TUTORIAL III:
One-parameter bifurcation analysis of limit cycles with MATCONT

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This session was tested on MatCont 7.3 with MATLAB 2020b, R9.9. It is devoted to the numerical initialization and continuation of limit cycles in systems of autonomous ODEs depending on one parameter

\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}, \]

and detection of their bifurcations. We will also show how to start the continuation of a limit cycle from the Hopf bifurcation and a doubled cycle from the period-doubling bifurcation.

1 Cycle initialization from a converging orbit

We consider again the Rössler system that was introduced in TUTORIAL I:

\[
\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + Ay \\
\dot{z} &= Bx - Cz + xz,
\end{align*}
\]

where \((x, y, z)\) are the phase variables, and \((A, B, C)\) are the parameters.

Input system (1) in MatCont, generating the derivatives of order 1, 2 and 3 symbolically. We start with computing by time integration an orbit, using the Starter and Integrator windows presented in Figure 1.

![Figure 1: The Starter and Integrator windows for the initial time integration.](image)

The computed orbit is presented in the 3D-plot in Figure 2 in \((x, y, z)\) - space within the region

\[-9 \leq x \leq 9, \quad -9 \leq y \leq 9, \quad -2 \leq z \leq 11.\]

It is visually clear that the orbit has converged to a (presumably stable) limit cycle. We select the last point of the orbit using Select|Initial Point|Last Point|Select Point and clear the Plot3D window.

It is readily seen that time integration from the selected point over a time interval 5 does not close the orbit, see the Starter and Integrator windows in Figure 3, and the orbit displayed in Figure 4.

Now perform a time integration over an interval 8 and check that it closes the limit cycle. This implies that 8 is larger than the period of the limit cycle but smaller than twice that period, which is the condition for starting the continuation of limit cycles from an orbit that is computed along the limit cycle. That is why time integrations over an interval 5 or 500 are of no use.
Figure 2: 3D-plot of the computed orbit.

Figure 3: **Starter** and **Integrator** windows for time integration from a selected point on the orbit.
Now press Control|View Result and double-click at Select Cycle (we note that a single-click at Select Cycle followed by clicking Select Point also works). This opens a small Choose tolerance ... input window with two input fields. The top one allows to change the tolerance for detecting a limit cycle along the orbit. The bottom one allows to change the number of mesh intervals that will be used to discretize the limit cycle. Make sure that it is set to 40; the window should now look like Figure 5.

Now press OK. The main MatCont window is then prepared for the computation of a limit cycle, see Figure 6(left). In the corresponding Starter window activate the parameter $A$ and the Period. Also, check all singularity monitoring boxes as in Figure 6(right). We note that the initial value of the Period is estimated by the initializer and equal to 6.27..., indeed a value between 5 and 8.

In the corresponding Continuer window only default values are to be used. Open a Numeric window and check Multipliers in its Layout menu for the multipliers of the periodic orbits to be displayed in the Numeric window during continuation.

Press Compute|Forward (depending on details of the previous computations it may be neces-
sary to press Compute|Backward instead). The continuation of limit cycles now starts. Observe that there is period-doubling bifurcation for $A = 0.27191773$. The Plot3D window and the Numeric window are displayed in Figures 7. Observe that at the PD-point there is a multiplier equal to $-1$ (i.e., absolute value 1 and argument equal to $180^\circ$).

In the MATLAB command line the output

Period Doubling (period = 6.148967e+00, parameter = 2.719177e-01)
Normal form coefficient = -4.283828e-03

is generated, including the normal form coefficient of the PD bifurcation.

Stop the computation and end the session by closing all output windows.

Figure 7: The Plot3D and the Numeric windows at the period-doubling bifurcation.
2 Fold and Neimark-Sacker bifurcations of cycles in a chemical model

Consider the following chemical model by Steinmetz and Larter:

\[
\begin{align*}
\dot{A} &= -k_1 ABX - k_3 ABD + k_7 - k_{-7} A, \\
\dot{B} &= -k_1 ABX - k_3 ABD + k_8, \\
\dot{X} &= k_1 ABX - 2k_2 X^2 + 2k_3 ABD - k_4 X + k_6, \\
\dot{Y} &= -k_3 ABD + 2k_2 X^2 - k_5 Y.
\end{align*}
\]

(2)

We will fix all parameters but \(k_7\) and study periodic solutions (limit cycles) of (2) when this parameter varies.

2.1 System specification

Specify a new ODE system – say StLar – in MatCont

\[
\begin{align*}
A' &= -k_1 ABX - k_3 ABD + k_7 - k_{-7} A, \\
B' &= -k_1 ABX - k_3 ABD + k_8, \\
X' &= k_1 ABX - 2k_2 X^2 + 2k_3 ABD - k_4 X + k_6, \\
Y' &= -k_3 ABD + 2k_2 X^2 - k_5 Y.
\end{align*}
\]

where \((A, B, X, Y)\) are the coordinates and \((k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_{-7}, k_8)\) are the parameters. Use (default) \(t\) for time and generate symbolical derivatives of order 1, 2 and 3.

2.2 Fold bifurcation of limit cycles

2.2.1 Continuation of an equilibrium

We will continue an equilibrium of (2) and detect its Hopf bifurcation, from which the continuation of a limit cycle can be started.

Input \textbf{Type} | \textbf{Initial point} | \textbf{Equilibrium} and \textbf{Type} | \textbf{Curve} | \textbf{Equilibrium} in the main MatCont window.

Input the following numerical data in the appeared \textbf{Starter} window:

\[
\begin{align*}
A &= 31.78997 \\
B &= 1.45468 \\
X &= 0.01524586 \\
Y &= 0.1776113 \\
k_1 &= 0.1631021 \\
k_2 &= 1250 \\
k_3 &= 0.046875 \\
k_4 &= 20 \\
k_5 &= 1.104 \\
k_6 &= 0.001 \\
k_7 &= 4.235322 \\
k_{-7} &= 0.1175 \\
k_8 &= 0.5
\end{align*}
\]

These values correspond to an unstable equilibrium \((A, B, X, Y)\) in the system. Activate the parameter \(k_7\).

Open \textbf{Window} | \textbf{Output} | \textbf{Numeric} and change its appearance via the \textbf{Layout} menu. Namely, check \textbf{Eigenvalues} to be also shown in the window.

Use \textbf{Window} | \textbf{Output} | \textbf{Graphic} | \textbf{2D plot} to open the corresponding window and select the coordinates \((A, B)\) as abscissa and ordinate, respectively, with the visibility limits:
respectively.

Start **Compute|Forward**. The equilibrium curve will be continued and you get a *Hopf bifurcation point* labeled by $H$. The message in the MATLAB Command Window

```
lable = H , x = ( 34.808899 1.328517 0.015246 0.177611 4.590046 )
First Lyapunov coefficient = 1.527549e-02
```

at $k_7 = 4.590046\ldots$ indicates a *subcritical* Hopf bifurcation. Indeed, there are two eigenvalues of the equilibrium with Re $\lambda_{1,2} \approx 0$ at this parameter value visible in the **Numeric** window. The critical frequency Im $\lambda_1 \neq 0$, while the first Lyapunov coefficient is positive. Thus, there should exist an unstable limit cycle, bifurcating from the equilibrium. Resume computations and terminate them when the curve leaves the graphic window.

For later use, rename the computed curve via **Select|Curve|Rename** into **Equilibrium(+)**.

### 2.2.2 Cycle continuation

Click **Select|Initial point|Hopf|Select Point** to select the H: Hopf point in the **Equilibrium(+)** curve as initial point. The **Starter** and **Continuer** windows for the continuation of the limit cycle from the Hopf point will appear.

Ensure that all boxes in the **Monitor Singularities** field of the **Starter** window are checked. Increase the **MaxStepsize** to 1.0 in the **Continuer** window to allow larger steps along the curve and set **MaxNumPoints** to 50. The **Starter** and **Continuer** windows should look like in Figure 8.

Change the layout of the **Numeric** window via **Numeric|Layout** by selecting all **Multipliers** to be shown. The absolute values (*modulae*) and arguments in *angular grads* will be displayed.

**Compute|Forward** will produce a family of cycles with a *cycle limit point* labeled by **LPC** at $k_7 = 4.74838\ldots$ (The initial Hopf point may also be detected as an **LPC** right after starting the continuation; in that case ignore it and resume the computation). The **Numeric** window corresponding to **LPC** is shown in Figure 9. Recall that exact numerical values could deviate slightly.

In the MATLAB Command Window, the following message appears:

```
Limit point cycle (period = 1.036108e+01, parameter = 4.748384e+00)
Normal form coefficient = -2.714838e-01
```

The critical cycle has (approximately) a double multiplier $\mu = 1$ and the normal form coefficient is nonzero. Thus, the limit cycle manifold has a fold here. Resume the computations. The continuation algorithm will automatically follow the second (stable) cycle branch after the **LPC** point. Verify this by looking at the nontrivial multipliers in the **Numeric** window ($|\mu| < 1$ for all such multipliers). The computations produce Figure 10.

To visualize **LPC** in another way, open another **Plot2D** window and select parameter $k_7$ and **Period** of the cycle as abscissa and ordinate, respectively. Use the visibility limits:

```
Abscissa: 4.5 4.8
Ordinate: 7 12
```

After **MatCont|Redraw Diagram** in the **Plot2D** window, you will get a curve with a limit point, clearly indicating the presence of two limit cycles with different periods for $k_7 < 4.74838$ near **LPC** (see Figure 11). Close both **Plot2D** windows.
Figure 8: The **Starter** and **Continuer** windows for the cycle continuation.

Figure 9: The **Numeric** window at the LPC-point.
Figure 10: The family of limit cycles bifurcating from the Hopf point $H$: LPC is a fold bifurcation of the cycle.

Figure 11: Period of the cycle versus $k_7$. 
2.3 Neimark-Sacker bifurcation

2.3.1 Backward continuation of the equilibrium

Load the first computed equilibrium curve Equilibrium(+) in the Data Browser window appearing after the Select|Curve command. Be sure to select Type|Curve|Equilibrium for continuation.

Open a new Plot2D window but select now B and X as abscissa and ordinate, respectively, with the visibility limits:

Abscissa: 25 27  
Ordinate: 0 0.03

Compute|Backward and Extend several times until you get another Hopf bifurcation at $k_7 = 0.712475 \ldots$ with the message

```
label = H , x = ( 1.808301 25.573303 0.015246 0.177611 0.712475 )
```

First Lyapunov coefficient = $-2.371880e-02$

in the MATLAB Command Window. The first Lyapunov coefficient is negative now. This means that a stable limit cycle bifurcates from the equilibrium, when it loses stability. Resume computations and terminate them when the equilibrium curve leaves the graphic window. Rename the computed curve into Equilibrium(-).

2.3.2 Cycle continuation

Select the Hopf point in the curve Equilibrium(-) as initial. MatCont will prepare to continue a limit cycle curve from the Hopf point (curve type H_LC). Check all boxes in the Monitor Singularities section of the Starter window. Activate $k_7$ and the Period in the Starter window. Set

MaxStepsize = 1
MaxNumPoints = 25

in the Continuer window. Check Multipliers to be shown in the Numeric window.

Click Compute|Forward to start the continuation of the limit cycle. At $k_7 = 0.716434 \ldots$ the message Neimark-Sacker indicates a torus bifurcation. (The initial Hopf point may also be detected as an LPC right after starting the continuation; in that case ignore it and resume the computation). In the MATLAB Command Window, the following message appears:

```
Neimark-Sacker (period = 1.091213e+01, parameter = 7.164336e-01)
Normal form coefficient = -4.912065e-08.
```

Indeed, there are two complex multipliers with (approximately) $|\mu| = 1$ and one trivial at $\mu = 1$. This can be seen in the Numeric window that stays open, see Figure 12. The normal form coefficient is negative and small but nonzero, indicating that a stable two-dimensional invariant torus bifurcates from the limit cycle.

Resume computations further to see that after the NS-point the cycle becomes unstable (with two multipliers satisfying $|\mu| > 1$. You should get a family of limit cycles bifurcating from the Hopf point as in Figure 13.

Close the Numeric and all output windows.

2.3.3 Dynamics on a stable torus

Click Select|Initial point... and select the NS: Neimark-Sacker point in the computed limit cycle curve. The critical parameter values will be read in from the archive. Immediately after that select Type|Initial point|Point and Type|Curve|Orbit. A point on the critical cycle will thus be selected as the initial for orbit integration.

Increase slightly the value of the parameter $k_7$ and perturb the initial point by altering the value of B, namely set:
Figure 12: The **Numeric** window at the NS-point.

Figure 13: Family of limit cycles bifurcating from the second Hopf point.
k7 0.7165
B 26.2

Change Method of integration to ode23s and alter

Interval 1000
Rel.Toletance 1e-6
Abs.Tolerance 1e-9

in the Integrator window.

To speed up visualization, click Options|Output Interval and set it to 1000, see Figure 14

![Output Interval window.](image)

Figure 14: **Output Interval** window.

Clear the graphic window and Compute|Forward followed by Compute|Extend. The integration will take some time. After a transient, the orbit will exhibit modulated oscillations with two frequencies near the limit cycle. This is a motion on a stable two-dimensional torus born via the Neimark-Sacker bifurcation, see Figure 15. Open another 2Dplot window with time $t$ and coordinate $B$ as abscissa and ordinate, with the visibility limits:

Abscissa: 1000 2000
Ordinate: 25.0 26.5

Figure 15: Dynamics on a stable 2-torus.

After MatCont|Redraw Curve, you will see the high-frequency oscillations with low-frequency modulation as in Figure 16.

Restore the original setting (i.e., 1) of Options|Output Interval via the main MatCont window.
Consider the following adaptive control system of Lur’e type
\[
\begin{align*}
  \dot{x} &= y, \\
  \dot{y} &= z, \\
  \dot{z} &= -\alpha z - \beta y - x + x^2.
\end{align*}
\] (3)

We will fix $\alpha = 1$ and continue limit cycles of (3) using $\beta$ as a bifurcation parameter.

### 3.1 System specification
As usual, define a new system in MatCont with the coordinates $x,y,z$, parameters $\alpha,\beta$, and select derivatives of order 1, 2, and 3 to be generated symbolically.

### 3.2 Equilibrium continuation
Select Type|Initial point|Equilibrium and Type|Curve|Equilibrium. In the Starter window, set $\alpha$ equal to 1 and activate $\beta$.

Open a Plot3D window and select $\beta,x,y$ as variables along the coordinate axes with the visibility limits:

- Abscissa: 0.4 1.2
- Ordinate: -0.5 1.2
- Applicate: -0.6 0.7

Compute|Forward results in an equilibrium curve with a Hopf bifurcation (labeled H) detected with the following message in the MATLAB Command Window:

- label = H, x = ( 0.000000 0.000000 0.000000 1.000000 )
- First Lyapunov coefficient = -3.000000e-01

The Hopf bifurcation occurs at $\beta = 1$ and is supercritical, thus generating a stable limit cycle. Resume the continuation and eventually stop it, when the curve leaves the plotting region.
3.3 Cycle continuation starting from the H-point

Click Control|View Result and double-click at the H: Hopf point. The Starter and Continuer windows appear, corresponding to the limit cycle curve $H_{LC}$. Make sure that beta and Period are activated.

In the Starter window, use the number of mesh points $ntst$ equal to 40 and check all boxes in the Monitor Singularities field. In the Continuer window, set MaxStepsize to 0.7 and MaxNumPoints to 80.

With Window/Output|Numeric, open a Numeric window and make the cycle Multipliers visible by checking them in the Numeric|Layout menu.

Compute|Forward will produce a branch of limit cycles of increasing amplitude with two period-doubling bifurcations (labeled PD). The Numeric windows at both PD bifurcations are shown in Figures 17 and 18.

![Figure 17: The Numeric window at the first PD bifurcation. The bifurcation parameter value is $\beta = 0.6273246\ldots$.](image)

Resume the computations at both PD points. The MATLAB Command Window will contain the messages

Period Doubling (period = 8.386068e+00, parameter = 6.273245e-01)
Normal form coefficient = -3.405330e-03
Period Doubling (period = 9.864971e+00, parameter = 5.417461e-01)
Normal form coefficient = -7.65596e-04
Figure 18: The **Numeric** window at the second PD bifurcation. The bifurcation parameter value is $\beta = 0.5417461\ldots$. 

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<th>Value</th>
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<tr>
<td>Arg[3]</td>
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</tbody>
</table>
Figure 19: The cycle branch in system (3) rooted at the Hopf point. Two critical cycles undergoing the period-doubling bifurcations are labeled by PD.

Figure 20: The cycle period near the homoclinic bifurcation.
indicating that in both cases the normal form coefficients are negative, so that stable double-period cycles are involved.

You get a cycle manifold shown in Figure 19. The period of the cycle rapidly increases when $\beta$ approaches $\beta_{\text{hom}} = 0.515489\ldots$. To see this, open a Plot2D window (see Figure 20) with the axes beta and Period and the plotting region

\[
\begin{array}{c|c}
\text{Abscissa:} & 0.51 & 0.53 \\
\text{Ordinate:} & 0.0 & 40.0 \\
\end{array}
\]

and apply MatCont|Redraw Curve.

Actually, the limit cycle approaches a homoclinic orbit to the second equilibrium $(x, y, z) = (1, 0, 0)$ of (3), which is a global bifurcation. Below $\beta_{\text{hom}}$ no cycle exists. Near the homoclinic bifurcation, the computation of the cycle and its multipliers becomes inaccurate. The loss of accuracy can be seen in the Numeric window, where no multiplier close to 1 is present at the last computed point.

### 3.4 Continuation of the double cycle from the PD-point

Using Select|Initial point, take the first PD: Period Doubling point in the computed above curve as initial.

Adjust the Plot3D window by setting the visibility limits for beta as follows:

\[
\begin{array}{c|c}
\text{Abscissa:} & 0.5 & 0.63 \\
\end{array}
\]

1. At $\beta = \beta_{\text{hom}}$, this equilibrium is a saddle-focus with the saddle quantity $\sigma < 0$. Thus, according to Shilnikov’s Theorem, from this homoclinic orbit only one limit cycle bifurcates - the one we computed.

Figure 21: The branch of stable doubled cycles connects two PD points.
In the **Starter** window, set the number of mesh points `ntst` to 80, increase the **amplitude** to 0.005, and check only **Branch Point of Cycles** (BPC) in the **Monitor Singularities** section.

In the **Continuer** window, increase `MaxTestIters` to 30, and make sure that `MaxStepsize` remains equal to 0.7.

Click **Compute|Forward** and continue the doubled cycle bifurcating from the first PD-point at $\beta = \beta_{PD,1}$. This cycle remains stable until at (approximately) $\beta = \beta_{PD,2}$ a **branching point cycle** (BPC) is detected. This is not surprising, since the PD-bifurcation for the original limit cycles corresponds to a branching point for the doubled cycles. Stop computations at BPC, clear **Plot3D** window and make **MatCont|Redraw Curve** to get Figure 21.

Exit **MatCont**.

# 4 Additional Problems

A. Study numerically limit cycles in the system

$$\begin{align*}
\dot{x} &= \mu x - y - xz, \\
\dot{y} &= x + \mu y, \\
\dot{z} &= -z + y^2 + x^2 z,
\end{align*}$$

that is another model of a feedback control system.

B. Consider the following **predator-prey** model

$$\begin{align*}
\dot{x} &= x - \frac{xy}{1 + \alpha x}, \\
\dot{y} &= -y + \frac{xy}{1 + \alpha x} - \delta y^2.
\end{align*}$$

1. Prove that the following polynomial system

$$\begin{align*}
\dot{x} &= x(1 + \alpha x) - xy, \\
\dot{y} &= -(y + \delta y^2)(1 + \alpha x) + xy,
\end{align*}$$

has the same orbits in the positive quadrant as the original model. Derive an explicit condition on $(\alpha, \delta)$ for system (4) to have a positive equilibrium with one zero eigenvalue. Derive a similar condition for (4) to have a positive equilibrium with a pair of purely imaginary eigenvalues.

2. Introduce new variables

$$\begin{align*}
\xi &= \ln x, \\
\eta &= \ln y,
\end{align*}$$

in which the original system becomes:

$$\begin{align*}
\dot{\xi} &= 1 - \frac{\exp(\eta)}{1 + \alpha \exp(\xi)}, \\
\dot{\eta} &= -1 + \frac{\exp(\xi)}{1 + \alpha \exp(\xi)} - \delta \exp(\eta).
\end{align*}$$

Fix $\alpha = 0.3$ and study numerically limit cycles of (5), when the parameter $\delta$ varies. **Hint:** Begin with finding an equilibrium at $\delta = 0.35$ by integration.

3. Relate your analytical and numerical results.

C. Consider the following **predator-double prey** system:

$$\begin{align*}
\dot{x} &= x(2.4 - x - 6y - 4z), \\
\dot{y} &= y(\beta - x - y - 10z), \\
\dot{z} &= -z(1 - 0.25x - 4y + z),
\end{align*}$$

modelling dynamics of two prey populations affected by a predator.
1. Starting from $\beta = 1.77$, find its positive equilibrium by orbit integration.

2. Continue the equilibrium with respect to $\beta$ until it exhibits a Hopf bifurcation.

3. Continue a limit cycle from the Hopf point and monitor the dependence of its period upon $\beta$. Hint: Set $\texttt{ntst}$ to a high value.

4. Plot the cycle at different period values and try to understand its asymptotic shape. Hint: Compute and analyze the equilibria located on the coordinate planes.

D. Consider the famous Lorenz system

$$\begin{cases} \dot{x} &= \sigma(-x + y), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= -bz + xy, \end{cases}$$

with the standard parameter values $\sigma = 10, b = \frac{8}{3}$ and $r > 0$. Use MatCont to analyse its simplest limit cycles numerically.

1. Compute a critical parameter value $r_{BP}$, at which the trivial equilibrium $(0, 0, 0)$ undergoes a symmetric pitchfork bifurcation.

2. Compute a critical parameter value $r_{H}$, corresponding to a Hopf bifurcation of nontrivial equilibria.

3. Continue the limit cycle born at the Hopf bifurcation with respect to $r$ and find out what happens to it. Plot the cycle period as function of $r$, and monitor stability.

4. Demonstrate by numerical integration that for $r = 400$ the system has a stable symmetric limit cycle and determine its period approximately. Continue this cycle with respect to $r$ and find the critical parameter value $r_{BPC}$ corresponding to a branch point of cycles (symmetry breaking). Hint: Use the procedure to start LC continuation from a simulation as done in Section 1.

Derive explicit formulas for $r_{BP}$ and $r_{H}$ as functions of $(\sigma, b)$ and verify the numerical values obtained with MatCont.