} = h_{30} + 2b_3q_1 + 2a_2h_{02} + 2b_2h_{11} - 2B(h_{11}, q_0) - B(h_{20}, q_1) - C(q_0, q_0, q_1),$$

which solvability implies

$$\begin{aligned} b_3 &= \frac{1}{2}\langle p_1, C(q_0, q_0, q_1) + 2B(h_{11}, q_0) + B(h_{20}, q_1)\rangle \\ &- \frac{1}{2}\langle p_1, h_{30} + 2a_2h_{02} + 2b_2h_{11}\rangle. \end{aligned}$$

- The singular linear systems resulting from the $w_0 w_1^2$- and w_1^3-terms,

$$Ah_{12} = 2h_{21} + 2b_2 h_{02} - B(h_{02}, q_0) - 2B(h_{11}, q_1) - C(q_0, q_1, q_1)$$

  and

$$Ah_{03} = 3h_{12} - 3B(h_{02}, q_1) - C(q_1, q_1, q_1),$$

  can be made solvable for any h_{02} by substituting $h_{30} \mapsto h_{30} + \delta_1 q_0$ and then $h_{21} \mapsto h_{21} + \delta_2 q_0$ with properly selected δ_1 and δ_2. This does not change b_3.

Fourth-order terms

- The w_0^4-terms imply

$$\begin{aligned} a_4 &= \frac{1}{24}\langle p_1, D(q_0,q_0,q_0,q_0) + 6C(h_{20},q_0,q_0)\rangle \\ &+ \frac{1}{24}\langle p_1, 4B(h_{30},q_0) + 3B(h_{20},h_{20})\rangle \\ &- \frac{1}{2}a_2\langle p_1, h_{21}\rangle - a_3\langle p_1, h_{11}\rangle. \end{aligned}$$

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a_4 &= \frac{1}{24}\langle p_1, D(q_0,q_0,q_0,q_0) + 6C(h_{20},q_0,q_0)\rangle \\
&+ \frac{1}{24}\langle p_1, 4B(h_{30},q_0) + 3B(h_{20},h_{20})\rangle \\
&- \frac{1}{2}a_2\langle p_1, h_{21}\rangle - a_3\langle p_1, h_{11}\rangle.
\end{aligned}
$$

- The $w_0^3 w_1$-terms imply

$$
\begin{aligned}
b_4 &= \frac{1}{6}\langle p_1, D(q_0,q_0,q_0,q_1) + 3C(h_{20},q_0,q_1) + 3C(h_{11},q_0,q_0)\rangle \\
&+ \frac{1}{6}\langle p_1, 3B(h_{21},q_0) + 3B(h_{11},h_{20}) + B(h_{30},q_1)\rangle \\
&- \frac{1}{6}\langle p_1, h_{40}\rangle - \frac{1}{2}b_2\langle p_1, h_{21}\rangle \\
&- \langle p_1, a_2h_{12} + a_3h_{02} + b_3h_{11}\rangle.
\end{aligned}
$$

Some simplifications

- Since $\langle p_1, h_{20} \rangle = -\langle p_0, B(q_0, q_0) \rangle$, we obtain

$$b_2 = \langle p_0, B(q_0, q_0) \rangle + \langle p_1, B(q_0, q_1) \rangle.$$

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$$b_2 = \langle p_0, B(q_0,q_0)\rangle + \langle p_1, B(q_0,q_1)\rangle.$$

- Since $\langle p_1, h_{11}\rangle = \frac{1}{2}\langle p_1, B(q_1,q_1)\rangle$, we obtain

$$a_3 = \frac{1}{6}\langle p_1, C(q_0,q_0,q_0)\rangle + \frac{1}{2}\langle p_1, B(h_{20},q_0)\rangle - \frac{1}{2}a_2\langle p_1, B(q_1,q_1)\rangle.$$

Some simplifications

- Since $\langle p_1, h_{20}\rangle = -\langle p_0, B(q_0,q_0)\rangle$, we obtain

$$b_2 = \langle p_0, B(q_0,q_0)\rangle + \langle p_1, B(q_0,q_1)\rangle.$$

- Since $\langle p_1, h_{11}\rangle = \frac{1}{2}\langle p_1, B(q_1,q_1)\rangle$, we obtain

$$a_3 = \frac{1}{6}\langle p_1, C(q_0,q_0,q_0)\rangle + \frac{1}{2}\langle p_1, B(h_{20},q_0)\rangle - \frac{1}{2}a_2\langle p_1, B(q_1,q_1)\rangle.$$

- Similarly, we obtain

$$\begin{aligned} b_3 &= \tfrac{1}{2}\langle p_1, C(q_0,q_0,q_1) + 2B(h_{11},q_0) + B(h_{20},q_1)\rangle \\ &+ \tfrac{1}{2}\langle p_0, C(q_0,q_0,q_0) + 3B(h_{20},q_0)\rangle \\ &- \tfrac{1}{2}b_2\langle p_1, B(q_1,q_1)\rangle + a_2\langle p_0, B(q_1,q_1)\rangle \\ &- 5a_2\langle p_0, h_{11}\rangle. \end{aligned}$$

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6D-model of two coupled Faraday disk homopolar dynamos

The ODE system (Moroz, Hilde & Soward [1998]):

$$\begin{cases} \dot{x}_1 &= m x_4 x_2 - x_1 - \beta x_3, \\ \dot{x}_2 &= \alpha - \alpha m x_1 x_4 - k x_2, \\ \dot{x}_3 &= x_1 - \lambda x_3, \\ \dot{x}_4 &= x_1 x_5 - x_4 - \beta x_6, \\ \dot{x}_5 &= \alpha - \alpha x_1 x_4 - k x_5, \\ \dot{x}_6 &= x_4 - \lambda x_6, \end{cases}$$

where $(\alpha, \beta, k, \lambda, m)$ are positive parameters. The system is invariant under the transformation

$$(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (-x_1, x_2, -x_3, -x_4, x_5, -x_6).$$

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For $(\alpha^0, \beta^0) = \left(\frac{(1+\lambda)k}{\sqrt{m}}, \lambda^2\right)$ the equilibrium $x^0 = \left(0, \frac{\alpha}{k}, 0, 0, \frac{\alpha}{k}, 0\right)$ has Jacobian matrix

$$A = \begin{pmatrix} -1 & 0 & -\lambda^2 & (1+\lambda)\sqrt{m} & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 \\ 1 & 0 & -\lambda & 0 & 0 & 0 \\ \frac{1+\lambda}{\sqrt{m}} & 0 & 0 & -1 & 0 & -\lambda^2 \\ 0 & 0 & 0 & 0 & -k & 0 \\ 0 & 0 & 0 & 1 & 0 & -\lambda \end{pmatrix}$$

with one double zero eigenvalue, i.e. an *equivariant BT bifurcation* occurs.

$$q_0 = \begin{pmatrix} \sqrt{m}\lambda \\ 0 \\ \sqrt{m} \\ \lambda \\ 0 \\ 1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} \sqrt{m} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$p_1 = \frac{1}{2\sqrt{m}} \begin{pmatrix} 1 \\ 0 \\ -\lambda \\ \sqrt{m} \\ 0 \\ -\sqrt{m}\lambda \end{pmatrix}, \quad p_0 = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \sqrt{m} \end{pmatrix}.$$

- Bilinear form $B : \mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}^6$,

$$B(v,w) = \begin{pmatrix} m(v_4 w_2 + v_2 w_4) \\ \\ -k\sqrt{m}(1+\lambda)(v_4 w_1 + v_1 w_4) \\ \\ 0 \\ \\ v_4 w_1 + v_1 w_5 \\ \\ -\dfrac{(1+\lambda)k}{\sqrt{m}}(v_4 w_1 + v_1 w_4) \\ \\ 0 \end{pmatrix}.$$

- Since no cubic term is present, the 3-form C vanishes identically.
- Due to the symmetry, we have $a_2 = b_2 = 0$, so that

$$a_3 = \frac{1}{2}\langle p_1, B(h_{20}, q_0)\rangle$$

and

$$\begin{aligned} b_3 &= \langle p_1, 2B(h_{11}, q_0)\rangle \\ &+ \frac{1}{2}\langle p_1, B(h_{20}, q_1)\rangle \\ &+ \frac{3}{2}\langle p_0, B(h_{20}, q_0)\rangle. \end{aligned}$$

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Solving the corresponding singular linear systems, we obtain

$$h_{20} = -2\lambda^2(1+\lambda)\begin{pmatrix} 0 \\ m \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; h_{11} = -\frac{2m\lambda(1+\lambda)(k-\lambda)}{k}\begin{pmatrix} 0 \\ m \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Here h_{20} is fixed to assure the solvability of the system for h_{02}, while h_{11} is an arbitrary solution of the corresponding system. Since $a_2 = 0$, its choice does not affect the value of b_3.

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- Using the above specified quantities, we easily compute

$$\begin{aligned} a_3 &= -\frac{1}{2}\sqrt{m}(m+1)\lambda^3(1+\lambda), \\ b_3 &= -\frac{1}{2k}\sqrt{m}(m+1)\lambda^2(1+\lambda)(3k-2\lambda). \end{aligned}$$

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- Since the coefficients are defined to within a nonzero multiple corresponding to the scaling of the normal form variables, they can be harmlessly divided by $-\frac{1}{2}\sqrt{m}(m+1)\lambda^2(1+\lambda)$, which leads to

$$a_3 = \lambda, \;\; b_3 = \frac{1}{k}(3k-2\lambda).$$

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$$a_3 = \lambda, \;\; b_3 = \frac{1}{k}(3k-2\lambda).$$

- A codim 3 bifurcation occurs at $\lambda = \frac{3}{2}k$, since then $b_3 = 0$.

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OPEN QUESTIONS

- Other bifurcations with cycle “blow-up”, e.g. ZH ?
- Higher codimension ?
- Parameter-dependent normalization ?

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