TUTORIAL IV:
Two-parameter bifurcation analysis of equilibria and limit cycles with matcont

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This session was tested on MatCont 7.3 with MATLAB2020b, R9.9. It is devoted to the numerical continuation of codim 1 bifurcations of equilibria and limit cycles in systems of autonomous ODEs depending on two parameters

\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^2, \]

and detection of their codim 2 bifurcations. We will also switch at some codim 2 equilibrium bifurcations to the continuation of codim 1 bifurcation curves rooted there.

1 Bifurcations of equilibria in the Bykov–Yablonskii–Kim model

We will use MatCont to continue equilibria and their bifurcations in the following chemical model that describes CO-oxidation on platinum:

\[
\begin{align*}
\dot{x} &= 2k_1z^2 - 2k_{-1}x^2 - k_3xy \\
\dot{y} &= k_2z - k_{-2}y - k_3xy \\
\dot{s} &= k_4z - k_{-4}s,
\end{align*}
\]

where \( z = 1 - x - y - s \). The ratio

\[ K = \frac{k_{-4}}{k_4} \]

and the parameter \( k_2 \) will be used as two bifurcation parameters.

1.1 Specify the model in MatCont

Specify a new ODE system in MatCont with coordinates \((x, y, s)\) and time \((t)\):

\[
\begin{align*}
z &= 1 - x - y - s \\
x' &= 2*Q1*z^2 - 2*Q5*x^2 - Q3*x*y \\
y' &= Q2*z - Q6*y - Q3*x*y \\
s' &= Q4*z - K*Q4*s
\end{align*}
\]

The reaction rates \( k_1, k_2, k_3, k_4, k_{-1}, k_{-2} \) are denoted by \( Q1, Q2, Q3, Q4, Q5, Q6 \), respectively, while \( K \) stands for \( K \). Generate the derivatives of order 1, 2, 3, 4 and 5 symbolically. Note that \( z \) is an auxiliary variable which is introduced to simplify the formal form of the equations and to improve the readability. The choice of the name is irrelevant but it should not conflict with the internal names in the MATLAB symbolic toolbox if that toolbox is used.

1.2 Equilibrium continuation

Select both Type|Initial point|Equilibrium and Type|Curve|Equilibrium. In the Starter window input the following numerical values, corresponding to a stable equilibrium in the system:

\[
\begin{align*}
x &= 0.001137 \\
y &= 0.891483 \\
s &= 0.062345 \\
Q1 &= 2.5 \\
Q2 &= 2.204678 \\
Q3 &= 10 \\
Q4 &= 0.0675 \\
Q5 &= 1 \\
Q6 &= 0.1 \\
K &= 0.4
\end{align*}
\]
Activate the parameter $Q_2$.

In the **Continuer** window, decrease the maximal step size along the curve to

\[
\text{MaxStepsize} = 0.025
\]

Open a **Plot2D** window with $(Q_2, x)$ on the axes and input the following visibility limits:

| Abscissa: | 0.5 | 2.0 |
| Ordinate: | 0.0 | 0.16 |

Also open a **Numeric** window and make **Eigenvalues** visible using **Numeric|Layout**.

Start the continuation of the equilibrium curve with the **Compute|Forward** command. Monitor the eigenvalues in the **Numeric** window. There will be four bifurcation points detected: two **Hopf points** (labeled by $H$) and two **fold points** (labeled by $LP$). Resume computations at each bifurcation point, and stop them when the curve leaves the window. You should get the central S-shaped curve presented in Figure 1. In a narrow interval of $Q_2$-values (between two LP-points) the system has three equilibria.

![S-shaped curve](image)

**Figure 1:** Equilibrium curves with $K = 0.15, 0.4$, and 2.0

The normal form coefficients for the Hopf and limit point bifurcations can be read in the MATLAB Command Window:

- **Hopf point**
  
  \[
  \text{label = H, x = ( 0.016357 0.523973 0.328336 1.051558 )}
  \]

- **First Lyapunov coefficient**
  
  \[
  \text{First Lyapunov coefficient = 1.070259e+01}
  \]

- **Limit point**
  
  \[
  \text{label = LP, x = ( 0.024717 0.450257 0.375018 1.042049 )}
  \]

  \[
  \text{a = 1.166509e-01}
  \]

  \[
  \text{label = LP, x = ( 0.054030 0.302241 0.459807 1.052200 )}
  \]

  \[
  \text{a = -1.346534e-01}
  \]
First Lyapunov coefficient = 4.332247e+00

Note that both first Lyapunov coefficients are positive (implying that unstable limit cycles bifurcate there), while the LP-coefficients are both nonzero (their sign is irrelevant and depends on the choice of a direction).

Rename the computed equilibrium curve via Select\|Curve\|Rename into

Equilibrium1(+)

Change the value of the parameter $K$ in the Starter window:

$K 0.15$

and repeat the computations with Compute\|Forward. You will get a monotone curve without bifurcation points (the left curve in Figure 1). There is only one equilibrium for all parameter values $Q2$.

Finally, change the value of $K$ to

$K 2.0$

and click on Compute\|Forward again. Another equilibrium curve will come out with the order of the singularities reversed (the right curve in Figure 1). There is now a big interval of the parameter values, in which there are three equilibria in the system.

In the MATLAB Command Window, the following messages appeared:

indicating that the LP-bifurcations are still nondegenerate ($a \neq 0$), while – instead of two Hopf points – two neutral saddles have appeared, which are not bifurcation points. Note that they are still marked by $H$.

Store the last two computed curves permanently by renaming them into

Equilibrium2(+)
Equilibrium3(+)

via Select\|Curve\|Rename. Close the Data Browser window.

1.3 Continuation of fold and Hopf bifurcations

This section explains how to continue fold and Hopf bifurcations with respect to two control parameters. While doing so, you will see how MatCont detects some codim 2 bifurcations and reports their normal form coefficients.

1.3.1 Fold continuation

Select the second LP: Limit point in the first computed curve Equilibrium1(+) via Select\|Curve. The new Starter window corresponding to the continuation of a limit point curve LP_LP is to appear, in which you should keep parameter $Q2$ active and activate parameter $K$, see Figure 2.

Compute the limit point curve in both directions to get Figure 3. Notice that both parameters
Figure 2: **Starter** window for the continuation of the limit point curve.
Figure 3: Limit point curve in Bykov’s model: BT - Bogdanov-Takens points; CP - cusp.
(Q2,K) are varied along the curve. Each point on the curve is a limit point for the equilibrium curve at the corresponding value of K (or Q2).

Three points were detected, corresponding to codim 2 bifurcations: two Bogdanov-Takens (BT) and one cusp (CP). At each BT point the system has an equilibrium with a double zero eigenvalue, while at the CP point there is an equilibrium with a simple zero eigenvalue but the normal form coefficient of the fold is zero. The normal form coefficients (a,b,c) of these bifurcations are reported in the MATLAB Command Window and they are all nonzero:

label = BT , x = ( 0.115909 0.315467 0.288437 1.417628 0.971397 )
(a,b)=(-8.378442e-02, -2.136280e+00)

label = CP , x = ( 0.035940 0.352008 0.451368 1.006408 0.355991 )
c=3.627844e-01
label = BT , x = ( 0.016337 0.638410 0.200456 1.161199 0.722339 )
(a,b)=(-4.822563e-02, -1.937632e+00)

Rename the obtained limit point branches as following:
Fold(+)
Fold(-)

1.3.2 Hopf continuation

The Bogdanov-Takens points are common points for the limit point curves and curves corresponding to equilibria with eigenvalues $\lambda_1 + \lambda_2 = 0, \lambda_3 \neq 0$. Actually, at each BT point, the Hopf bifurcation curve (with $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$) turns into the neutral saddle curve (with real $\lambda_1 = -\lambda_2$). Thus, we can start a Hopf curve from a Bogdanov-Takens point.

Select the BT: Bogdanov-Takens point in the curve Fold(+) as initial. Then choose Type|Curve|Hopf to prepare for the continuation of the Hopf curve. The new Starter window for the continuation of a Hopf curve will appear automatically, see Figure 4. Check that the type of the curve to be computed is changed in the MatCont window. Activate the parameters Q2,K.

Compute|Forward (resuming the computation at each special point and using Compute|Extend) produces a closed Hopf curve as in Figure 5. Actually, only the part of it to the left of two BT points corresponds to a Hopf bifurcation; the other part represents a neutral saddle. As it has been mentioned, this transition happens at the Bogdanov-Takens points. There are two more codim 2 bifurcation points at the Hopf point: Two Bautin or generalized Hopf (GH) points, where the first Lyapunov coefficient $l_1$ vanishes.

The Command Window of MATLAB shows a message about the starting BT point and the following messages related to the GH points:

label = GH, x = ( 0.018022 0.368238 0.497968 0.891319 0.232487 0.003324 )
12=-7.768996e+02
label = GH, x = ( 0.064311 0.211095 0.554870 0.924255 0.305879 0.003512 )
12=-2.4022563e+02

indicating that both GH points are nondegenerate, since the second Lyapunov coefficients 12 are nonzero (in fact, negative).

1.3.3 Two-parameter bifurcation diagram

Rename the obtained Hopf curve into
Hopf(+)

and delete all computed equilibrium curves.

Open a new Plot2D window to plot the bifurcation diagram in the (Q2,K)-plane with the visibility limits
Figure 4: **Starter** window for the continuation of the Hopf curve from the BT-point.

\[ Q2 \quad 0.7 \quad 1.5 \]
\[ K \quad 0.1 \quad 1.1 \]

Upon redrawing the diagram in the \((Q2,K)\)-window, you will get Figure 6.

The edge between two fold curves is a parameter region where the system has three equilibria. Crossing a Hopf curve results in the appearance of either stable or unstable periodic orbit.

**Warning:** There are other bifurcations in the model.
Figure 5: The parameter plot with the added Hopf curve in Bykov's model: GH - generalized Hopf points
Figure 6: Fold and Hopf curves in Bykov's model: BT - Bogdanov-Takens, CP - cusp, GH - generalized Hopf points
2 Fold and torus bifurcations of cycles in the Steinmetz–Larter model

Consider the following chemical model:

\[
\begin{align*}
\dot{A} &= -k_1ABX - k_3ABY + k_7 - k_{-7}A, \\
\dot{B} &= -k_1ABX - k_3ABY + k_8, \\
\dot{X} &= k_1ABX - 2k_2X^2 + 2k_3ABY - k_4X + k_6, \\
\dot{Y} &= -k_3ABY + 2k_2X^2 - k_5Y.
\end{align*}
\]

We will study bifurcations of limit cycles of this model when parameters \(k_7, k_8\) vary. Note: If you have worked out TUTORIAL III, the system was already introduced in Section 2.3. In this case, recompile the system with symbolic derivatives up to order 5 and delete all previously computed curves.

2.1 System specification

Specify a new ODE system – say STLAR – in MATCONT

\[
\begin{align*}
A' &= -k_1A*B*X-k_3*A*B*Y+k_7-k_{-7}A \\
B' &= -k_1A*B*X-k_3*A*B*Y+k_8 \\
X' &= k_1A*B*X-2k_2X^2+2k_3*A*B*Y-k_4X+k_6 \\
Y' &= -k_3ABY+2k_2X^2-k_5Y
\end{align*}
\]

where \((A, B, X, Y)\) are the coordinates and \((k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_{-7}, k_8)\) are the parameters. Use (default) \(t\) for time and generate symbolically partial derivatives of order 1, 2, 3, 4 and 5.

2.2 Preliminary one-parameter analysis

2.2.1 Continuation of an equilibrium

To begin with, we continue an equilibrium of the model and detect its Hopf bifurcation.

Input Type|Initial point|Equilibrium in the main MATCONT window.

Input the following numerical data in the appearing Starter window:

\[
\begin{align*}
A &= 31.78997 \\
B &= 1.45468 \\
X &= 0.01524586 \\
Y &= 0.1776113 \\
k_1 &= 0.1631021 \\
k_2 &= 1250 \\
k_3 &= 0.046875 \\
k_4 &= 20 \\
k_5 &= 1.104 \\
k_6 &= 0.001 \\
k_7 &= 4.235322 \\
k_{-7} &= 0.1175 \\
k_8 &= 0.5
\end{align*}
\]

These values correspond to an equilibrium \((A, B, X, Y)\) in the system. Activate the parameter \(k_7\).

Open Window/Output|Numeric and change its appearance via the Numeric/Layout command. Namely, select Eigenvalues and stepsize to be shown in the window.

Start Compute|Forward. The equilibrium curve will be continued and you get a Hopf bifurcation. The message in the MATLAB Command Window

\[
\text{label} = \text{H} , \quad x = (34.808899 \ 1.328517 \ 0.015246 \ 0.177611 \ 4.590046 )
\]

First Lyapunov coefficient = 1.527549e-02
at $k_7 = 4.590046\ldots$ indicates a subcritical Hopf bifurcation. Indeed, there are two eigenvalues of the equilibrium with $\text{Re} \lambda_{1,2} \approx 0$ at this parameter value visible in the Numeric window. The critical frequency $\text{Im} \lambda_1 \neq 0$, while the first Lyapunov coefficient is positive. Thus, there should exist an unstable limit cycle, bifurcating from the equilibrium.

Note also that in the Hopf point, as in all other bifurcation points, the stepsize is indicated as 0 in the Numeric window. The reason for this is that bifurcation points are not located by continuation but by a variant of the bisection method.

Stop computations and rename the computed curve via Select|Curve|Rename into Equilibrium(+).

Compute|Backward with Compute|Extend once until you get a Hopf bifurcation at $k_7 = 0.712475\ldots$ with the message

```
label = H , x = ( 1.808301 25.573303 0.015246 0.177611 0.712475 )
First Lyapunov coefficient = -2.371880e-02
```

in the MATLAB Command Window. The first Lyapunov coefficient is negative now. This means that a stable limit cycle bifurcates from the equilibrium, when it looses stability. Stop computations and rename the computed curve into Equilibrium(-).

### 2.2.2 Cycle continuation

Select the H: Hopf point in the computed Equilibrium(-) as initial. MATCONT will prepare to continue a limit cycle curve from the Hopf point (curve type $H_{LC}$). Choose the yes-option to monitor all singularities in the Starter window and set amplitude to 0.001. Choose $k_7$ and Period as the free parameters. Set

<table>
<thead>
<tr>
<th>MaxStepsize</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>MaxNumPoints</td>
<td>25</td>
</tr>
</tbody>
</table>

in the Continuer window. Change the appearance of the Numeric window via the Window|Layout command by selecting Multipliers to be shown.

Click Compute|Forward to start the continuation of the limit cycle. At $k_7 = 0.716434\ldots$ the message Neimark-Sacker indicates a torus bifurcation. In the MATLAB Command Window, the following message appears:

```
Neimark-Sacker (period = 1.091213e+01, parameter = 7.164336e-01)
Normal form coefficient = -4.912065e-08
```

Indeed, there are two complex multipliers with (approximately) $|\mu| = 1$ and one trivial multiplier (approximately) equal to 1. This can be seen in the Numeric window. The normal form coefficient is small but negative, indicating that a stable two-dimensional invariant torus bifurcates from the limit cycle.

Rename the computed curve into cycle.

### 2.3 Two-parameter analysis

#### 2.3.1 Continuation of the Hopf bifurcation curve

Select the H: Hopf point in the computed Equilibrium(+), curve starting with the Select|Curve command. Then navigate in the Data Browser to the Equilibrium(+), curve, double-click it and then select the Hopf point.

Select Type|Curve|Hopf. Activate two parameters, namely $k_7$ and $k_8$, in the Starter window.

Open a new Plot2D window with the parameters $k_7$ and $k_8$ as abscissa and ordinate, respectively, and the visibility limits:
Continue the Hopf bifurcation curve with Compute|Forward followed by Compute|Extend. At $(k_7, k_8) = (6.336084\ldots, 0.413039\ldots)$ a generalized Hopf bifurcation will be found, where the first Lyapunov coefficient vanishes (label GH). Resume the continuation and terminate it when the Hopf curve leaves the window. In the MATLAB Command Window, the following message has appeared:

```
label = GH, x = ( 50.40856 0.97900 0.01342 0.13184 6.33604 0.41303 0.59436 )
l2=3.325910e-03
```

from which it follows that this codim 2 bifurcation is nondegenerate (the second Lyapunov coefficient 12 is nonzero). Rename the computed curve to Hopf(+).

The backward continuation of the Hopf bifurcation curve reveals one more generalized Hopf point at $(k_7, k_8) = (0.999480\ldots, 0.645896\ldots)$. Resume and extend until the curve leaves the window. You should get Figure 7.

![Figure 7: The Hopf bifurcation curve with two generalized Hopf points GH.](image)

The MATLAB Command Window contains the message:

```
label = GH, x = ( 3.00921 14.18442 0.01797 0.26026 0.99947 0.64589 0.29229 )
l2=-4.044531e-03
```

indicating that this GH is also nondegenerate. Rename this curve to Hopf(-).
2.3.2 Continuation of the LPC-bifurcation starting at the GH-point

Using Select|Curve and the Data Browser select GH: Generalized Hopf point in the Hopf(+) curve as initial point. Since it is known that an LPC curve originates there, select Type|Curve|Limit point of cycles to continue. The Starter and Continuer windows for the continuation of the fold bifurcation of cycles from the generalized Hopf point will appear.

In the Starter window, set yes to monitor for all singularities except the Cusp bifurcation of cycles (LPC) and increase the amplitude to 0.01. To increase speed set ntst to 20 and check that ncol is 4. In the Continuer window, set MaxStepsize equal to 2.0 and MaxNumPoints equal to 300. The Starter and Continuer windows should look like in Figure 8.

Compute|Forward the LPC-bifurcation curve (with Compute|Extend) and observe that it actually connects the two found GH points in the Hopf curve, see Figure 9. The continuation (which takes some time) stops with the message **Current step size too small** near the left GH point.

The computed curve will also contain two extra codim 2 points labeled by R1. These are strong resonance 1:1 points, where the cycle has a triple multiplier 1 (counting the trivial multiplier). In the Numeric window, you can read the corresponding values of \((k_7, k_8)\). The MATLAB Command Window gives the following messages:

Resonance 1:1 (period = 1.40019e+01, parameters = 1.85767e+00, 9.30422e-01)
\[ ab = 1.432638e-01 \]
Resonance 1:1 (period = 1.23945e+01, parameters = 1.17955e+00, 7.23957e-01)
\[ ab = -2.686129e-03 \]
Figure 9: Hopf and limit point of cycles curves connecting the GH points.
where the critical cycle period and parameter values, as well as the product of the normal form coefficients for \( R_1 \), are reported.

Rename the computed \( G_H \_LPC \) curve to \( cyclefold \).

2.3.3 Continuation of the \( NS \)-curve in two parameters

Take the \( NS: Neimark-Sacker \) point in the limit cycle curve \( cycle \) as initial point. By default, the Curve Type is \( Neimark-Sacker \). Activate \( k_7, k_8 \) as free parameters.

To speed up the continuation, untick the boxes for monitoring any singularity. Compute the multipliers, though. Set \( MaxNumPoints \) to 300.

Change the plotting region of the \( Plot2D \) window to:

- **Abscissa**: 1, 2.2
- **Ordinate**: 0.6, 1

and redraw the diagram.

To display the \( NS \) curve in red click MatCont|Options|Plot Properties. In the \( NS: \) line of the \( Plot Properties \) window replace the default ‘blue’ by ‘red’ and close the \( Plot Properties \) window.

Click Compute|Forward and wait until the computed \( NS \)-curve enters the window (this takes some time). Extend the computation several times to obtain (after redrawing the Hopf curve in green in the same way) Figure 10.

![plot](image.png)

Figure 10: Hopf (green), limit point of cycles (blue) and Neimark-Sacker (red) curves: The \( NS \)-curve between two \( R_1 \)-points corresponds to neutral saddle cycles.

Restore the default fields in the \( Plot Properties \) window.

The computed \( NS \)-curve passes through the \( R_1 \) points of strong resonance 1:1, where it tangentially meets the \( LPC \)-curve and turns from the Neimark-Sacker bifurcation curve into the non-bifurcation neutral saddle cycle curve (both are characterised by the presence of two multipliers with \( \mu_1\mu_2 = 1 \)).
Warning: There are many other bifurcations in the model and Figure 10 shows only a few of them.

3 Additional Problems

A. Consider the following system by Lorenz [1984]:
\[
\begin{aligned}
\dot{x} &= -y^2 - z^2 - ax + aF, \\
\dot{y} &= xy - bxz - y + G, \\
\dot{z} &= bxy + xz - z,
\end{aligned}
\] (1)
where \((a, b, F, G)\) are parameters.

1. Using MatCont, compute fold and Hopf bifurcation curves for equilibria of (1) with \(a = \frac{1}{4}, b = 4\), in the parameter domain \(\{(G, F) : 0 \leq G \leq 3, 0 \leq F \leq 3\}\).

Find numerical parameter values \((G_{ZH}, F_{ZH})\) at which (1) exhibits a fold-Hopf bifurcation, i.e. has an equilibrium with eigenvalues \(\lambda_1 = 0, \lambda_{2,3} = \pm i\omega_0, \omega_0 > 0\). Find numerical parameter values \((G_{CP}, F_{CP})\) at which (1) exhibits a cusp bifurcation, i.e. has an equilibrium with a triple zero.

2. Derive analytical expressions for \((G_{ZH}, F_{ZH})\) and \((G_{CP}, F_{CP})\) as functions of \((a, b)\) and verify the numerical results above.

Hints: At the fold-Hopf both the trace and the determinant of the Jacobian matrix of (1) vanish. To find a triple equilibrium, reduce the equilibrium system for (1) to one cubic equation for the \(x\)-coordinate of the equilibrium and look for its triple root.

B. Consider the adaptive control system of Lur’e type
\[
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -\alpha z - \beta y - x + x^2.
\end{aligned}
\] (2)

1. Continue the equilibrium \((x, y, z) = (0, 0, 0)\) of (2) with respect to parameter \(\alpha\) starting at \(\alpha = 2, \beta = 1\) and detect its Hopf bifurcation.

2. Compute the Hopf bifurcation curve in the \((\alpha, \beta)\)-plane and plot it using the visibility limits

\[
\begin{aligned}
\text{alpha} &\quad 0 &\quad 1.5 \\
\text{beta} &\quad 0 &\quad 2
\end{aligned}
\]

3. Find the period-doubling (PD) of the cycle bifurcating from the Hopf point and continue it in two parameters.

4. Find the period-doubling of the doubled cycle and continue it in the \((\alpha, \beta)\)-plane.

C. Consider the following prey-predator system from mathematical ecology by Bazykin and Khibnik [1981]:
\[
\begin{aligned}
\dot{x} &= \frac{x^2(1 - x)}{n + x} - xy, \\
\dot{y} &= -y(m - x),
\end{aligned}
\] (3)
where \(n > 0, 0 < m < 1\).

1. Compare its bifurcation diagrams with respect to parameter \(m\) for \(n_1 = \frac{1}{4}\) and \(n_2 = \frac{1}{16}\) using MatCont.
2. Compute the two-parameter bifurcation diagram in the \((m,n)\)-plane and sketch all qualitatively different phase portraits.

*Hint:* The diagram should include a Hopf bifurcation curve \((H)\) and a cycle fold curve \((LPC)\) that connects a generalized Hopf point \((GH)\) on \(H\) with the point \((m,n) = (0,0)\).

3. Derive an analytic expression for the Hopf bifurcation curve in (3). *Hint:* Consider the orbitally-equivalent polynomial system

\[
\begin{align*}
\dot{x} &= x^2(1-x) - xy(n+x), \\
\dot{y} &= -y(m-x)(n+x).
\end{align*}
\] (4)

4. Verify the numerically found \(GH\)-point by proving that the first Lyapunov coefficient \(l_1\) vanish at \((m_{GH}, n_{GH}) = \left(\frac{1}{4}, \frac{1}{8}\right)\).

*Hints:*

(a) At Hopf parameter values, translate the origin in (4) to the equilibrium and scale the variables to obtain a system in the form

\[
\begin{align*}
\dot{\xi} &= -\omega \eta + P(\xi, \eta), \\
\dot{\eta} &= \omega \xi + Q(\xi, \eta),
\end{align*}
\]

where \(P, Q\) contain terms of order two and higher in \((\xi, \eta)\).

(b) Introduce complex variables \(z = \xi + i\eta\) and \(\bar{z} = \xi - i\eta\), and derive the equation

\[
\dot{z} = i\omega z + \sum_{2 \leq j+k \leq 3} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4).
\]

(c) Compute

\[l_1 = \frac{1}{2\omega^2} \text{Re}(ig_{20}g_{11} + \omega g_{21}).\]