Continuation of point-to-cycle connections in 3D ODEs

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joint work with E.J. Doedel, B.W. Kooi, and G.A.K. van Voorn

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- Previous works
- Truncated BVP's with projection BC's
- The defining BVP in 3D
- Finding starting solutions with homopoty
- Examples
- Open questions

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Previous works

- W.-J. Beyn, [1994], “On well-posed problems for connecting orbits in dynamical systems.”, In *Chaotic Numerics (Geelong, 1993)*, volume 172 of *Contemp. Math.*, 131–168. Amer. Math. Soc., Providence, RI.
- T. Pampel, [2001], “Numerical approximation of connecting orbits with asymptotic rate,” *Numer. Math.*, **90**, 309–348.
- L. Dieci and J. Rebaza, [2004], “Point-to-periodic and periodic-to-periodic connections,” *BIT Numerical Mathematics*, **44**, 41–62.
- L. Dieci and J. Rebaza, [2004], “Erratum: “Point-to-periodic and periodic-to-periodic connections”,” *BIT Numerical Mathematics*, **44**, 617–618.

2. Truncated BVP's with projection BC's

- Some notations
- Isolated families of connecting orbits
- Truncated BVP
- Error estimate

Some notations

- Consider the (local) flow φ^t generated by a smooth ODE

$$\frac{du}{dt} = f(u,\alpha), \quad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n.$$

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- Let O^+ be a hyperbolic *limit cycle* with $\dim W^s_+ = m^+_s$.

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- Consider the (local) flow φ^t generated by a smooth ODE

$$\frac{du}{dt} = f(u,\alpha), \quad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n.$$

- Let $O^- = \xi$ be a hyperbolic *equilibrium* with $\dim W_-^u = n_u^-$.
- Let O^+ be a hyperbolic *limit cycle* with $\dim W_+^s = m_s^+$.
- If $x^+(t)$ is a periodic solution (with minimal period T^+) corresponding to O^+, then $m_s^+ = n_s^+ + 1$, where n_s^+ is the number of eigenvalues μ^+ of the *monodromy matrix*

$$M^+ = D_x \varphi^{T^+}(x)\Big|_{x=x^+(0)},$$

satisfying $|\mu^+| < 1$.

Isolated families of connecting orbits

- Necessary condition: $p = n - m_s^+ - n_u^- + 2$ (Beyn, 1994).

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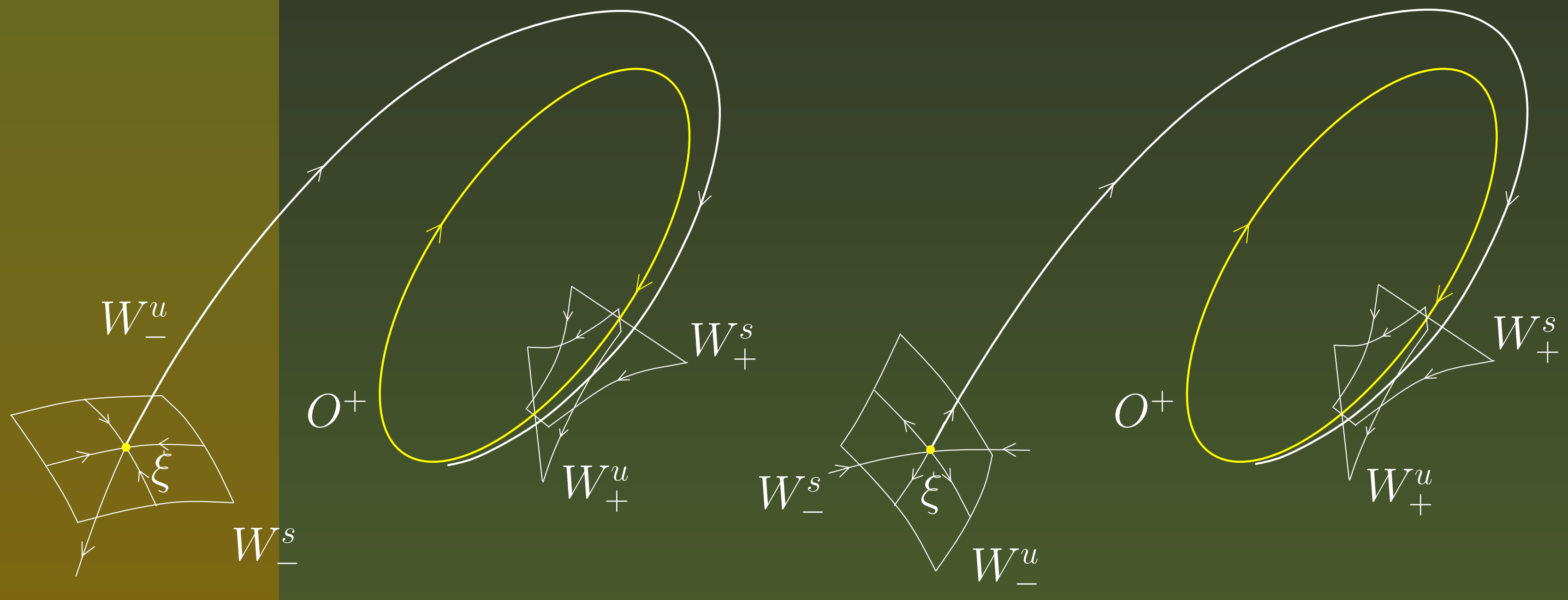
Isolated families of connecting orbits

- Necessary condition: $p = n - m_s^+ - n_u^- + 2$ (Beyn, 1994).
- Two types of point-to-cycle connections in $\mathbb{R}^3$:



(a) $\dim W_-^u = 1$ (b) $\dim W_-^u = 2$

Truncated BVP

- The connecting solution $u(t)$ is *truncated* to an interval $[\tau_-, \tau_+]$.

Truncated BVP

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- The points $u(\tau_-)$ and $u(\tau_+)$ are required to belong to the linear subspaces that are tangent to the unstable and stable invariant manifolds of O^- and O^+, respectively:

$$\begin{cases} L^-(u(\tau_-) - \xi) & = & 0, \\ L^+(u(\tau_+) - x^+(0)) & = & 0. \end{cases}$$

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- Generically, the truncated BVP composed of the ODE, the above *projection BC's*, and a *phase condition* on u, has a unique solution family $(\hat{u}, \hat{\alpha})$, provided that the ODE has a connecting solution family satisfying the pahase condition and Beyn's equality.

Error estimate

If u is a generic connecting solution to the ODE at parameter value α, then the following estimate holds:

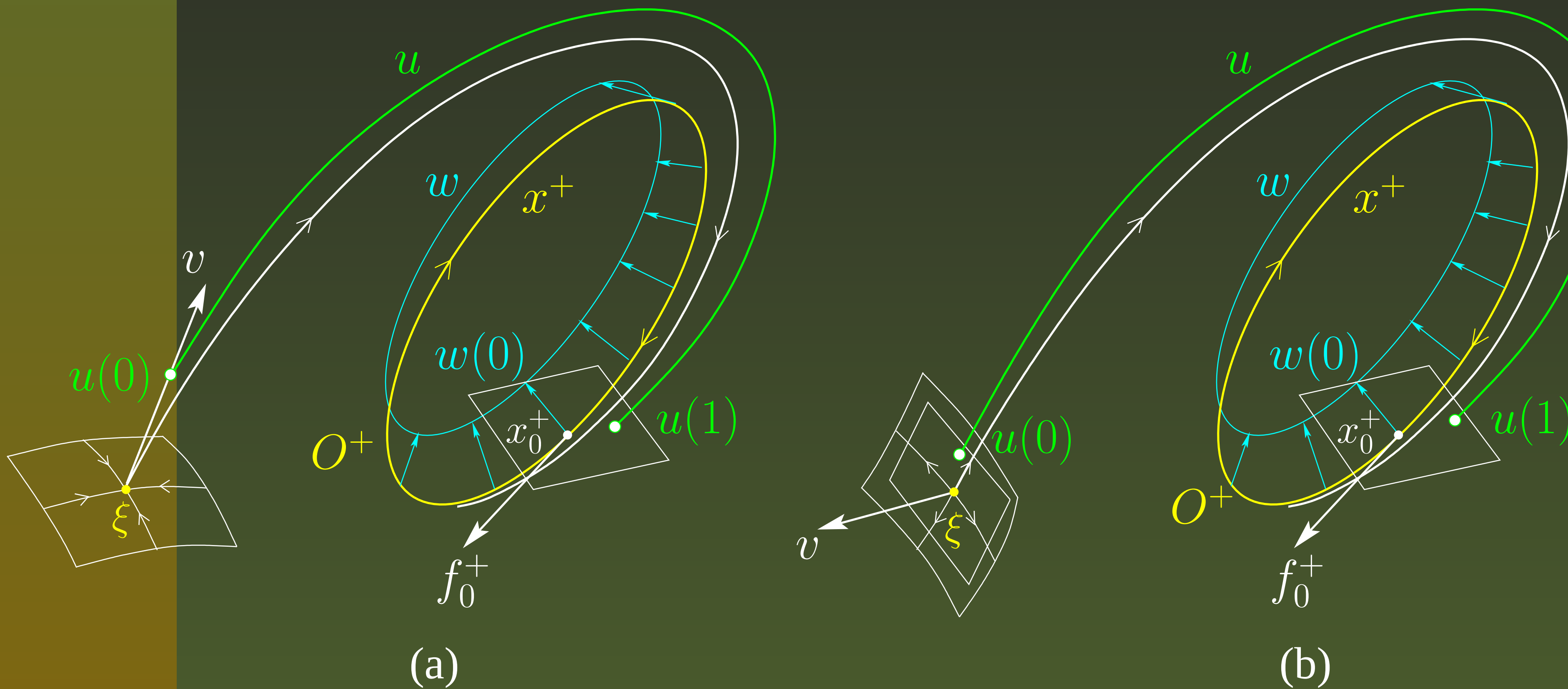
$$\|(u|_{[\tau_-,\tau_+]}, \alpha) - (\hat{u}, \hat{\alpha})\| \leq C\mathrm{e}^{-2\min(\mu_-|\tau_-|,\mu_+|\tau_+|)},$$

where

- $\| \cdot \|$ is an appropriate norm in the space $C^1([\tau_-, \tau_+], \mathbb{R}^n) \times \mathbb{R}^p$,
- $u|_{[\tau_-,\tau_+]}$ is the restriction of u to the truncation interval,
- $\mu_\pm$ are determined by the eigenvalues of the Jacobian matrix $D_u f$ at ξ and the monodromy matrix M^+.

(Pampel, 2001; Dieci and Rebaza, 2004)

3. The defining BVP in 3D



It has equilibrium-, cycle-, and connection-related parts.

Equilibrium-related equations

- If $n_u^- = 1$, we use $u(\tau_-) = \xi + \varepsilon v$, where

$$\begin{cases} f(\xi,\alpha) &= 0\,, \\ f_\xi(\xi,\alpha)v - \lambda_u v &= 0\,, \\ \langle v,v\rangle - 1 &= 0. \end{cases}$$

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$$\begin{cases} f(\xi,\alpha) = 0\ , \\ f_\xi(\xi,\alpha)v - \lambda_u v = 0\ , \\ \langle v, v\rangle - 1 = 0. \end{cases}$$

- If $n_u^- = 2$, we use $\langle v, u(\tau_-) - \xi\rangle = 0$, where

$$\begin{cases} f(\xi,\alpha) = 0\ , \\ f_\xi^{\mathrm{T}}(\xi,\alpha)v - \lambda_s v = 0\ , \\ \langle v, v\rangle - 1 = 0, \end{cases}$$

together with $\langle u(\tau_-) - \xi, u(\tau_-) - \xi\rangle - \varepsilon^2 = 0$.

Cycle-related equations

- Periodic solution:

$$\begin{cases} \dot{x}^+ - f(x^+, \alpha) & = & 0, \\ x^+(0) - x^+(T^+) & = & 0. \end{cases}$$

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Cycle-related equations

- Periodic solution:

$$\begin{cases} \dot{x}^+ - f(x^+, \alpha) &= 0, \\ x^+(0) - x^+(T^+) &= 0. \end{cases}$$

- Adjoint eigenfunction: $\mu = \frac{1}{\mu_u^+}$

$$\begin{cases} \dot{w} + f_u^{\mathrm{T}}(x^+, \alpha) w &= 0\,, \\ w(T^+) - \mu w(0) &= 0\,, \\ \langle w(0), w(0) \rangle - 1 &= 0\,. \end{cases}$$

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- Projection BC: $\langle w(0), u(\tau_+) - x^+(0) \rangle = 0.$

Connection-related equations

- We need a phase condition to select a unique periodic solution, *i.e.*, to fix a *base point*

$$x_0^+ = x^+(0)$$

on the cycle O^+.

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Connection-related equations

- We need a phase condition to select a unique periodic solution, *i.e.*, to fix a *base point*

$$x_0^+ = x^+(0)$$

  on the cycle O^+.

- Usually, an integral phase condition is used.
- For the point-to-cycle connection, we require the end point of the connection to belong to a plane orthogonal to the vector $f_0^+ = f(x^+(0), \alpha)$:

$$\begin{cases} \dot{u} - f(u, \alpha) & = & 0, \\ \langle f(x^+(0), \alpha), u(\tau_+) - x^+(0) \rangle & = & 0. \end{cases}$$

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The defining BVP in 3D: $\lambda = \ln|\mu|, \quad s = \mathrm{sign}\mu = \pm 1.$ lor eco

$$\left\{\begin{array}{rcl} u(0) - \xi - \varepsilon v & = & 0, \\ f(\xi, \alpha) & = & 0, \\ f_\xi(\xi, \alpha) v - \lambda_u v & = & 0, \\ \langle v, v \rangle - 1 & = & 0. \end{array}\right.$$

or

$$\left\{\begin{array}{rcl} \langle v, u(0) - \xi \rangle & = & 0, \\ \langle u(0) - \xi, u(0) - \xi \rangle - \varepsilon^2 & = & 0, \\ f(\xi, \alpha) & = & 0, \\ f_\xi^{\mathrm{T}}(\xi, \alpha) v - \lambda_s v & = & 0, \\ \langle v, v \rangle - 1 & = & 0, \end{array}\right.$$

$$\left\{\begin{array}{rl} \dot{x}^+ - T^+ f(x^+, \alpha) & = 0, \\ x^+(0) - x^+(1) & = 0, \\ \langle w(0), u(1) - x^+(0) \rangle & = 0, \\ \dot{w} + T^+ f_u^{\mathrm{T}}(x^+, \alpha) w + \lambda w & = 0, \\ w(1) - s w(0) & = 0, \\ \langle w(0), w(0) \rangle - 1 & = 0, \\ \dot{u} - T f(u, \alpha) & = 0, \\ \langle f(x^+(0), \alpha), u(1) - x^+(0) \rangle & = 0. \end{array}\right.$$

4. Finding starting solutions with homopoty

- Adjoint scaled eigenfunction.
- Homotopies to connecting orbits.

References to homotopy techniques for point-to-point connections:

- E.J. Doedel, M.J. Friedman, and A.C. Monteiro, [1994], “On locating connecting orbits", *Appl. Math. Comput.*, **65**, 231–239.
- E.J. Doedel, M.J. Friedman, and B.I. Kunin, [1997], “Successive continuation for locating connecting orbits", *Numer. Algorithms*, **14** , 103–124.

Adjoint scaled eigenfunction

- For fixed α and any λ, $x^+(\tau) = x^+_{old}(\tau), w(\tau) \equiv 0$, and $h = 0$ satisfy

$$\left\{\begin{array}{rcl} \dot{x}^+ - f(x^+,\alpha) &=& 0, \\ x^+(0) - x^+(T^+) &=& 0, \\ \int_0^1 \langle \dot{x}^+_{old}(\tau), x^+(\tau) \rangle &=& 0, \\ \dot{w} + T^+ f_u^{\mathrm{T}}(x^+,\alpha) w + \lambda w &=& 0, \\ w(1) - s w(0) &=& 0, \\ \langle w(0), w(0) \rangle - h &=& 0, \end{array}\right.$$

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- A *branch point* at λ_1 corresponds to the adjoint multiplier $\mu = s e^{\lambda_1}$. Branch switching and continuation towards $h = 1$ gives the eigenfunction w.

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Continuation in (T, h_1) for fixed α $(\dim W_-^u = 1)$ lor

$$\left\{\begin{array}{rcl} u(0) - \xi - \varepsilon v &=& 0, \\ f(\xi,\alpha) &=& 0, \\ f_\xi(\xi,\alpha)v - \lambda_u v &=& 0, \\ \langle v, v\rangle - 1 &=& 0. \end{array}\right. \quad \left\{\begin{array}{rl} \dot{x}^+ - T^+ f(x^+,\alpha) &= 0, \\ x^+(0) - x^+(1) &= 0, \\ \Psi[x^+] &= 0, \\ \dot{w} + T^+ f_u^{\mathrm{T}}(x^+,\alpha)w + \lambda w &= 0, \\ w(1) - sw(0) &= 0, \\ \langle w(0), w(0)\rangle - 1 &= 0, \\ \dot{u} - Tf(u,\alpha) &= 0, \\ \langle f(x^+(0),\alpha), u(1) - x^+(0)\rangle - h_1 &= 0. \end{array}\right.$$

Here, e.g. $\Psi[x^+] = x_j^+(0) - a_j$ and the initial connection $u(\tau) = \xi + \varepsilon v e^{\lambda_u T\tau}$.

Continuation in (α_1, h_2) for fixed T $(\dim W_-^u = 1)$ lor

$$\left\{\begin{array}{rcl} u(0) - \xi - \varepsilon v & = & 0, \\ f(\xi, \alpha) & = & 0, \\ f_\xi(\xi, \alpha) v - \lambda_u v & = & 0, \\ \langle v, v \rangle - 1 & = & 0. \end{array}\right. \qquad \left\{\begin{array}{rl} \dot{x}^+ - T^+ f(x^+, \alpha) & = 0, \\ x^+(0) - x^+(1) & = 0, \\ \langle w(0), u(1) - x^+(0) \rangle - h_2 & = 0, \\ \dot{w} + T^+ f_u^{\mathrm{T}}(x^+, \alpha) w + \lambda w & = 0, \\ w(1) - s w(0) & = 0, \\ \langle w(0), w(0) \rangle - 1 & = 0, \\ \dot{u} - T f(u, \alpha) & = 0, \\ \langle f(x^+(0), \alpha), u(1) - x^+(0) \rangle & = 0. \end{array}\right.$$

When $h_2 = 0$ is located, improve connection by the continuation in (α_1, T) and then continue in (α_1, α_2) with fixed T (using the primary BVP).

Continuation in (T, h_1) or (c_1, c_2, h_k) $(\dim W_-^u = 2)$ eco

The equilibrium-related part is replaced by the explicit BC

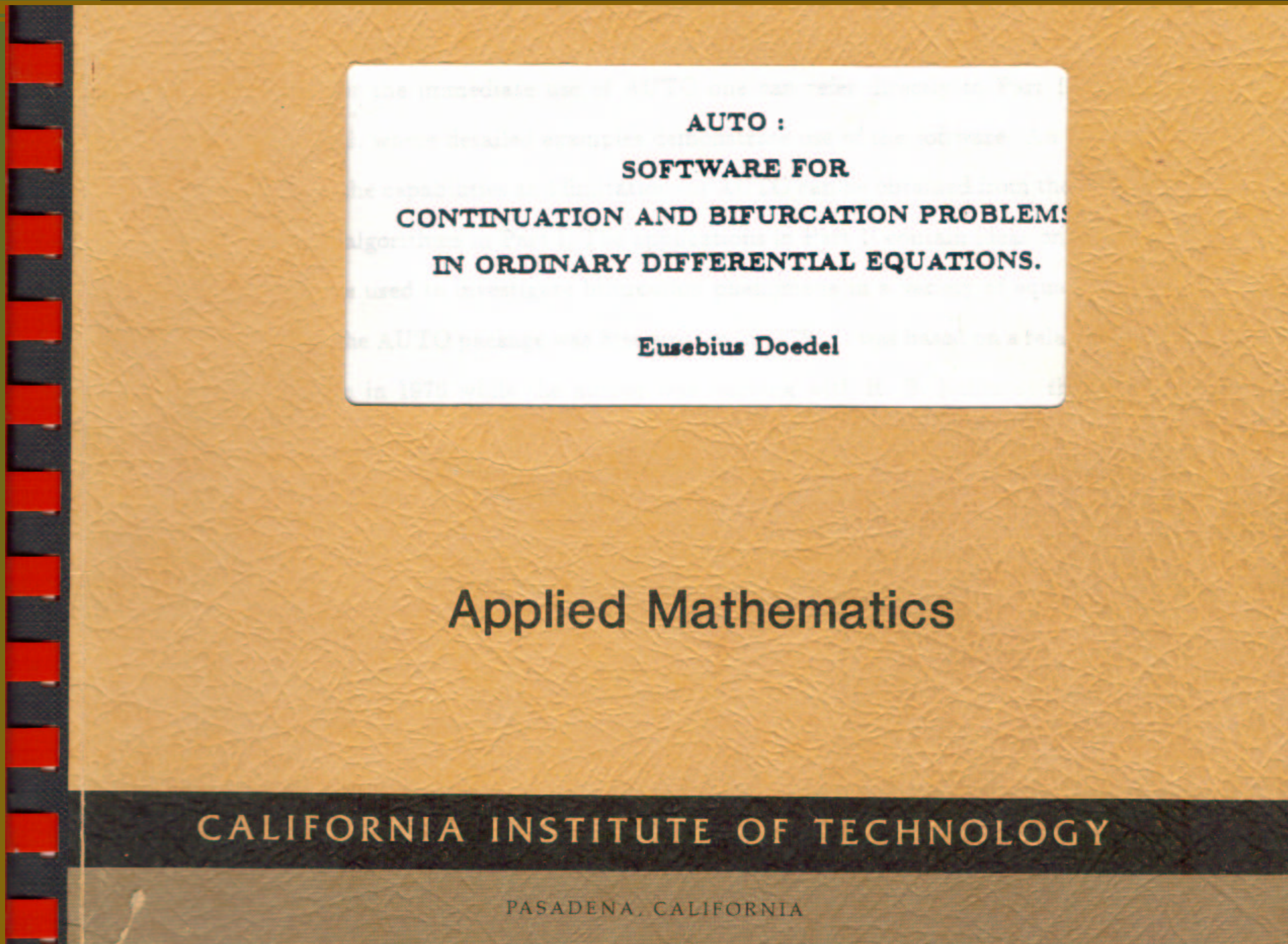
$$\left\{ \begin{array}{rcl} u(0) - \xi - \varepsilon(c_1 v^{(1)} + c_2 v^{(2)}) & = & 0, \\ c_1^2 + c_2^2 - 1 & = & 0, \\ f(\xi, \alpha) & = & 0, \\ f_\xi(\xi, \alpha)v - \lambda_u v & = & 0, \\ \langle v, v \rangle - 1 & = & 0, \end{array} \right.$$

where $v^{(1)}$ and $v^{(2)}$ are independent unit vectors tangent to W_-^u at ξ.

The initial connection

$$u(\tau) = \xi + \varepsilon e^{\tau T f_u(\xi,\alpha)} v^{(1)}, \;\; c_1 = 1, \;\; c_2 = 0.$$

Implementation in AUTO



Implementation in AUTO

$$
\begin{aligned}
\dot{U}(\tau) - F(U(\tau), \beta) &= 0, \quad \tau \in [0,1], \\
b(U(0), U(1), \beta) &= 0, \\
\int_0^1 q(U(\tau), \beta) d\tau &= 0,
\end{aligned}
$$

where

$$
U(\cdot), F(\cdot,\cdot) \in \mathbb{R}^{n_d},\; b(\cdot,\cdot) \in \mathbb{R}^{n_{bc}},\; q(\cdot,\cdot) \in \mathbb{R}^{n_{ic}},\; \beta \in \mathbb{R}^{n_{fp}},
$$

The number n_{fp} of *free parameters* β is

$$
n_{fp} = n_{bc} + n_{ic} - n_d + 1.
$$

In our primary BVPs: $n_d = 9,\; n_{ic} = 0$, and $n_{bc} = 19$ or 18

Example: $\dim W_-^u = 1$

- Lorenz system:

$$\begin{cases} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - b x_3, \end{cases}$$

with the standard value $b = \dfrac{8}{3}$.

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- The bifurcation curve in the (r, σ)-plane corresponding to the point-to-cycle connection is first presented by L.P. Shilnikov (1980).

Homotopy to eigenfunction

- At $(r, \sigma) = (21, 10)$, there is a *saddle limit cycle* with

$$x^{+}(0) = (9.265335, 13.196014, 15.997250), \quad T^{+} = 0.816222,$$

that has

$$\mu_s^{+} = 0.0000113431, \quad \mu_u^{+} = 1.26094.$$

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- Continuation in (λ, h) of the trivial solution of the BVP for the scaled adjoint eigenfunction $w(\tau)$ detects a *branch point* at

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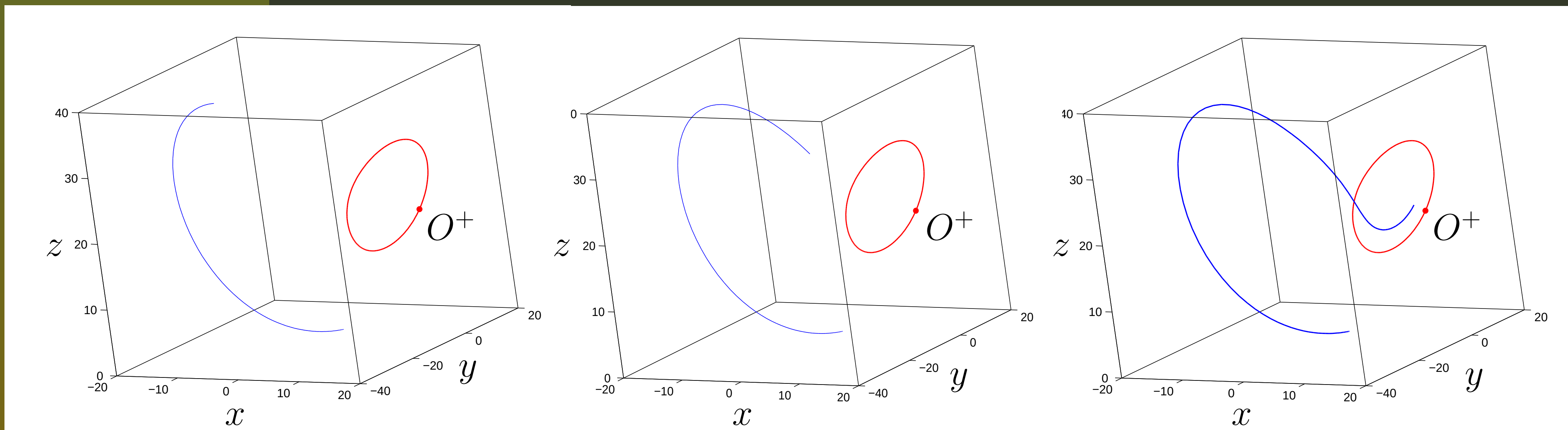
$$\lambda = \ln(\mu_u^+) = 0.231854.$$

- From it a nontrivial branch is followed until the value $h = 1$ is reached. This gives a nontrivial eigenfunction $w(t)$ with

$$w(0) = (0.168148, 0.877764, -0.448616)^{\mathrm{T}}, \quad \|w(0)\| = 1.$$

Homotopy to connection

- Continue in (T, h_1) until $h_1 = 0$:



(a) $T = 1.43924$ (b) $T = 1.54543$ (c) $T = 2.00352$

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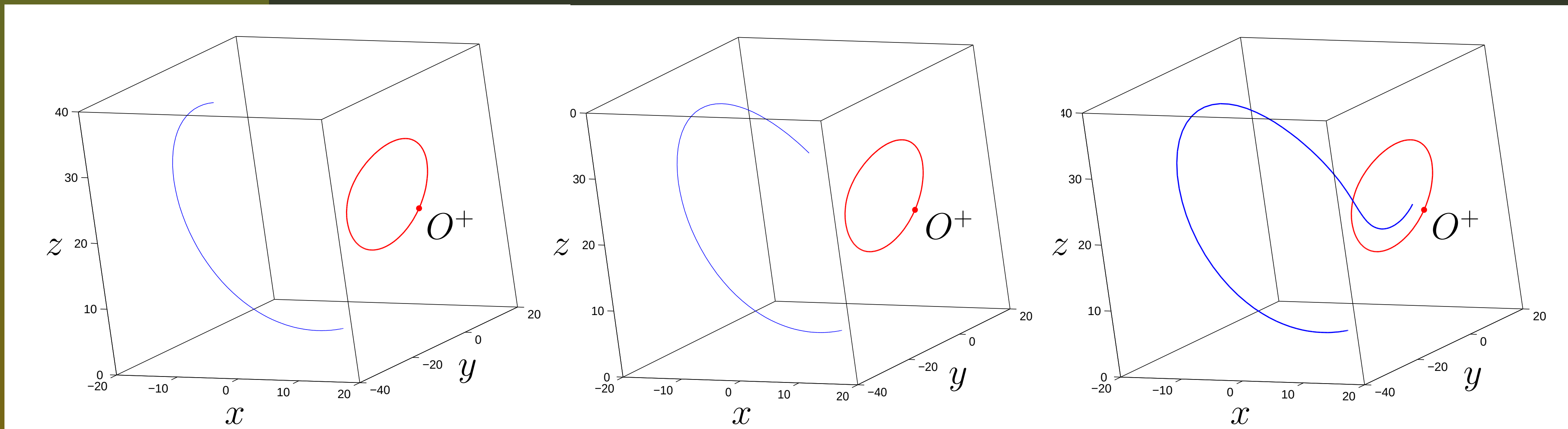
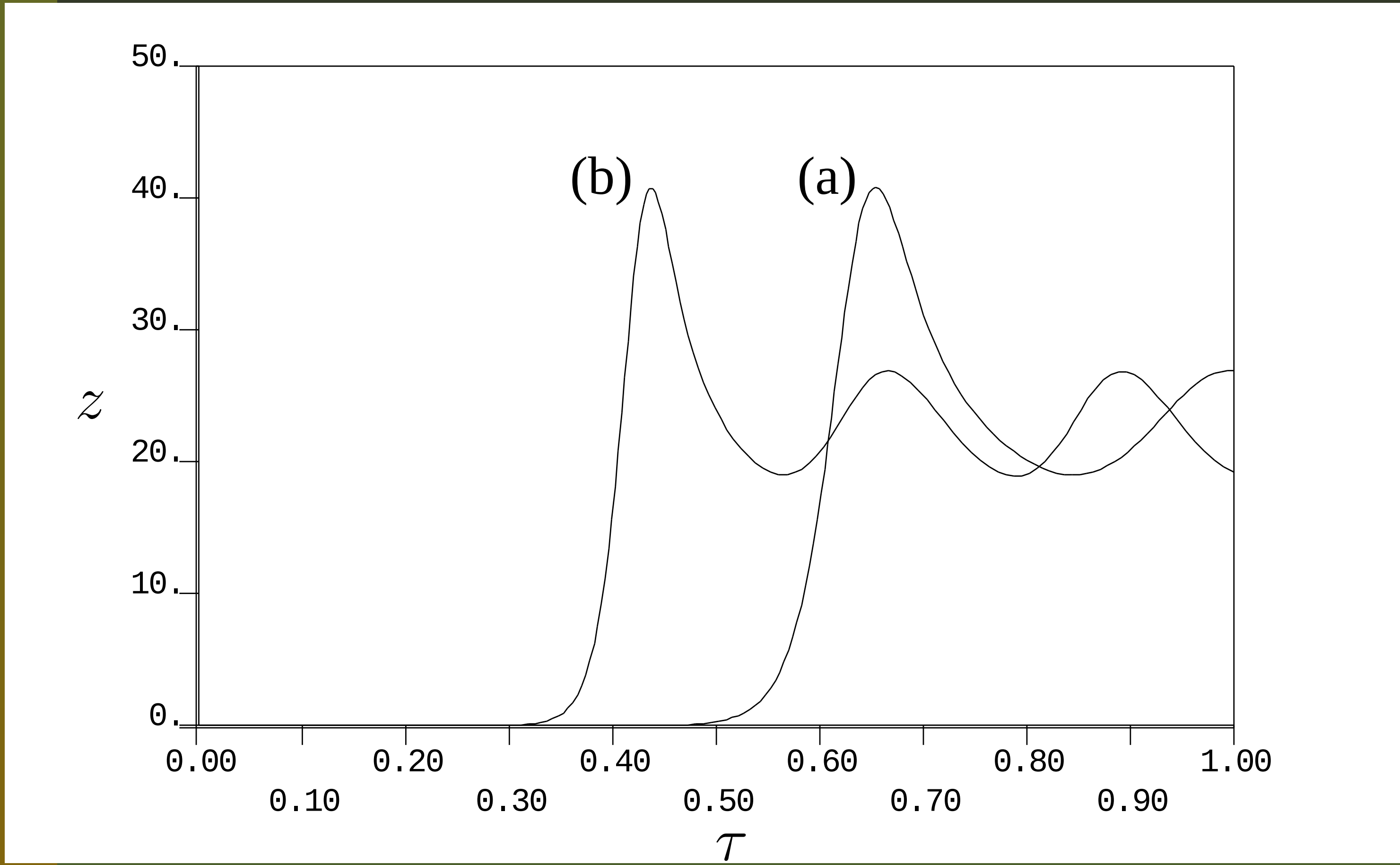


(a) $T = 1.43924$ (b) $T = 1.54543$ (c) $T = 2.00352$

- Continue in (r, h_2) until $h_2 = 0$, that occurs at $r = 24.0720$.

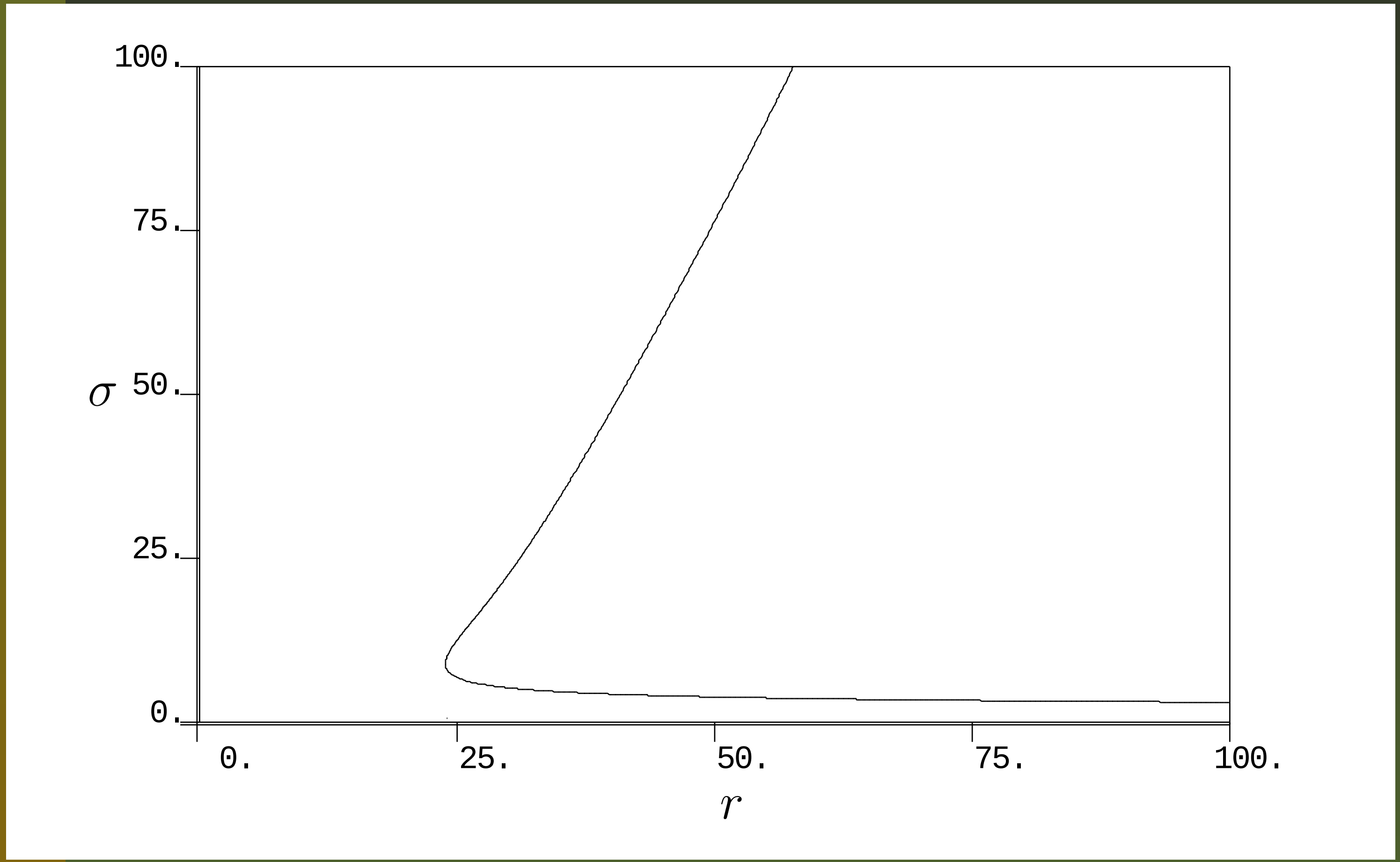
Continuation of the connection

- Improve connection by the continuation in (r, T):



(a) $(r, T) = (21.0, 2.00352)$; (b) $(r, T) = (24.0579, 3.0)$

- Continue the point-to-cycle bifurcation curve in (r, σ):



Example: $\dim W^u_- = 2$

- The standard tri-trophic food chain model:

$$\begin{cases} \dot{x}_1 &= x_1(1 - x_1) - \dfrac{a_1 x_1 x_2}{1 + b_1 x_1}, \\ \dot{x}_2 &= \dfrac{a_1 x_1 x_2}{1 + b_1 x_1} - \dfrac{a_2 x_2 x_3}{1 + b_1 x_2} - d_1 x_2, \\ \dot{x}_3 &= \dfrac{a_2 x_2 x_3}{1 + b_1 x_2} - d_2 x_3, \end{cases}$$

with $a_1 = 5,\ a_2 = 0.1,\ b_1 = 3$, and $b_2 = 2$.

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  with $a_1 = 5,\ a_2 = 0.1,\ b_1 = 3$, and $b_2 = 2$.

- Point-to-cycle connections in this model were first studied by M.P. Boer, B.W. Kooi, and S.A.L.M. Kooijman, [1999], “Homoclinic and heteroclinic orbits to a cycle in a tri-trophic food chain,” *J. Math. Biol.*, **39**, 19–38.

Homotopy to eigenfunction

- At $d_1 = 0.25,\ d_2 = 0.0125$, we have an *equilibrium*

$$\xi = (0.74158162, 0.16666666, 11.997732)$$

and a *saddle limit cycle* with the period $T^+ = 24.282248$ and

$$x^+(0) = (0.839705, 0.125349, 10.55289)$$

Its nontrivial multipliers are $\mu_s^+ = 0.6440615, \mu_u^+ = 6.107464 \cdot 10^2$.

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- Continuation in (λ, h) of the secondary barnch from the *branch point*

$$\lambda = \ln(\mu_s^+) = -0.439961.$$

gives at $h = 1$ a nontrivial eigenfunction $w(t)$ with $\|w(0)\| = 1$:

$$w(0) = (0.09306, -0.87791, -4.69689)^{\mathrm{T}}.$$

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Homotopy to connection

- The initial solution $u(\tau)$ is found by integration in CONTENT from a point in the plane tangent to W^u_- at distance $\varepsilon = 0.001$ to ξ:

$$u(0) = (0.742445, 0.166163, 11.997732).$$

Integration interval $T = 155.905$.

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- Continue in (T, h_1) towards a minimum of h_1.

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- Continue in (T, h_1) towards a minimum of h_1.
- Continue in (c_1, c_2, h_1) to get $h_1 = 0$;

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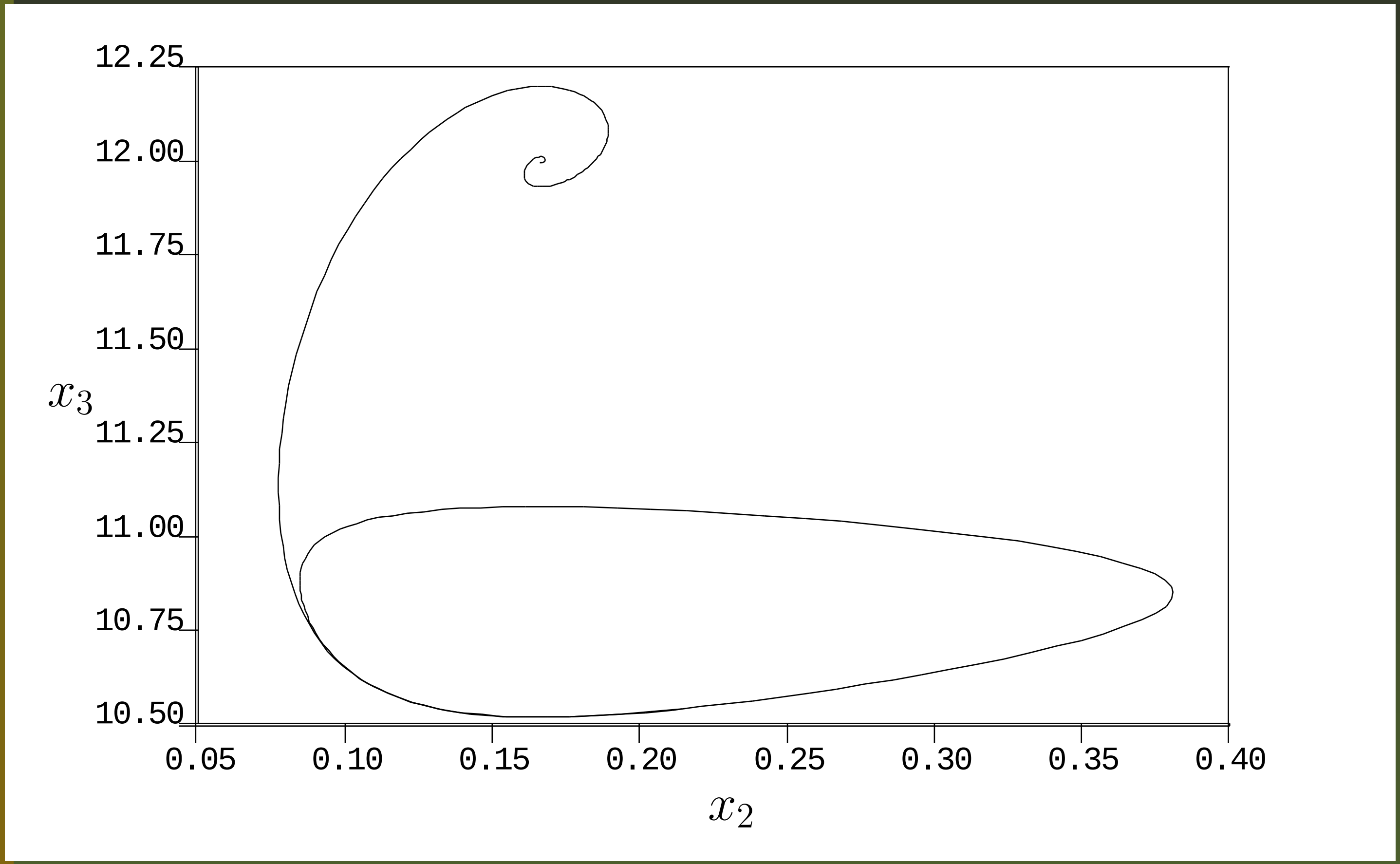
- Continue in (T, h_1) towards a minimum of h_1.
- Continue in (c_1, c_2, h_1) to get $h_1 = 0$;
- Continue in (c_1, c_2, h_2) to get $h_2 = 0$.

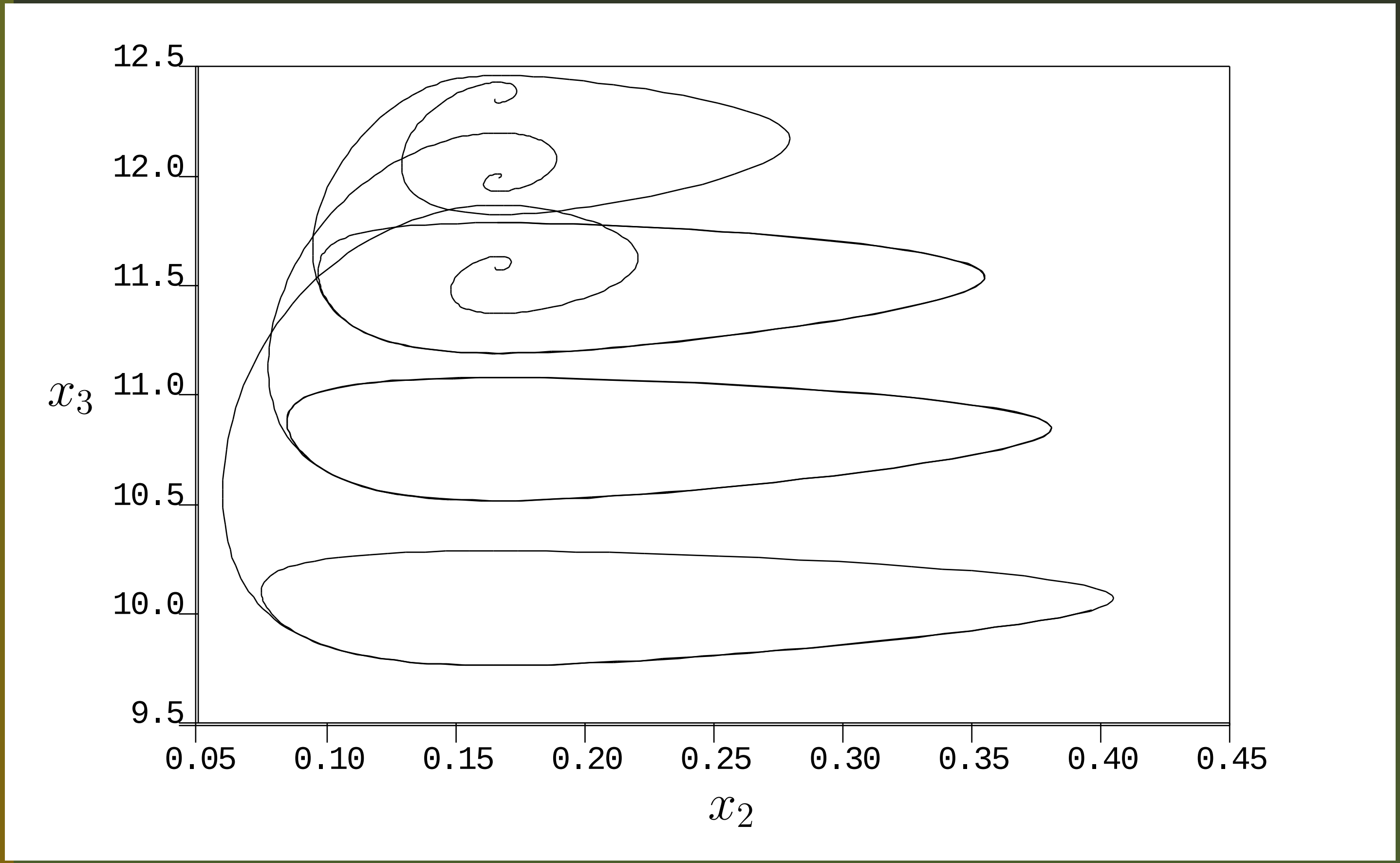
Continuation of the connection

- Improve connection by the continuations in T (and then in ε):



The connection with $T = 180.0, \quad \varepsilon^2 = 10^{-5}$.

- Continuation in $\alpha_1 = d_1$:



LP: $d_1 = 0.280913$ and $d_1 = 0.208045$ (LPC).

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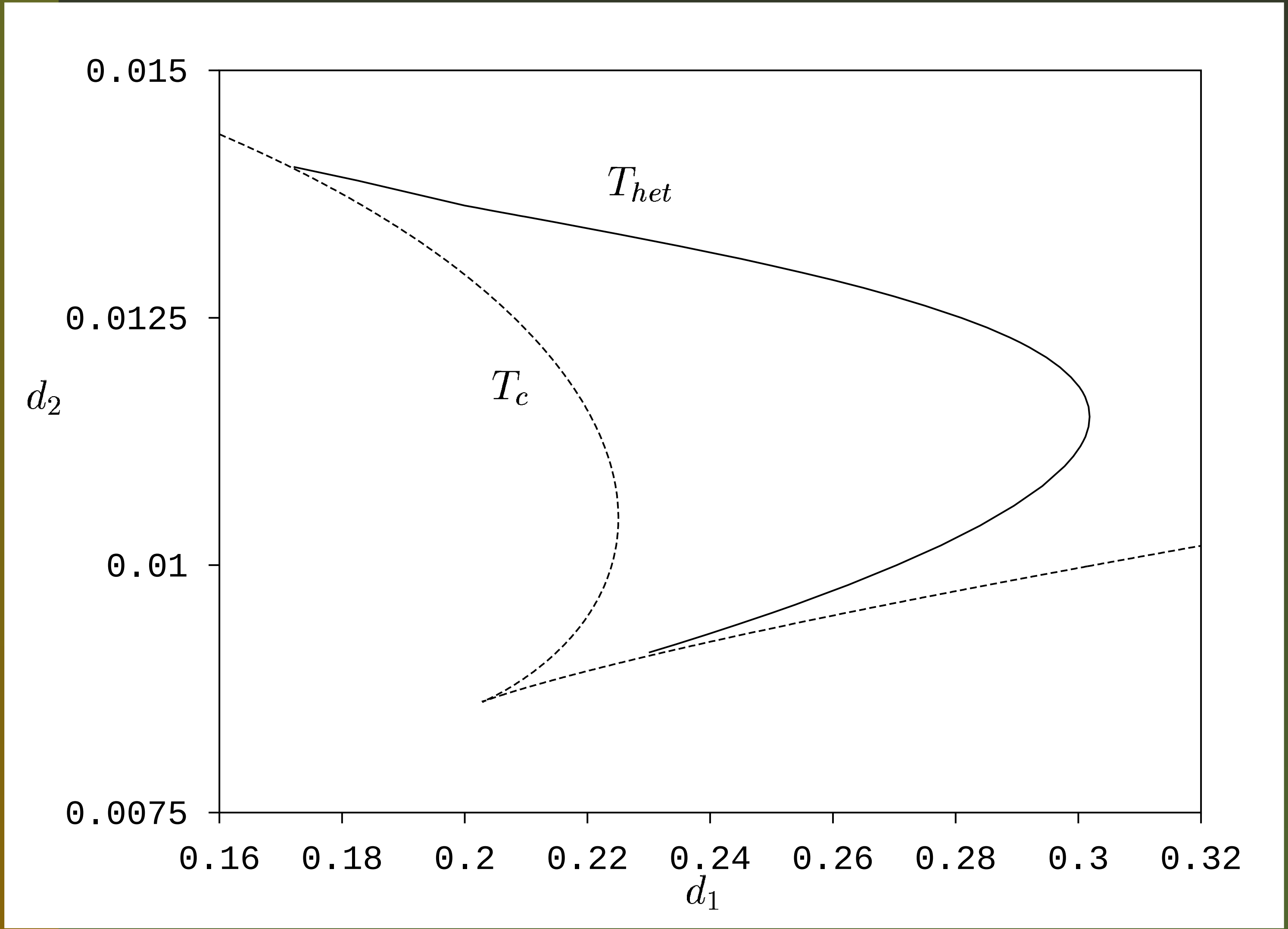
- Continue the point-to-cycle LP-bifurcation curve T_{het} in (d_1, d_2):



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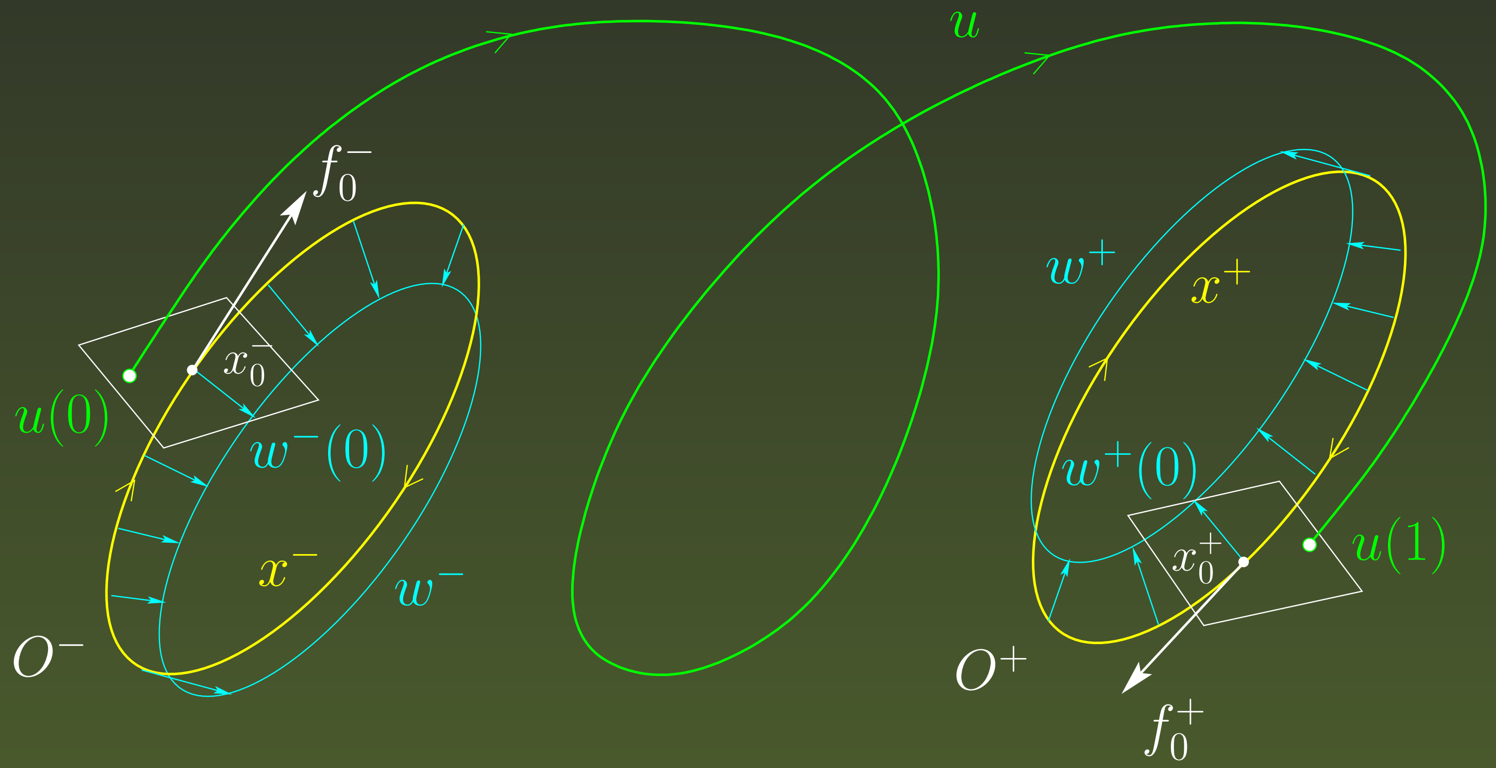
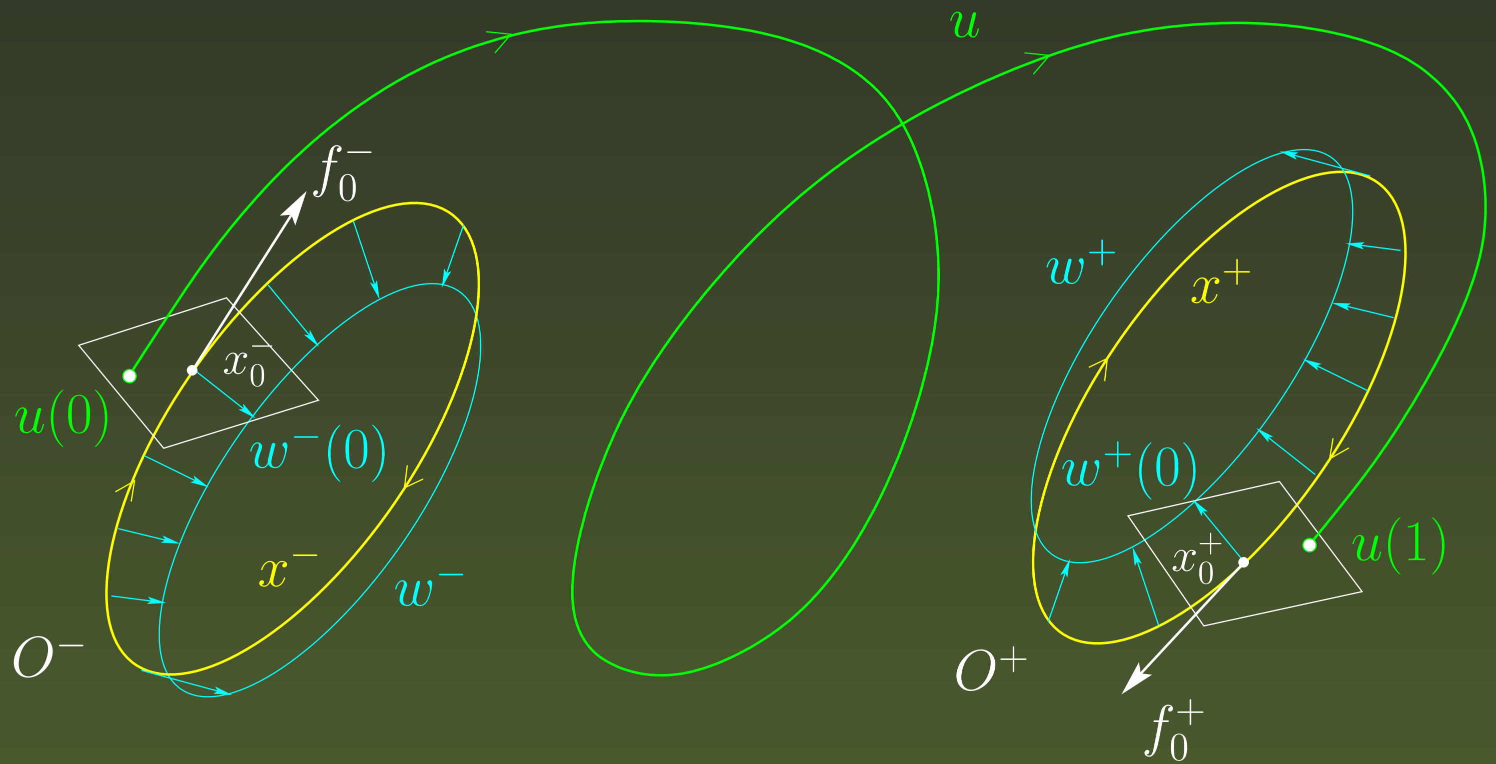
Open questions

- Cycle-to-cycle connections ?



Open questions

- Cycle-to-cycle connections ?



- Should all this be integrated in AUTO ?

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To be continued