Lecture 4

Numerical local bifurcation analysis of iterated maps

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Contents

1. Fixed points and cycles

- Consider a family of smooth maps

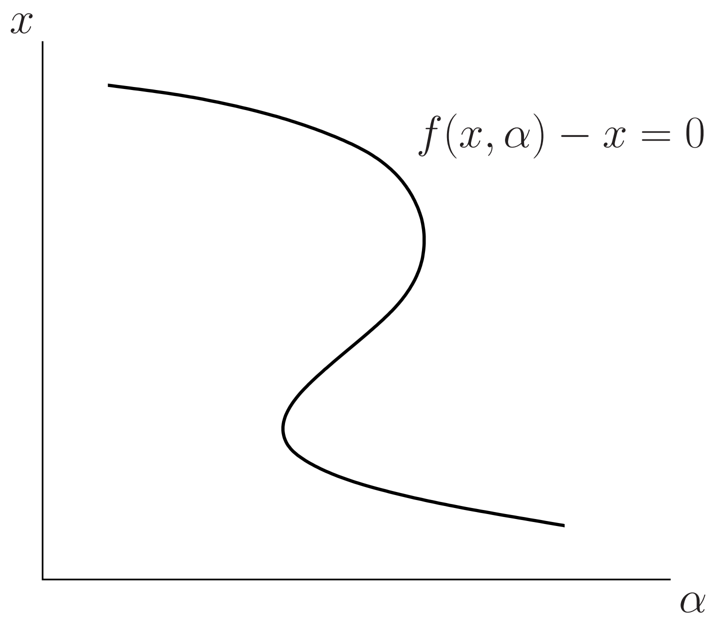
$$x \mapsto f(x,\alpha) \equiv f_{(\alpha)}(x), \qquad x \in \mathbb{R}^n, \alpha \in \mathbb{R},$$

- A **fixed point** x_0 satisfies $f(x_0,\alpha) = x_0$.
  A k-**cycle** $\{x_0, x_1, x_2, \ldots, x_{k-1}\}$ satisfies

$$\left.\begin{array}{rcl} f(x_0,\alpha_0) & = & x_1 \\ f(x_1,\alpha_0) & = & x_2 \\ \cdots & & \\ f(x_{k-1},\alpha_0) & = & x_0 \end{array}\right\} \Rightarrow \; f^k_{(\alpha)}(x_0) = x_0$$

  All points $x_j, j = 0, 1, \ldots, k-1$ of the cycle are assumed to be different.

- Fixed point manifold:

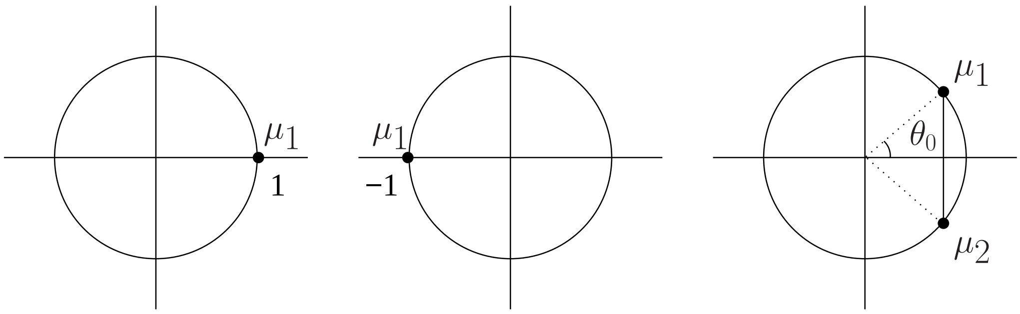


Critical stability cases

- Let $x_0 \in \mathbb{R}^n$ be a fixed point at parameter value α_0 and $A_0 = f_x(x_0, \alpha_0)$.

  If $|\mu| < 1$ for each eigenvalue (**multiplier**) μ of A_0, x_0 is stable.
  If $|\mu| > 1$ for at least one eigenvalue μ of A_0, x_0 is unstable.

- Critical cases:



  - **Fold** (LP): $\mu_1 = 1$;
  - **Flip** (PD): $\mu_1 = -1$;
  - **Neimark-Sacker** (NS): $\mu_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$.

2. Detection of codim 1 bifurcations

- Test functions:

$$\begin{aligned}
\psi_1 &= \det\begin{pmatrix} f_x & f_\alpha \\ v^{\mathsf{T}} & \end{pmatrix} \\
\psi_2 &= v_{n+1} \\
\psi_3 &= \det(A(x,\alpha) + I_n) \\
\psi_4 &= \det(A(x,\alpha) \odot A(x,\alpha) - I_m),
\end{aligned}$$

  where $A(x,\alpha) = f_x(x,\alpha)$, $v \in \mathbb{R}^{n+1}$ is the normalized tangent vector to the fixed point manifold,

$$m = \frac{n(n-1)}{2},$$

  and $\odot$ stands for the **bialternate matrix product**.

- Singularities:
  - branching point `BP` ($\psi_1 = 0$)
  - `LP`: $\mu_1 = 1$ ($\psi_2 = 0,\ \psi_1 \neq 0$)
  - `PD`: $\mu_1 = -1$ ($\psi_3 = 0$)
  - `NS`: $\mu_1\mu_2 = 1$ ($\psi_4 = 0$)

3. Continuation of codim 1 bifurcations

LP and PD curves:

- ALCP with $(x, \alpha) \in \mathbb{R}^{n+2}$

$$\begin{cases} f(x,\alpha) - x &= 0 \\ \det(A(u,\alpha) \mp I_n) &= 0 \end{cases}$$

have disadvantages. Consider an equivalent defining ALCP:

$$\begin{cases} f(x,\alpha) - x &= 0 \\ g(x,\alpha) &= 0 \end{cases}$$

where $g(x,\alpha)$ is obtained from the **bordered system**:

$$\begin{pmatrix} B(x,\alpha) & w_0 \\ v_0^{\mathsf{T}} & 0 \end{pmatrix} \begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $B(x,\alpha) = A(x,\alpha) \mp I_n$. The vectors $v_0, w_0 \in \mathbb{R}^n$ are selected to make

$$M(x,\alpha) = \begin{pmatrix} B(x,\alpha) & w_0 \\ v_0^{\mathsf{T}} & 0 \end{pmatrix}$$

nonsingular. Then

$$g(x,\alpha) = \frac{\det B(x,\alpha)}{\det M(x,\alpha)}$$

- Let z be a component of x or α. The derivative g_z can be expressed explicitly as

$$g_z = -w^\top B_z(x, \alpha)v,$$

where $(w \quad h)^\top$ is the solution to

$$M^\top \begin{pmatrix} w \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- Alternative ALCPs:

$$(x, v, \alpha) \in \mathbb{R}^{2n+2}$$

Fold (`LP`):

$$\begin{cases} f(x,\alpha) - x &= 0 \\ A(x,\alpha)v - v &= 0 \\ \langle v_0, v\rangle - 1 &= 0 \end{cases}$$

Flip:

$$\begin{cases} f(x,\alpha) - x &= 0 \\ A(x,\alpha)v + v &= 0 \\ \langle v_0, v\rangle - 1 &= 0 \end{cases}$$

Neimark-Sacker (NS) curve:

The $n \times n$-matrix

$$C(x,\alpha,\kappa) = A^2(x,\alpha) - 2\kappa A(x,\alpha) + I_n$$

with $\kappa = \cos\theta_0$ has rank $n-2$ at the Neimark-Sacker point, where $A = f_x$ has two simple complex eigenvalues $\mu_{1,2} = e^{\pm i\theta_0}$.

- ALCP with $(x,\alpha,\kappa) \in \mathbb{R}^{n+3}$

$$\begin{cases} f(x,\alpha) - x &= 0, \\ g_{11}(x,\alpha,\kappa) &= 0, \\ g_{22}(x,\alpha,\kappa) &= 0, \end{cases}$$

where $g_{kk}(x,\alpha,\kappa)$ are obtained by solving the **double-bordered system**:

$$\begin{pmatrix} C(x,\alpha,\kappa) & W_1 & W_2 \\ V_1^{\mathsf{T}} & 0 & 0 \\ V_2^{\mathsf{T}} & 0 & 0 \end{pmatrix} \begin{pmatrix} H_1 & H_2 \\ g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here the vectors V_j, W_j are selected to make this linear system nonsingular. At a Neimark-Sacker point, all $g_{ij} = 0$.

Alternative defining ALCPs:

- Single bordered: $(x, \alpha) \in \mathbb{R}^{n+2}$

$$\begin{cases} f(x,\alpha) - x &= 0 \\ g(x,\alpha) &= 0, \end{cases}$$

where $g(x, \alpha)$ is obtained from the **bordered system**:

$$\begin{pmatrix} A(x,\alpha) \odot A(x,\alpha) - I_m & W_0 \\ V_0^{\mathsf{T}} & 0 \end{pmatrix} \begin{pmatrix} V \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $V \in \mathbb{R}^m$ and $V_0, W_0 \in \mathbb{R}^m$ are selected to make the system nonsingular.

- Extended system: $(x, v, \kappa, \alpha) \in \mathbb{R}^{2n+3}$

$$\begin{cases} f(x,\alpha) - x &= 0 \\ (A^2(x,\alpha) - 2\kappa A(x,\alpha) + I_n)v &= 0 \\ \langle v_0, v \rangle - 1 &= 0 \\ \langle v_1, v \rangle &= 0, \end{cases}$$

where $v_{0,1} \in \mathbb{R}^n$ are not orthogonal to

$$\mathcal{N}(A^2(u,\alpha) - 2\kappa A(u,\alpha) + I_n).$$

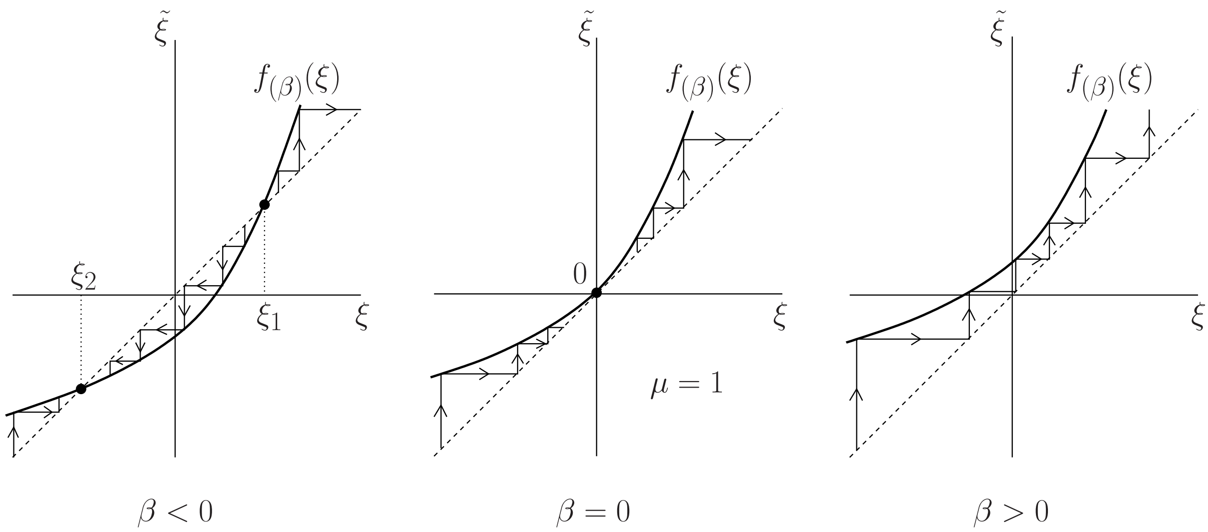
4. Normal forms for codim 1 bifurcations

- **Fold (LP) normal form:**

$$\xi \mapsto \beta + \xi + b(\beta)\xi^2 \equiv f_{(\beta)}(\xi), \quad \xi, \beta \in \mathbb{R},$$

where $b(0) > 0$.

- At $\beta = 0$ this map has fixed point $\xi_0 = 0$ with multiplier $\mu = 1$.



For $\beta < 0$ there are two fixed points

$$\xi_{1,2}(\beta) = \pm\sqrt{-\frac{\beta}{b(\beta)}}.$$

For $\beta > 0$ the map has no fixed points.

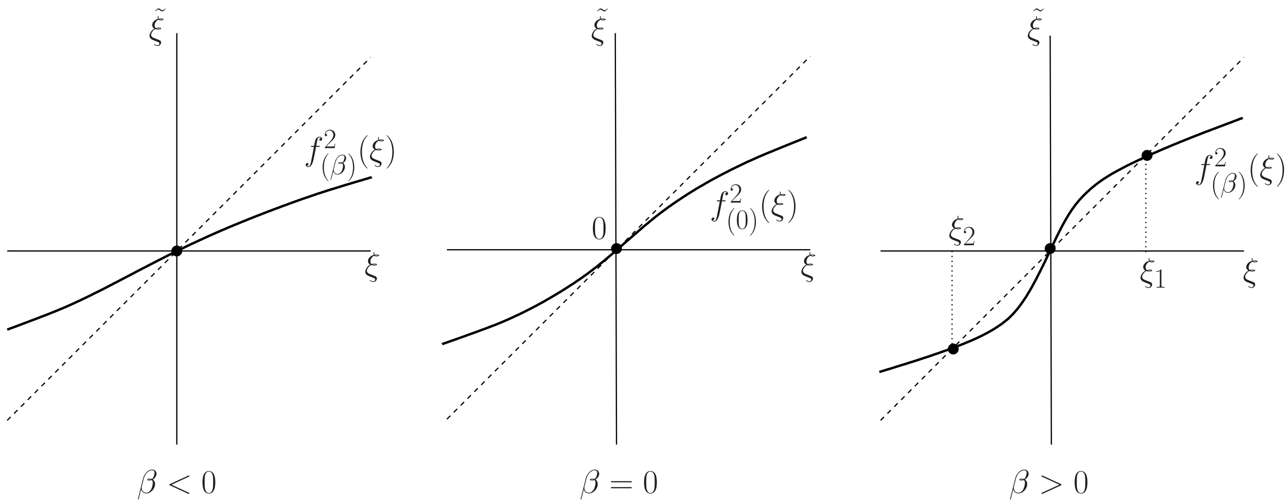
- **Flip (PD) normal form**

$$\xi \mapsto -(1+\beta)\xi + c(\beta)\xi^3 \equiv f_{(\beta)}(\xi), \quad \xi, \beta \in \mathbb{R}.$$

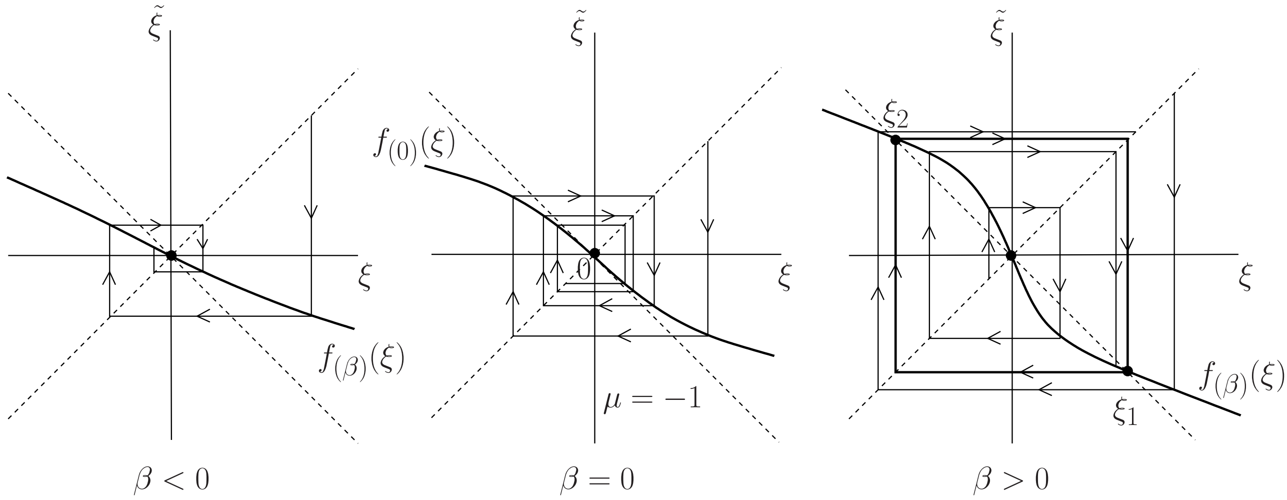
where $c(0) \neq 0$. At $\beta = 0$ this map has fixed point $\xi_0 = 0$ with multiplier $\mu = -1$.

If $c(0) > 0$, the **second iterate** $f^2_{(\beta)}$ has two stable fixed points for $\beta > 0$, namely

$$\xi_{1,2} = \pm\sqrt{\frac{\beta}{c(\beta)}}.$$



The map $f_{(\beta)}$ has the **stable 2-cycle** $\{\xi_1, \xi_2\}$.



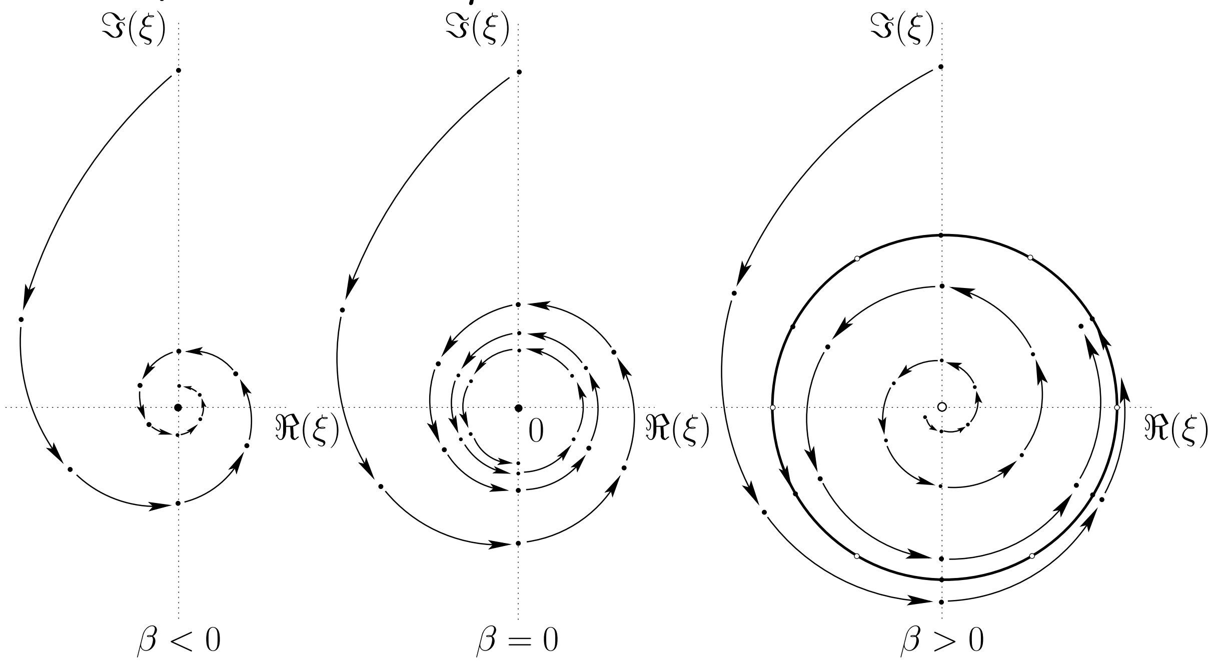
- **Neimark-Sacker (NS) normal form**

  $$\xi \mapsto e^{i\theta(\beta)}\xi(1+\beta+d(\beta)|\xi|^2)+O(|\xi|^4), \ \ \xi \in \mathbb{C}, \beta \in \mathbb{R},$$

  where $0 < \theta < \pi$ and $a(0) = \Re(d(0)) \neq 0$. At $\beta = 0$ the corresponding real planar map has fixed point $(0, 0)$ with multipliers $\mu_{1,2} = e^{\pm i\theta(0)}$. Using $\xi = \rho e^{i\varphi}$ we obtain

  $$\begin{cases} \rho & \mapsto & \rho(1+\beta+a(\beta)\rho^2)+O(\rho^4) \\ \varphi & \mapsto & \varphi+\theta(\beta)+O(\rho^2), \end{cases}$$

  where $a(\beta) = \Re(d(\beta))$ and the O-terms are 2π-periodic in φ.



  If $a(0) < 0$, the real planar map has for $\beta > 0$ a **stable closed invariant curve** near

  $$\rho_0(\beta) = \sqrt{-\frac{\beta}{a(\beta)}}$$

Remarks on the normal forms

1. The LP and PD normal forms are **topological normal forms**, i.e. any one-dimensional map near the corresponding bifurcation can be transformed to them plus higher-order terms in ξ, which are irrelevant for the orbit topology.

2. In the NS-case:

   - Only planar maps without **strong resonances**, i.e. for which

     $$e^{i\nu\theta(0)} \neq 1 \quad \text{for} \quad \nu = 1, 2, 3, 4,$$

     can be transformed near the NS-bifurcation to the above given normal form.

   - Even in the absence of strong resonances, the **orbit topology depends on the $O(|\xi|^4)$-terms**. Generically, there are either two k-cycles with some $k \geq 5$ in the closed invariant curve or all orbits are dense there.

5. Computation of the critical NF-coefficients

- Write the critical map with $F(0) = 0$ as

$$\tilde{x} = F(x), \quad x \in \mathbb{R}^n,$$

and restrict it to its n_c-dimensional **center manifold**:

$$x = H(\xi), \quad H : \mathbb{R}^{n_c} \to \mathbb{R}^n, \tag{6}$$

- Assume that the restricted map is put into the **normal form**

$$\tilde{\xi} = G(\xi), \quad G : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}.$$

The invariancy of CM, $\tilde{x} = H(\tilde{\xi})$, gives the **homological equation**:

$$F(H(\xi)) = H(G(\xi)).$$

- Substitute the Taylor expansions:

$$F(x) = Ax + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\|x\|^4),$$

$$G(\xi) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} g_\nu \xi^\nu, \quad H(\xi) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} h_\nu \xi^\nu,$$

and collect ξ^ν-terms with the multi-index ν. All appearing linear systems for h_ν are **solvable**.

Fold (LP) bifurcation

Let $q,p \in \mathbb{R}^n$ satisfy

$$Aq = q, \quad A^{\mathsf{T}}p = p, \quad \langle q,q\rangle = \langle p,q\rangle = 1.$$

Expand

$$F(H) = AH + \frac{1}{2}B(H,H) + O(\|H\|^3)$$

and parametrize the center manifold:

$$H(\xi) = \xi q + \frac{1}{2}h_2\xi^2 + O(\xi^3), \quad \xi \in \mathbb{R},\ h_2 \in \mathbb{R}^n.$$

The critical normal form is

$$\tilde{\xi} = G(\xi) = \xi + b\xi^2 + O(\xi^3).$$

The equation $F(H(\xi)) = H(G(\xi))$ reads as

$$A(\xi q + \frac{1}{2}h_2\xi^2 + \cdots) + \frac{1}{2}B(\xi q + \cdots, \xi q + \cdots) + \cdots$$
$$= \ (\xi + b\xi^2 + \cdots)q + \frac{1}{2}h_2(\xi + \cdots)^2 + \cdots$$

The ξ^2-terms give the equation for h_2:

$$(A - I_n)h_2 = -B(q,q) + 2bq.$$

It is singular but solvable, thus

$$b = \frac{1}{2}\langle p, B(q,q)\rangle$$

Flip (PD) bifurcation

Let $q, p \in \mathbf{R}^n$ satisfy

$$Aq = -q, \;\; A^T p = -p, \;\; \langle q, q\rangle = \langle p, q\rangle = 1.$$

Expand

$$F(H) = AH + \frac{1}{2}B(H,H) + \frac{1}{6}C(H,H,H) + O(\|H\|^4),$$

and parametrize the center manifold as

$$H(\xi) = \xi q + \frac{1}{2}h_2\xi^2 + \frac{1}{6}h_3\xi^3 + O(\xi^4),$$

where $\xi \in \mathbb{R}$, $h_{2,3} \in \mathbb{R}^n$. The critical normal form is

$$\tilde{\xi} = G(\xi) = -\xi + c\xi^3 + O(\xi^4).$$

The ξ^2-terms in the homological equation give for h_2:

$$(A - I_n)h_2 = -B(q,q).$$

Since $\mu = 1$ is not an eigenvalue of A, the matrix $(A - I_n)$ is nonsingular. Thus,

$$h_2 = -(A - I_n)^{-1}B(q,q).$$

The ξ^3-terms in the homological equation give the linear system for h_3:

$$(A + I_n)h_3 = 6cq - C(q, q, q) - 3B(q, h_2).$$

This system is singular, since $(A + I_n)q = 0$, so it has a solution only if

$$\langle p, 6cq - C(q, q, q) - 3B(q, h_2)\rangle = 0,$$

which implies

$$c = \frac{1}{6}\langle p, C(q, q, q)\rangle + \frac{1}{2}\langle p, B(q, h_2)\rangle.$$

Taking into account $h_2 = -(A - I_n)^{-1}B(q, q)$, we get the invariant formula for the flip normal form coefficient:

$$c = \frac{1}{6}\langle p, C(q, q, q)\rangle - \frac{1}{2}\langle p, B(q, (A - I_n)^{-1}B(q, q))\rangle.$$

Notice that all expressions can be evaluated in the original basis.

Neimark-Sacker (NS) bifurcation

Introduce two complex eigenvectors:

$$Aq = e^{i\theta_0}q, \;\; A^\top p = e^{-i\theta_0}p, \;\; \langle q,q\rangle = \langle p,q\rangle = 1,$$

The homological equation takes the form

$$F(H(\xi,\overline{\xi})) = H(G(\xi,\overline{\xi})),$$

where

$$\begin{aligned} H(\xi,\overline{\xi}) &= \xi q + \overline{\xi}\,\overline{q} \\ &+ \sum_{2\le j+k\le 3} \frac{1}{j!k!} h_{jk}\xi^j\overline{\xi}^k + O(|\xi|^4), \end{aligned}$$

$$F(H) = AH + \frac{1}{2}B(H,H) + \frac{1}{6}C(H,H,H) + O(\|H\|^4),$$

and

$$G(\xi,\overline{\xi}) = e^{i\theta_0}\xi + \frac{1}{2}G_{21}\xi|\xi|^2 + O(|\xi|^4).$$

Quadratic terms give

$$\begin{aligned} h_{20} &= (e^{2i\theta_0} I_n - A)^{-1} B(q,q), \\ h_{11} &= (I_n - A)^{-1} B(q,\overline{q}). \end{aligned}$$

While the $\xi^2\overline{\xi}$-terms give the singular system

$$\begin{aligned} (e^{i\theta_0} I_n - A) h_{21} &= C(q,q,\overline{q}) + B(\overline{q}, h_{20}) \\ &+ 2B(q, h_{11}) - G_{21} q. \end{aligned}$$

The solvability of this system is equivalent to

$$\langle p, C(q,q,\overline{q}) + B(\overline{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q \rangle = 0,$$

so the cubic normal form coefficient can be expressed as

$$\begin{aligned} G_{21} &= \langle p, C(q,q,\overline{q}) + B(\overline{q}, (e^{2i\theta_0} I_n - A)^{-1} B(q,q)) \\ &+ 2B(q, (I_n - A)^{-1} B(q,\overline{q})) \rangle, \end{aligned}$$

Then the direction of the Neimark-Sacker bifurcation is determined by

$$a = \frac{1}{2} \Re(e^{-i\theta_0} G_{21}).$$

Detection of codim 2 bifurcations

Codim 2 bifurcations along the LP-curve:

- Test functions:

$$\begin{aligned}\psi_1 &= 2a = \langle p, B(q,q)\rangle, \quad \langle q,q\rangle = \langle p,q\rangle = 1,\\ \psi_2 &= \langle p,q\rangle, \quad \langle q,q\rangle = \langle p,p\rangle = 1,\\ \psi_3 &= \det(A + I_n),\\ \psi_4 &= \det(A \odot A - I_m),\end{aligned}$$

  where $m = \frac{1}{2}n(n-1)$ and

$$Aq = q, \;\; A^{\mathsf{T}}p = p.$$

- Singularities:
  - cusp ($\psi_1 = 0$)
  - 1:1 resonance ($\psi_2 = 0$)
  - fold-flip ($\psi_3 = 0$)
  - fold-NS ($\psi_4 = 0, \;\psi_2 \neq 0$)

Codim 2 bifurcations along the PD-curve:

- Test functions:

$$\begin{aligned}
\psi_1 &= 6b = \langle p, C(q,q,q)\rangle \\
&\quad - 3\langle p, B(q,(A-I_n)^{-1}B(q,q))\rangle, \\
&\qquad\qquad \langle q,q\rangle = \langle p,q\rangle = 1, \\
\psi_2 &= \langle p,q\rangle, \quad \langle q,q\rangle = \langle p,p\rangle = 1, \\
\psi_3 &= \det(A-I_n), \\
\psi_4 &= \det(A \odot A - I_m),
\end{aligned}$$

  where $m = \frac{1}{2}n(n-1)$ and

$$Aq = -q, \ \ A^{\mathsf{T}}p = -p.$$

- Singularities:
  - generalized flip ($\psi_1 = 0$)
  - 1:2 resonance ($\psi_2 = 0$)
  - fold-flip ($\psi_3 = 0$)
  - flip-NS ($\psi_4 = 0, \ \psi_2 \neq 0$)

Codim 2 bifurcations along the NS-curve:

- Test functions:

$$\begin{aligned}
\psi_1 &= \Re(e^{-i\theta_0} G_{21}), \\
\psi_2 &= \kappa - 1, \\
\psi_3 &= \det(A - I_n), \\
\psi_4 &= \det(A + I_n), \\
\psi_5 &= \det(A^{\perp} \odot A^{\perp} - I_m), \\
\psi_6 &= \kappa + 1, \\
\psi_7 &= \kappa + \frac{1}{2}, \\
\psi_8 &= \kappa,
\end{aligned}$$

  where $A^{\perp}$ is the restriction of A to the orthogonal complement to the critical NS-eigenspace; $m = \frac{1}{2}(n-2)(n-3)$.

- Singularities:
  - generalized NS ($\psi_1 = 0$)
  - fold-NS ($\psi_3 = 0,\ \psi_2 \neq 0$)
  - flip-NS ($\psi_4 = 0,\ \psi_6 \neq 0$)
  - double NS ($\psi_5 = 0$)
  - 1:1 resonance ($\psi_2 = \psi_3 = 0$)
  - 1:2 resonance ($\psi_4 = \psi_6 = 0$)
  - 1:3 resonance ($\psi_7 = 0$)
  - 1:4 resonance ($\psi_8 = 0$)