Lecture 3

Bifurcations of limit cycles of ODEs and their numerical analysis using BVPs

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Contents

1. Codim 1 bifurcations of limit cycles

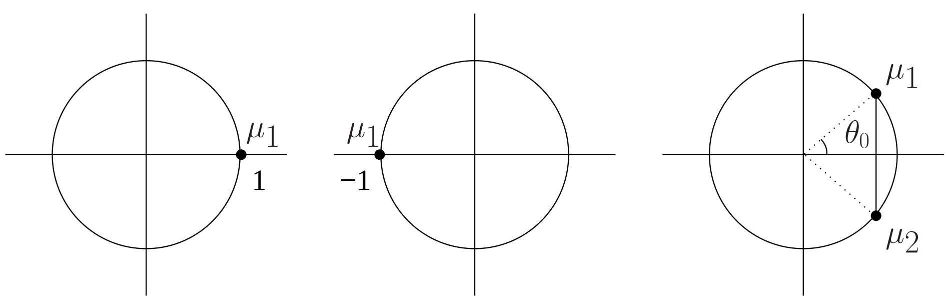
- Consider

$$\dot{x} = f(x, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

A **limit cycle** C_0 corresponds to a periodic solution $x_0(t + T_0) = x_0(t)$ and has **Floquet multipliers** $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 1$, the eigenvalues of $M(T_0)$:

$$\dot{M}(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \quad M(0) = I_n.$$

- Critical cases:



  - **Fold (LPC)**: $\mu_1 = 1$;
  - **Flip (PD)**: $\mu_1 = -1$;
  - **Torus (NS)**: $\mu_{1,2} = e^{\pm i\theta_0}, 0 < \theta_0 < \pi$.

Continuation of cycles in one parameter

- Defining system (BVCP):

$$\begin{cases} \dot{u}(\tau) - Tf(u(\tau), \alpha) &= 0, \quad \tau \in [0,1], \\ u(0) - u(1) &= 0, \\ \int_0^1 \langle \dot{v}(\tau), u(\tau) \rangle \, d\tau &= 0, \end{cases}$$

  where v is a reference periodic solution.

- Linearization of BVCP: $L \begin{pmatrix} u_1 \\ T_1 \\ \alpha_1 \end{pmatrix} = 0$ where

$$L = \begin{bmatrix} D - Tf_x(u,\alpha) & -f(u,\alpha) & -Tf_\alpha(u,\alpha) \\ \delta_0 - \delta_1 & 0 & 0 \\ \mathrm{Int}_{\dot{v}} & 0 & 0 \end{bmatrix}$$

  is an operator

  $L : \mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R}^2 \to \mathcal{C}^0([0,1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}.$

  At a generic solution (u, T, α) to BVCP:

$$\dim \mathcal{N}(L) = 1.$$

- The discretization of L via orthogonal collocation is a sparse $(mnN + n + 1) \times (mnN + n + 2)$ matrix that coincides with the matrix of the linearization of the discretization.

2. Detection of codim 1 bifurcations

- Test functions:

$$\begin{aligned}\psi_1 &= V_{mnN+n+2}, \\ \psi_2 &= \det(M(T_0)+I_n) \\ &= \frac{1}{\det(P_1)}\det(-P_0+P_1), \\ \psi_3 &= \det(M(T_0)\odot M(T_0) - I_{\frac{1}{2}n(n-1)}) \\ &= \frac{1}{\det(P_1\odot P_1)}\det(P_0\odot P_0 - P_1\odot P_1),\end{aligned}$$

where V is the tangent vector to the curve, $\odot$ is the bialternate product,

$$M(T_0) = -P_1^{-1}P_0$$

is the monodromy matrix.

- Bifurcations:

LPC : $\psi_1 = 0$

PD : $\psi_2 = 0$

NS : $\psi_3 = 0, \psi_1 \neq 0 \quad (\mu_1\mu_2 = 1)$

$$M^2(T_0) - 2\cos(\theta_0)M(T_0) + I_n$$

has rank defect 2.

3. Continuation of codim 1 bifurcations

- PD and LPC:

$$(u, T, \alpha) \in \mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2$$

$$\begin{cases} \dot{u}(\tau) - Tf(u(\tau), \alpha) &= 0, \quad \tau \in [0,1], \\ u(0) - u(1) &= 0, \\ \int_0^1 \langle \dot{v}(\tau), u(\tau) \rangle \, d\tau &= 0, \\ G[u, T, \alpha] &= 0. \end{cases}$$

- NS: $(u, T, \alpha, \kappa) \in \mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$

$$\begin{cases} \dot{u}(\tau) - Tf(u(\tau), \alpha) &= 0, \quad \tau \in [0,1], \\ u(0) - u(1) &= 0, \\ \int_0^1 \langle \dot{v}(\tau), u(\tau) \rangle \, d\tau &= 0, \\ G_{11}[u, T, \alpha, \kappa] &= 0, \\ G_{22}[u, T, \alpha, \kappa] &= 0. \end{cases}$$

PD-continuation

- There exist $v_{01}, w_{01} \in \mathcal{C}^0([0,1],\mathbb{R}^n)$, and $w_{02} \in \mathbb{R}^n$, such that

  $$N_1 : \mathcal{C}^1([0,1],\mathbb{R}^n)\times\mathbb{R} \to \mathcal{C}^0([0,1],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R},$$

  $$N_1 = \left[\begin{array}{cc} D - Tf_x(u,\alpha) & w_{01} \\ \delta_0 + \delta_1 & w_{02} \\ \mathrm{Int}_{v_{01}} & 0 \end{array}\right],$$

  is one-to-one and onto near a generic PD bifurcation point.

- Define G by solving

  $$N_1 \begin{pmatrix} v \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- The BVP for (v, G) can be written in the "classical form"

  $$\left\{\begin{array}{rcl} \dot{v}(\tau) - Tf_x(u(\tau),\alpha)v(\tau) + Gw_{01}(\tau) & = & 0, \\ v(0) + v(1) + Gw_{02} & = & 0, \\ \int_0^1 \langle v_{01}(\tau), v(\tau)\rangle d\tau & = & 1. \end{array}\right.$$

LPC-continuation

- There exist $v_{01}, w_{01} \in \mathcal{C}^0([0,1],\mathbb{R}^n), w_{02} \in \mathbb{R}^n$, and $v_{02}, w_{03} \in \mathbb{R}$ such that

  $N_2 : \mathcal{C}^1([0,1],\mathbb{R}^n)\times\mathbb{R}^2 \to \mathcal{C}^0([0,1],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^2$,

  $$N_2 = \begin{bmatrix} D - Tf_x(u,\alpha) & -f(u,\alpha) & w_{01} \\ \delta_0 - \delta_1 & 0 & w_{02} \\ \mathrm{Int}_{f(u,\alpha)} & 0 & w_{03} \\ \mathrm{Int}_{v_{01}} & v_{02} & 0 \end{bmatrix},$$

  is one-to-one and onto near a generic LPC bifurcation point.

- Define G by solving

  $$N_2 \begin{pmatrix} v \\ S \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

- "Classical" form:

  $$\left\{ \begin{array}{rcl} \dot{v}(\tau) - Tf_x(u(\tau),\alpha)v(\tau) & & \\ -Sf(u(\tau),\alpha) + Gw_{01}(\tau) & = & 0, \\ v(0) - v(1) + Gw_{02} & = & 0, \\ \int_0^1 \langle f(u(\tau),\alpha), v(\tau)\rangle d\tau + Gw_{03} & = & 0, \\ \int_0^1 \langle v_{01}(\tau), v(\tau)\rangle d\tau + Sv_{02} & = & 1. \end{array} \right.$$

NS-continuation

- There exist $v_{01}, v_{02}, w_{11}, w_{12} \in \mathcal{C}^0([0,2], \mathbb{R}^n)$, and $w_{21}, w_{22} \in \mathbb{R}^n$, such that

  $N_3 : \mathcal{C}^1([0,2], \mathbb{R}^n) \times \mathbb{R}^2 \to \mathcal{C}^0([0,2], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2,$

  $$N_3 = \left[\begin{array}{ccc} D - Tf_x(u,\alpha) & w_{11} & w_{12} \\ \delta_0 - 2\kappa\delta_1 + \delta_2 & w_{21} & w_{22} \\ \mathrm{Int}_{v_{01}} & 0 & 0 \\ \mathrm{Int}_{v_{02}} & 0 & 0 \end{array} \right],$$

  is one-to-one and onto near a generic NS bifurcation point.

- Define G_{jk} by solving

  $$N_3 \begin{pmatrix} r & s \\ G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- At the NS-cycle: $\kappa = \cos\theta$.

Remarks on continuation of bifurcations

- After discretization via orthogonal collocation, all linear BVPs for G's have sparsity structure that is identical to that of the linearization of the BVP for limit cycles.

- For each defining system holds: *Simplicity of the bifurcation + Transversality ⇒ Regularity of the defining BVP*.

- Jacobian matrix of each (discretized) defining BVP can be efficiently computed using adjoint linear BVP.

- Border adaptation using solutions of the adjoint linear BVPs.

- Actually implemented in MATCONT.

5. Periodic normal forms

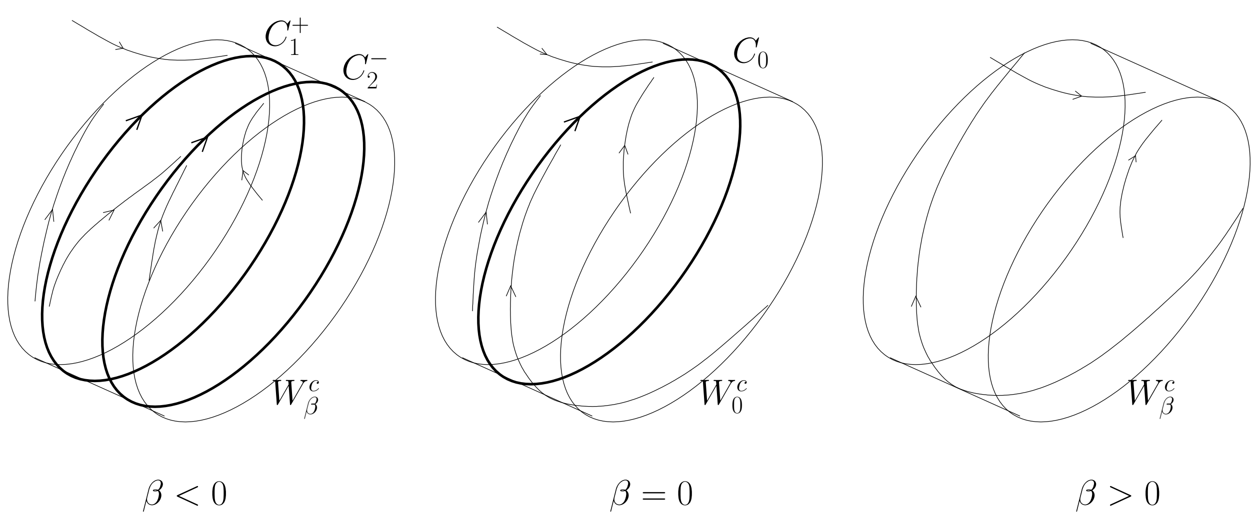
LPC bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\beta) - \xi + a(\beta)\xi^2 + \mathcal{O}(\xi^3), \\ \dfrac{d\xi}{dt} = \beta + b(\beta)\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

  where $a, b \in \mathbb{R}$ and the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

- Phase portraits ($b(0) > 0$):



  Collision and disappearance of two limit cycles: $C_1^- + C_2^+ \to \emptyset$

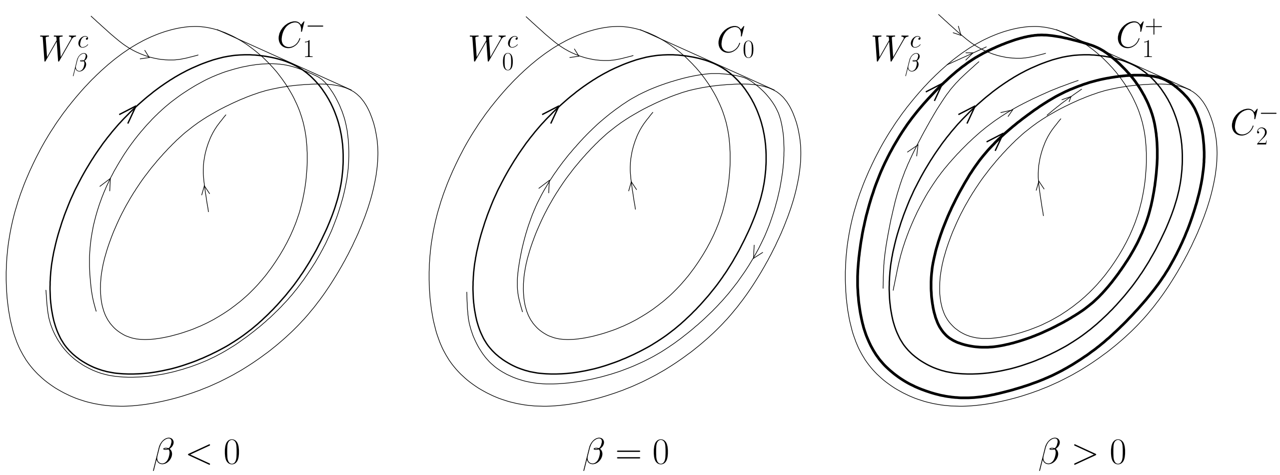
PD bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} & = & 1 + \nu(\beta) + a(\beta)\xi^2 + \mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} & = & \beta\xi + c(\beta)\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$ and the $\mathcal{O}(\xi^3)$-terms are $2T_0$-periodic in τ.

- Phase portraits ($c(0) < 0$):



Period-doubling: $C_1^- \to C_1^+ + C_2^-$

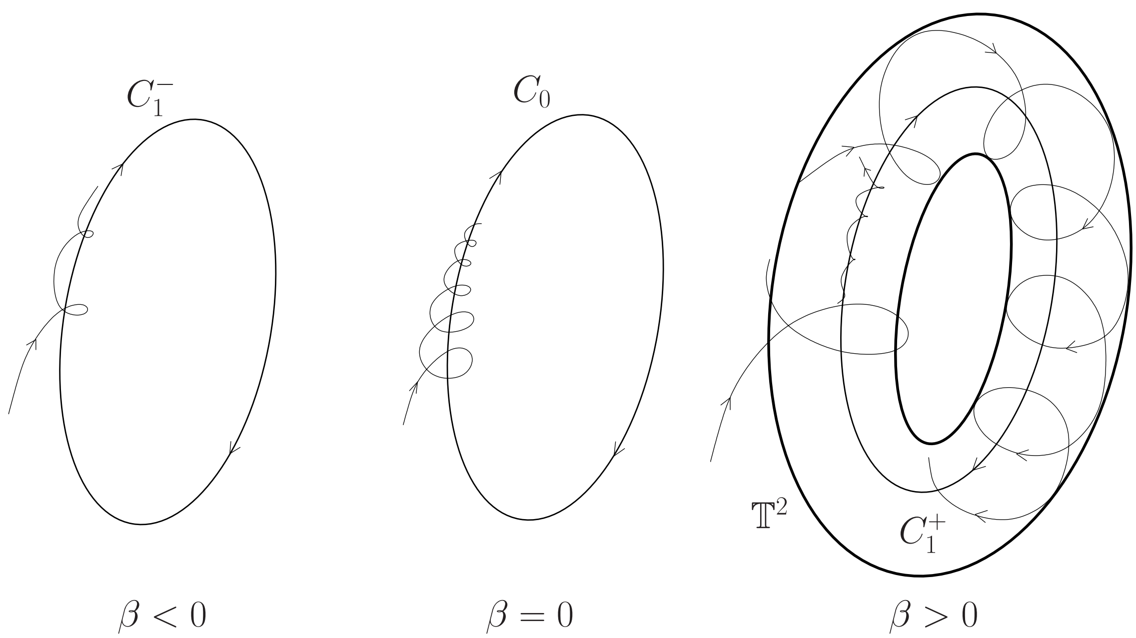
NS bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1+\nu(\beta)+a(\beta)|\xi|^2+\mathcal{O}(|\xi|^4), \\ \dfrac{d\xi}{dt} &= \left(\beta+\dfrac{i\theta(\beta)}{T(\beta)}\right)\xi+d(\beta)\xi|\xi|^2+\mathcal{O}(|\xi|^4), \end{cases}$$

where $a\in\mathbb{R}, d\in\mathbb{C}$ and the $\mathcal{O}(\|\xi\|^4)$-terms are T_0-periodic in τ

- Phase portraits ($\Re(d(0))<0$):



Torus generation: $C_1^- \to C_1^+ + \mathbb{T}^2$

5. Critical normal form coefficients

- **Fredholm Alternative for BVPs**

Assume $\varphi, \varphi^* \in \mathcal{C}^1([0, T_0], \mathbb{R}^n)$ satisfy

$$\begin{cases} \dot{\varphi}(\tau) - A(\tau)\varphi(\tau) &= 0, \ \tau \in [0, T_0], \\ \varphi(0) - \varphi(T_0) &= 0, \\ \int_0^{T_0} \langle \varphi(\tau), \varphi(\tau) \rangle d\tau - 1 &= 0, \end{cases}$$

and

$$\begin{cases} \dot{\varphi}^*(\tau) + A^{\mathsf{T}}(\tau)\varphi^*(\tau) &= 0, \ \tau \in [0, T_0], \\ \varphi^*(0) - \varphi^*(T_0) &= 0. \end{cases}$$

If $h \in \mathcal{C}^1([0, T_0], \mathbb{R}^n)$ is a solution to

$$\begin{cases} \dot{h}(\tau) - A(\tau)h(\tau) &= g(\tau), \ \tau \in [0, T_0], \\ h(0) - h(T_0) &= 0, \end{cases}$$

with $g \in \mathcal{C}^0([0, T_0], \mathbb{R}^n)$, then

$$\int_0^{T_0} \langle \varphi^*(\tau), g(\tau) \rangle \ d\tau = 0$$

(**Fredholm solvability condition**). When it holds, there is a unique solution h satisfying

$$\int_0^{T_0} \langle \varphi^*(\tau), h(\tau) \rangle \ d\tau = 0.$$

Multilinear forms

At a codimension-one point write

$$\begin{aligned} f(x_0(t)+v,\alpha_0) &= f(x_0(t),\alpha_0) \\ &+ A(t)v + \frac{1}{2}B(t;v,v) \\ &+ \frac{1}{6}C(t;v,v,v) + \mathcal{O}(\|v\|^4), \end{aligned}$$

where $A(t) = f_x(x_0(t),\alpha_0)$ and the components of the multilinear functions B and C are given by

$$B_i(t;u,v) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 f_i(x,\alpha_0)}{\partial x_j \partial x_k}\right|_{x=x_0(t)} u_j v_k$$

and

$$C_i(t;u,v,w) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 f_i(x,\alpha_0)}{\partial x_j \partial x_k \partial x_l}\right|_{x=x_0(t)} u_j v_k w_l,$$

for $i = 1, 2, \ldots, n$. These are T_0-periodic in t.

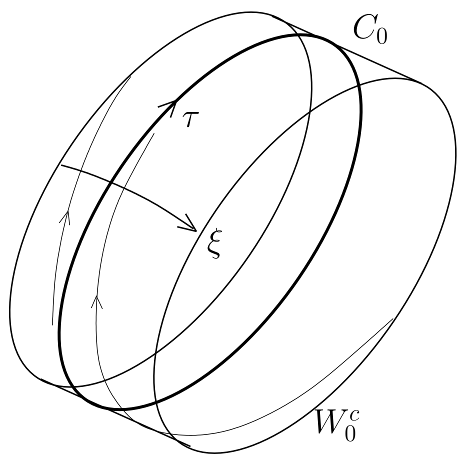
Fold (LPC): $\mu_1 = 1$

- Critical center manifold W_0^c:

$$x = x_0(\tau) + \xi v(\tau) + H(\tau, \xi),$$

where $\tau \in [0, T_0], \xi \in \mathbb{R}, H(T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2} h_2(\tau)\xi^2 + \mathcal{O}(\xi^3)$$



- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 - \xi + a\xi^2 + \mathcal{O}(\xi^3), \\ \dfrac{d\xi}{dt} = b\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$, while the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

LPC: Generalized and adjoint eigenfunctions

$$\left\{\begin{array}{rcl} \dot{v}(\tau) - A(\tau)v(\tau) - f(x_0(\tau),\alpha_0) & = & 0,\ \ \tau \in [0,T_0], \\ v(0) - v(T_0) & = & 0, \\ \int_0^{T_0} \langle v(\tau), f(x_0(\tau),\alpha_0)\rangle d\tau & = & 0, \end{array}\right.$$

implying

$$\int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau),\alpha_0)\rangle \, d\tau = 0,$$

where φ^* satisfies

$$\left\{\begin{array}{rcl} \dot{\varphi}^*(\tau) + A^{\mathsf{T}}(\tau)\varphi^*(\tau) & = & 0,\ \ \tau \in [0,T_0], \\ \varphi^*(0) - \varphi^*(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

LPC: Computation of b

- Substitute into

$$\frac{dx}{dt} = \frac{\partial x}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial x}{\partial \tau}\frac{d\tau}{dt}$$

- Collect

$$\begin{aligned}
\xi^0 &: \ \dot{x}_0 = f(x_0, \alpha_0), \\
\xi^1 &: \ \dot{v} - A(\tau)v = \dot{x}_0, \\
\xi^2 &: \ \dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) - 2af(x_0, \alpha_0) + \\
& \qquad\qquad\qquad\qquad 2\dot{v} - 2bv.
\end{aligned}$$

- Fredholm solvability condition

$$b = \frac{1}{2}\int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) + 2A(\tau)v(\tau)\rangle d\tau$$

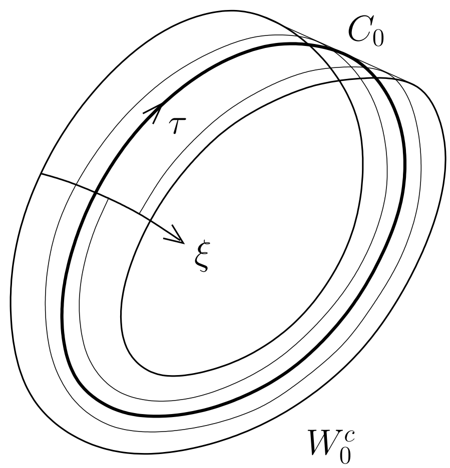
Flip (PD): $\mu_1 = -1$

- Critical center manifold W_0^c:

$$x = x_0(\tau) + \xi w(\tau) + H(\tau, \xi),$$

where $\tau \in [0, 2T_0], \xi \in \mathbb{R}, H(2T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2}h_2(\tau)\xi^2 + \frac{1}{6}h_3(\tau)\xi^3 + \mathcal{O}(\xi^4)$$



- Critical periodic normal form on W_0^c:

$$\begin{cases} \frac{d\tau}{dt} &= 1 + a\xi^2 + \mathcal{O}(\xi^4), \\ \frac{d\xi}{dt} &= c\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$, while the $\mathcal{O}(\xi^4)$-terms are $2T_0$-periodic in τ.

PD: Eigenfunctions

$$w(\tau) = \begin{cases} v(\tau), & \tau \in [0, T_0], \\ -v(\tau - T_0), & \tau \in [T_0, 2T_0], \end{cases} ,$$

$$w^*(\tau) = \begin{cases} v^*(\tau), & \tau \in [0, T_0], \\ -v^*(\tau - T_0), & \tau \in [T_0, 2T_0], \end{cases}$$

with

$$\begin{cases} \dot{v}(\tau) - A(\tau)v(\tau) = 0, \ \tau \in [0, T_0], \\ v(0) + v(T_0) = 0, \\ \int_0^{T_0} \langle v(\tau), v(\tau) \rangle d\tau - 1 = 0, \end{cases}$$

$$\begin{cases} \dot{v}^*(\tau) + A^{\mathsf{T}}(\tau)v^*(\tau) = 0, \ \tau \in [0, T_0], \\ v^*(0) + v^*(T_0) = 0, \\ \int_0^{T_0} \langle v^*(\tau), v(\tau) \rangle d\tau - 1/2 = 0. \end{cases}$$

PD: Quadratic terms

$$\xi^2 \ : \ \dot{h}_2 - A(\tau)h_2 = B(\tau; w, w) - 2a\dot{x}_0, \quad \tau \in [0, 2T_0].$$

Since $\mathcal{N}\left(\frac{d}{d\tau} - A(\tau)\right) = \text{span}\{w, \psi = \dot{x}_0\}$, we must have

$$\left\{ \begin{array}{rcl} \int_0^{2T_0} \langle w^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau)\rangle \, d\tau & = & 0, \\ \int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau)\rangle \, d\tau & = & 0, \end{array} \right.$$

where ψ^* satisfies

$$\left\{ \begin{array}{rcl} \dot{\psi}^*(\tau) + A^{\mathsf{T}}(\tau)\psi^*(\tau) & = & 0, \ \tau \in [0, T_0], \\ \psi^*(0) - \psi^*(T_0) & = & 0, \\ \int_0^{T_0} \langle \psi^*(\tau), f(x_0(\tau), \alpha_0)\rangle \, d\tau - 1/2 & = & 0, \end{array} \right.$$

and is extended to $[T_0, 2T_0]$ by periodicity.

PD: Computation of a and h_2

- The first Fredholm condition holds identically for all a, while the second gives

$$
\begin{aligned}
a &= \frac{1}{2}\int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)\rangle d\tau \\
&= \int_0^{T_0} \langle \psi^*(\tau), B(\tau; v(\tau), v(\tau)\rangle d\tau.
\end{aligned}
$$

- Define h_2 on $[0, T_0]$ as the unique solution to

$$
\left\{
\begin{array}{rcl}
\dot{h}_2(\tau) - A(\tau)h_2(\tau) & & \\
-B(\tau; v(\tau), v(\tau)) + 2af(x_0(\tau), \alpha_0) & = & 0, \\
h_2(0) - h_2(T_0) & = & 0, \\
\int_0^{T_0} \langle \psi^*(\tau), h_2(\tau)\rangle \, d\tau & = & 0,
\end{array}
\right.
$$

and extend it by periodicity to $[T_0, 2T_0]$.

PD: Computation of c

Cubic terms: ξ^3

$$\dot{h}_3 - A(\tau)h_3 = C(\tau; w, w, w) + 3B(\tau; w, h_2) - 6a\dot{w} - 6cw$$

The Fredholm solvability condition implies

$$\begin{aligned} 6c &= \int_0^{2T_0} \langle w^*(\tau), C(\tau; w(\tau), w(\tau), w(\tau)) + \\ &\qquad 3B(\tau; w(\tau), h_2(\tau))\rangle d\tau \\ &- \int_0^{2T_0} \langle w^*(\tau), 6aA(\tau)w(\tau)\rangle d\tau \end{aligned}$$

or

$$\begin{aligned} c &= \frac{1}{3}\int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), v(\tau)) + \\ &\qquad 3B(\tau; v(\tau), h_2(\tau)) - 6aA(\tau)v(\tau)\rangle d\tau \end{aligned}$$

Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}$

No **strong resonances**: $e^{i\nu\theta_0} \neq 1, \nu = 1, 2, 3, 4.$

- Critical center manifold $W_0^c : \tau \in [0, T_0],\ \xi \in \mathbb{C}$

  $$x = x_0(\tau) + \xi v(\tau) + \bar{\xi}\bar{v}(\tau) + H(\tau, \xi, \bar{\xi}),$$

  where $H(T_0, \xi, \bar{\xi}) = H(0, \xi, \bar{\xi})$,

  $$\begin{aligned} H(\tau, \xi, \bar{\xi}) &= \frac{1}{2}h_{20}(\tau)\xi^2 + h_{11}(\tau)\xi\bar{\xi} + \frac{1}{2}h_{02}(\tau)\bar{\xi}^2 \\ &+ \frac{1}{6}h_{30}(\tau)\xi^3 + \frac{1}{2}h_{21}(\tau)\xi^2\bar{\xi} \\ &+ \frac{1}{2}h_{12}(\tau)\xi\bar{\xi}^2 + \frac{1}{6}h_{03}(\tau)\bar{\xi}^3 \\ &+ \mathcal{O}(|\xi|^4). \end{aligned}$$

- Critical periodic normal form on W_0^c:

  $$\left\{ \begin{aligned} \frac{d\tau}{dt} &= 1 + a|\xi|^2 + \mathcal{O}(|\xi|^4), \\ \frac{d\xi}{dt} &= \frac{i\theta_0}{T_0}\xi + d\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{aligned} \right.$$

  where $a \in \mathbb{R}, d \in \mathbb{C}$, and the $\mathcal{O}(|\xi|^4)$-terms are T_0-periodic in τ.

NS: Complex eigenfunctions

$$\left\{\begin{array}{rcl}\dot{v}(\tau) - A(\tau)v(\tau) + \dfrac{i\theta_0}{T_0}v(\tau) & = & 0,\ \ \tau \in [0, T_0],\\ v(0) - v(T_0) & = & 0,\\ \int_0^{T_0} \langle v(\tau), v(\tau)\rangle d\tau - 1 & = & 0.\end{array}\right.$$

and

$$\left\{\begin{array}{rcl}\dot{v}^*(\tau) + A^{\mathsf{T}}(\tau)v^*(\tau) + \dfrac{i\theta_0}{T_0}v^*(\tau) & = & 0,\ \ \tau \in [0, T_0],\\ v^*(0) - v^*(T_0) & = & 0,\\ \int_0^{T_0} \langle v^*(\tau), v(\tau)\rangle d\tau - 1 & = & 0.\end{array}\right.$$

NS: Quadratic terms

- $\xi^2\bar{\xi}^0$:

$$\dot{h}_{20} - A(\tau)h_{20} + \frac{2i\theta_0}{T_0}h_{20} = B(\tau; v, v)$$

Since $e^{2i\theta_0}$ is not a multiplier of the critical cycle, the BVP

$$\left\{ \begin{array}{rcl} \dot{h}_{20} - A(\tau)h_{20} & & \\ +\frac{2i\theta_0}{T_0}h_{20} - B(\tau; v(\tau), v(\tau)) & = & 0, \\ h_{20}(0) - h_{20}(T_0) & = & 0. \end{array} \right.$$

has a unique solution on $[0, T_0]$.

- $|\xi|^2$:

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - a\dot{x}_0$$

Here

$$\mathcal{N}\left(\frac{d}{d\tau} - A(\tau)\right) = \text{span}(\varphi = \dot{x}_0).$$

NS: Computation of a and h_{11}

- Define φ^* as the unique solution of

$$\left\{\begin{array}{rcl} \dot{\varphi}^*(\tau) + A^{\mathsf{T}}(\tau)\varphi^*(\tau) & = & 0, \\ \varphi^*(0) - \varphi^*(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

- Fredholm solvability:

$$a = \int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), \bar{v}(\tau)\rangle \ d\tau$$

- Then find h_{11} on $[0, T_0]$ from the BVP

$$\left\{\begin{array}{rcl} \dot{h}_{11}(\tau) - A(\tau)h_{11}(\tau) & & \\ -B(\tau; v(\tau), \bar{v}(\tau)) + af(x_0(\tau), \alpha_0) & = & 0, \\ h_{11}(0) - h_{11}(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), h_{11}(\tau)\rangle \ d\tau & = & 0. \end{array}\right.$$

NS: Computation of d

- Cubic terms: $\xi^2\bar{\xi}$

$$\begin{aligned} \dot{h}_{21} - Ah_{21} + \frac{i\theta_0}{T_0}h_{21} &= 2B(\tau; h_{11}, v) \\ &+ B(\tau; h_{20}, \bar{v}) \\ &+ C(\tau; v, v, \bar{v}) \\ &- 2a\dot{v} - 2dv. \end{aligned}$$

- Fredholm solvability condition:

$$\begin{aligned} d &= \frac{1}{2}\int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), \bar{v}(\tau))\rangle \, d\tau \\ &+ \frac{1}{2}\int_0^{T_0} \langle v^*(\tau), B(\tau; h_{11}(\tau), v(\tau)) + \\ &\qquad B(\tau; h_{20}(\tau), \bar{v}(\tau))\rangle \, d\tau \\ &- a\int_0^{T_0} \langle v^*(\tau), A(\tau)v(\tau)\rangle \, d\tau - \frac{ia\theta_0}{T_0}. \end{aligned}$$

Remarks on numerical periodic normalization

- Only the derivatives of $f(x, \alpha_0)$ are used, not those of the Poincaré map.

- Detection of codim 2 points is easy.

- After discretization via orthogonal collocation, all linear BVPs involved have the standard sparsity structure.

- One can re-use solutions to linear BVPs appearing in the continuation to compute the normal form coefficients.

- Actually implemented in MATCONT.