

TUTORIAL IV:
Two-parameter bifurcation analysis of equilibria and limit
cycles with **matcont**

Yu.A. Kuznetsov
Department of Mathematics
Utrecht University
Budapestlaan 6
3508 TA, Utrecht

N. Neiryck
Department of Applied Mathematics,
Computer Science and Statistics
Ghent University
Krijgslaan 281, S9
B-9000, Ghent

May 13, 2019

This session was tested on MATCONT7.0 with MATLAB2018b. It is devoted to the numerical continuation of codim 1 bifurcations of equilibria and limit cycles in systems of autonomous ODEs depending on two parameters

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^2,$$

and detection of their codim 2 bifurcations. We will also switch at some codim 2 equilibrium bifurcations to the continuation of codim 1 bifurcation curves rooted there.

1 Bifurcations of equilibria in the Bykov–Yablonskii–Kim model

We will use MATCONT to continue equilibria and their bifurcations in the following chemical model that describes CO-oxidation on platinum:

$$\begin{cases} \dot{x} &= 2k_1z^2 - 2k_{-1}x^2 - k_3xy \\ \dot{y} &= k_2z - k_{-2}y - k_3xy \\ \dot{s} &= k_4z - k_{-4}s, \end{cases}$$

where $z = 1 - x - y - s$. The ratio

$$K = \frac{k_{-4}}{k_4}$$

and the parameter k_2 will be used as two bifurcation parameters.

1.1 Specify the model in MATCONT

Specify a new ODE system in MATCONT with coordinates (x,y,s) and time (t):

```
z=1-x-y-s
x'=2*Q1*z^2-2*Q5*x^2-Q3*x*y
y'=Q2*z-Q6*y-Q3*x*y
s'=Q4*z-K*Q4*s
```

The reaction rates $k_1, k_2, k_3, k_4, k_{-1}, k_{-2}$ are denoted by Q1, Q2, Q3, Q4, Q5, Q6, respectively, while K stands for K . Generate the derivatives of order 1, 2, 3, 4 and 5 symbolically. Note that z is an *auxiliary variable* which is introduced to simplify the formal form of the equations and to improve the readability. The choice of the name is irrelevant but it should not conflict with the internal names in the MATLAB symbolic toolbox if that toolbox is used.

1.2 Equilibrium continuation

Select both **Type|Initial point|Equilibrium** and **Type|Curve|Equilibrium**. In the **Starter** window input the following numerical values, corresponding to a stable equilibrium in the system:

x	0.001137
y	0.891483
s	0.062345
Q1	2.5
Q2	2.204678
Q3	10
Q4	0.0675
Q5	1
Q6	0.1
K	0.4

Activate the parameter Q2.

In the **Continuer** window, decrease the maximal step size along the curve to

MaxStepsize 0.025

Open a **Plot2D** window with (Q2,x) on the axes and input the following visibility limits:

Abscissa: 0.5 2.0
 Ordinate: 0.0 0.16

Also open a **Numeric** window and make **Eigenvalues** visible using **Numeric|Layout**.

Start the continuation of the equilibrium curve with the **Compute|Forward** command. Monitor the eigenvalues in the **Numeric** window. There will be four bifurcation points detected: two *Hopf points* (labeled by H) and two *fold points* (labeled by LP). Resume computations at each bifurcation point, and stop them when the curve leaves the window. You should get the central S-shaped curve presented in Figure 1. In a narrow interval of Q2-values (between two LP-points) the system has three equilibria.

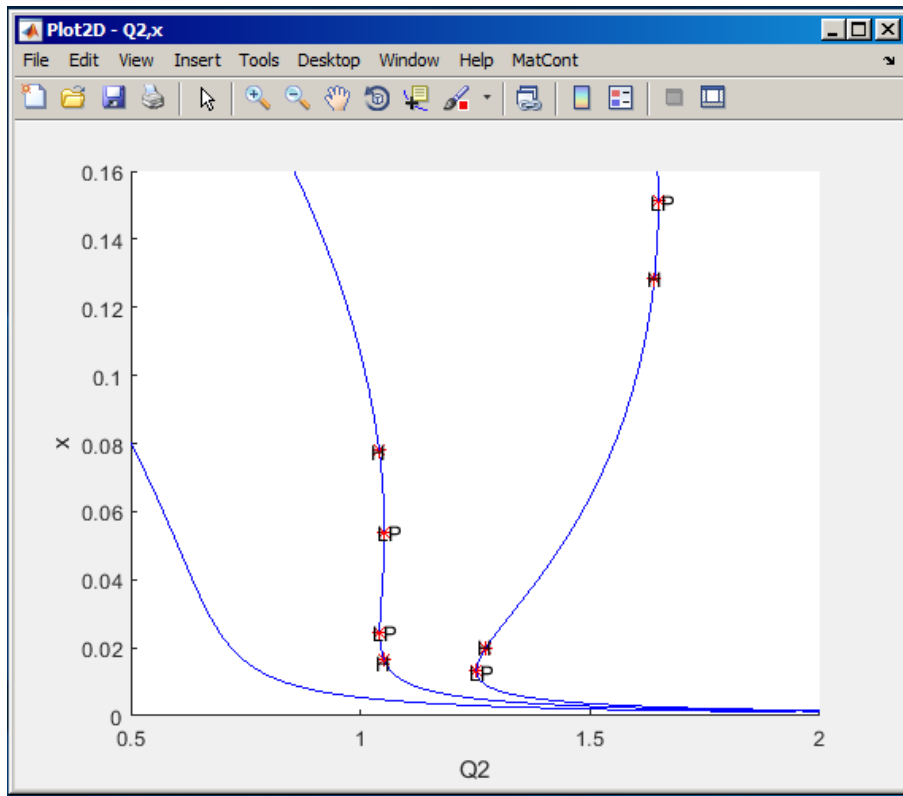


Figure 1: Equilibrium curves with $K = 0.15, 0.4,$ and 2.0

The normal form coefficients for the Hopf and limit point bifurcations can be read in the MATLAB Command Window:

```
label = H , x = ( 0.016357 0.523973 0.328336 1.051558 )
First Lyapunov coefficient = 1.070259e+01
label = LP, x = ( 0.024717 0.450257 0.375018 1.042049 )
a=1.166509e-01
label = LP, x = ( 0.054030 0.302241 0.459807 1.052200 )
a=-1.346534e-01
```

```
label = H , x = ( 0.077929 0.233063 0.492149 1.040991 )
First Lyapunov coefficient = 4.332247e+00
```

Note that both first Lyapunov coefficients are positive (implying that unstable limit cycles bifurcate there), while the LP-coefficients are both nonzero (their sign is irrelevant and depends on the choice of a direction).

Rename the computed equilibrium curve via **Select|Curve** and **Select|Curve|Rename** into **Equilibrium1(+)**

Change the value of the parameter K in the **Starter** window:

```
K          0.15
```

and repeat the computations with **Compute|Forward**. You will get a monotone curve without bifurcation points (the left curve in Figure 1). There is only one equilibrium for all parameter values $Q2$.

Finally, change the value of K to

```
K          2.0
```

and click on **Compute|Forward** again. Another equilibrium curve will come out with the order of the singularities reversed (the right curve in Figure 1). There is now a big interval of the parameter values, in which there are three equilibria in the system.

In the MATLAB Command Window, the following messages appeared:

```
label = LP, x = ( 0.013310 0.771372 0.071773 1.252630 )
a=1.705787e+00
label = H , x = ( 0.019639 0.726319 0.084681 1.271109 )
Neutral saddle
label = H , x = ( 0.128699 0.384106 0.162399 1.640254 )
Neutral saddle
label = LP, x = ( 0.151168 0.344167 0.168222 1.648672 )
a=-1.787355e+00
```

indicating that the LP-bifurcations are still nondegenerate ($a \neq 0$), while – instead of two Hopf points – two *neutral saddles* have appeared, which are not bifurcation points. Note that they are still marked by H.

Store the last two computed curves permanently by renaming them into

```
Equilibrium2(+)  
Equilibrium3(+)
```

via **Select|Curve|Rename**. Close the **Data Browser** window.

1.3 Continuation of fold and Hopf bifurcations

This section explains how to continue fold and Hopf bifurcations with respect to two control parameters. While doing so, you will see how MATCONT detects some codim 2 bifurcations and reports their normal form coefficients.

1.3.1 Fold continuation

Select the second LP: **Limit point** in the first computed curve **Equilibrium1(+)** via **Select|Curve**. The new **Starter** window corresponding to the continuation of a limit point curve LP_LP is to appear, in which you should keep parameter **Q2** active and activate parameter **K**, see Figure 2. Compute the limit point curve in both directions to get Figure 3. Notice that both parameters ($Q2, K$) are varied along the curve. Each point on the curve is a limit point for the equilibrium

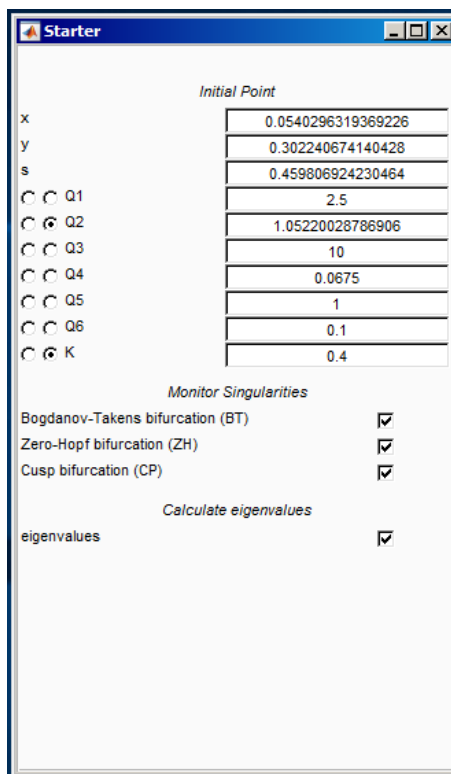


Figure 2: **Starter** window for the continuation of the limit point curve.

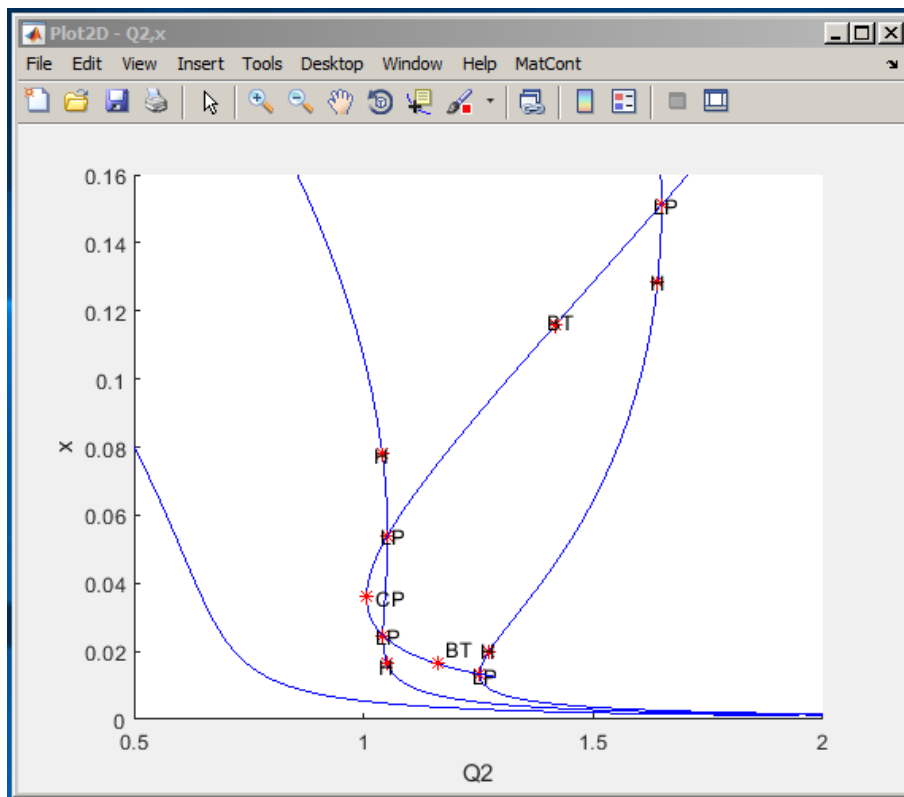


Figure 3: Limit point curve in Bykov's model: BT - Bogdanov-Takens points; CP - cusp.

curve at the corresponding value of K (or $Q2$).

Three points were detected, corresponding to codim 2 bifurcations: two *Bogdanov-Takens* (BT) and one *cusp* (CP). At each BT point the system has an equilibrium with a double zero eigenvalue, while at the CP point there is an equilibrium with a simple zero eigenvalue but the normal form coefficient of the fold is zero. The normal form coefficients (a, b, c) of these bifurcations are reported in the MATLAB Command Window and they are all nonzero:

```
label = BT , x = ( 0.115909 0.315467 0.288437 1.417628 0.971397 )
(a,b)=(-8.378442e-02, -2.136280e+00)
```

```
label = CP , x = ( 0.035940 0.352008 0.451368 1.006408 0.355991 )
c=3.627844e-01
```

```
label = BT , x = ( 0.016337 0.638410 0.200456 1.161199 0.722339 )
(a,b)=(-4.822563e-02, -1.937632e+00)
```

Rename the obtained limit point branches as following:

```
Fold(+)
```

```
Fold(-)
```

1.3.2 Hopf continuation

The Bogdanov-Takens points are common points for the limit point curves and curves corresponding to equilibria with eigenvalues $\lambda_1 + \lambda_2 = 0, \lambda_3 \neq 0$. Actually, at each BT point, the Hopf bifurcation curve (with $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$) turns into the neutral saddle curve (with real $\lambda_1 = -\lambda_2$). Thus, we can start a Hopf curve from a Bogdanov-Takens point.

Select the BT: **Bogdanov-Takens point** in the curve **Fold(+)** as initial. Then choose **Type|Curve|Hopf** to prepare for the continuation of the Hopf curve. The new **Starter** window for the continuation of a Hopf curve will appear automatically, see Figure 4. Check that the type of the curve to be computed is changed in the **MATCONT** window. Activate the parameters $Q2, K$.

Compute|Forward (resuming the computation at each special point and using **Compute|Extend**) produces a *closed* Hopf curve as in Figure 5. Actually, only the part of it to the left of two BT points corresponds to a Hopf bifurcation; the other part represents a neutral saddle. As it has been mentioned, this transition happens at the Bogdanov-Takens points. There are two more codim 2 bifurcation points at the Hopf point: Two *Bautin* or *generalized Hopf* (GH) points, where the first Lyapunov coefficient l_1 vanishes.

The Command Window of MATLAB shows a message about the starting BT point and the following messages related to the GH points:

```
label = GH, x = ( 0.018022 0.368238 0.497968 0.891319 0.232487 0.003324 )
l2=-7.768996e+02
```

```
label = GH, x = ( 0.064311 0.211095 0.554870 0.924255 0.305879 0.003512 )
l2=-2.401233e+02
```

indicating that both GH points are nondegenerate, since the second Lyapunov coefficients l_2 are nonzero (in fact, negative).

1.3.3 Two-parameter bifurcation diagram

Rename the obtained Hopf curve into

```
Hopf(+)
```

and delete all computed equilibrium curves.

Open a new **Plot2D** window to plot the bifurcation diagram in the $(Q2, K)$ -plane with the visibility limits

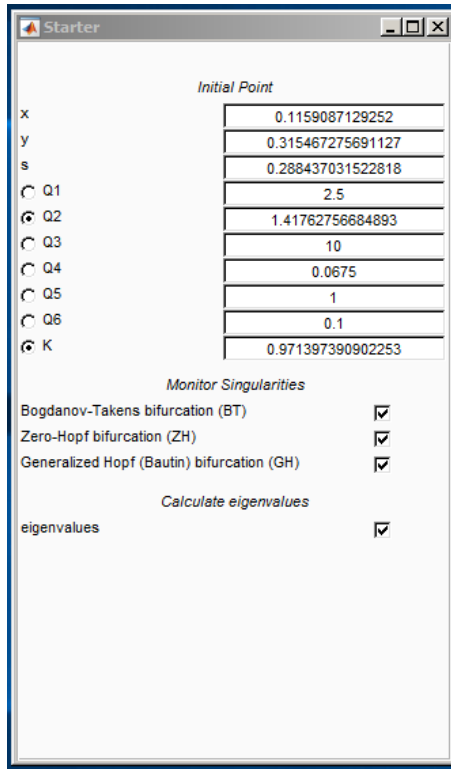


Figure 4: **Starter** window for the continuation of the Hopf curve from the BT-point.

Q2	0.7	1.5
K	0.1	1.1

Upon redrawing the diagram in the (Q2,K)-window, you will get Figure 6.

The edge between two fold curves is a parameter region where the system has three equilibria. Crossing a Hopf curve results in the appearance of either stable or unstable periodic orbit.

Warning: There are other bifurcations in the model.

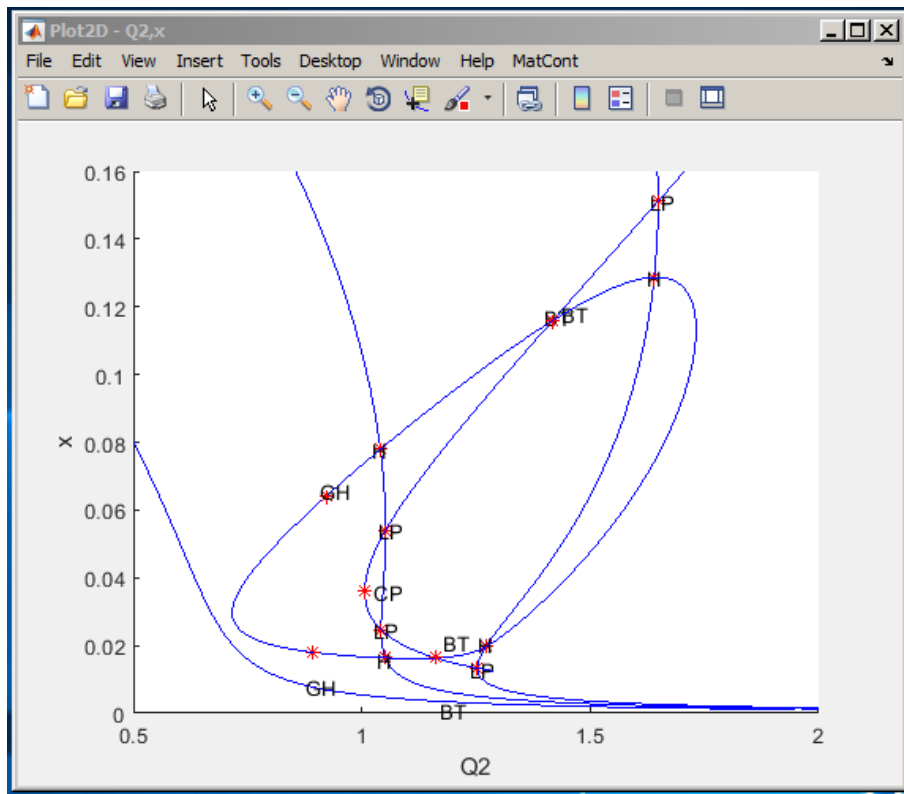


Figure 5: The parameter plot with the added Hopf curve in Bykov's model: GH - generalized Hopf points

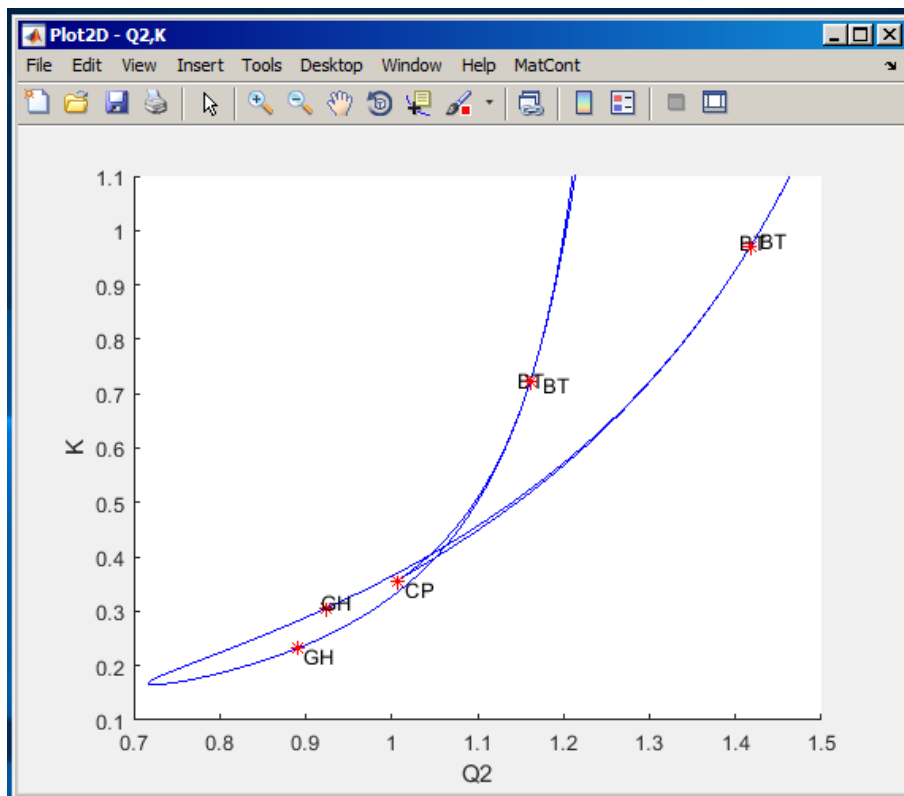


Figure 6: Fold and Hopf curves in Bykov's model: BT- Bogdanov-Takens, CP -cusp, GH - generalized Hopf points

2 Fold and torus bifurcations of cycles in the Steinmetz–Larter model

Consider the following chemical model:

$$\begin{cases} \dot{A} &= -k_1ABX - k_3ABY + k_7 - k_{-7}A, \\ \dot{B} &= -k_1ABX - k_3ABY + k_8, \\ \dot{X} &= k_1ABX - 2k_2X^2 + 2k_3ABY - k_4X + k_6, \\ \dot{Y} &= -k_3ABY + 2k_2X^2 - k_5Y. \end{cases}$$

We will study bifurcations of limit cycles of this model when parameters (k_7, k_8) vary. Note: If you have worked out TUTORIAL III, the system was already introduced in Section 2.3. In this case, recompile the system with symbolic derivatives up to order 5 and delete all previously computed curves.

2.1 System specification

Specify a new ODE system – say `STLAR` – in `MATCONT`

```
A'=-k1*A*B*X-k3*A*B*Y+k7-km7*A
B'=-k1*A*B*X-k3*A*B*Y+k8
X'=k1*A*B*X-2*k2*X^2+2*k3*A*B*Y-k4*X+k6
Y'=-k3*A*B*Y+2*k2*X^2-k5*Y
```

where (A, B, X, Y) are the coordinates and $(k_1, k_2, k_3, k_4, k_5, k_6, k_7, km_7, k_8)$ are the parameters. Use (default) `t` for time and generate symbolically partial derivatives of order 1, 2, 3, 4 and 5.

2.2 Preliminary one-parameter analysis

2.2.1 Continuation of an equilibrium

To begin with, we continue an equilibrium of the model and detect its Hopf bifurcation.

Input **Type|Initial point|Equilibrium** in the main `MATCONT` window.

Input the following numerical data in the appearing **Starter** window:

A	31.78997
B	1.45468
X	0.01524586
Y	0.1776113
k1	0.1631021
k2	1250
k3	0.046875
k4	20
k5	1.104
k6	0.001
k7	4.235322
km7	0.1175
k8	0.5

These values correspond to an equilibrium (A, B, X, Y) in the system. Activate the parameter `k7`.

Open **Window|Output|Numeric** and change its appearance via the **Numeric|Layout** command. Namely, select **Eigenvalues** and **stepsize** to be shown in the window.

Start **Compute|Forward**. The equilibrium curve will be continued and you get a Hopf bifurcation. The message in the MATLAB Command Window

```
label = H , x = ( 34.808899 1.328517 0.015246 0.177611 4.590046 )
First Lyapunov coefficient = 1.527549e-02
```

at $k_7 = 4.590046\dots$ indicates a *subcritical* Hopf bifurcation. Indeed, there are two eigenvalues of the equilibrium with $\text{Re } \lambda_{1,2} \approx 0$ at this parameter value visible in the **Numeric** window. The critical frequency $\text{Im } \lambda_1 \neq 0$, while the first Lyapunov coefficient is positive. Thus, there should exist an unstable limit cycle, bifurcating from the equilibrium.

Note also that in the Hopf point, as in all other bifurcation points, the **stepsize** is indicated as 0 in the **Numeric** window. The reason for this is that bifurcation points are not located by continuation but by a variant of the bisection method.

Stop computations and rename the computed curve via **Select|Curve|Rename** into

Equilibrium(+)

Compute|Backward with **Compute|Extend** once until you get a *Hopf bifurcation* at $k_7 = 0.712475\dots$ with the message

```
label = H , x = ( 1.808301 25.573303 0.015246 0.177611 0.712475 )
First Lyapunov coefficient = -2.371880e-02
```

in the MATLAB Command Window. The first Lyapunov coefficient is negative now. This means that a *stable* limit cycle bifurcates from the equilibrium, when it loses stability. Stop computations and rename the computed curve into **Equilibrium(-)**.

2.2.2 Cycle continuation

Select the **H: Hopf** point in the curve **Equilibrium(-)** as initial. MATCONT will prepare to continue a limit cycle curve from the Hopf point (curve type **H-LC**). Choose the **yes**-option to monitor all singularities in the **Starter** window and set **amplitude** to 0.001. Choose k_7 and **Period** as the free parameters. Set

```
MaxStepsize          1
MaxNumPoints         25
```

in the **Continuer** window. Change the appearance of the **Numeric** window via the **Window|Layout** command by selecting **Multipliers** to be shown.

Click **Compute|Forward** to start the continuation of the limit cycle. At $k_7 = 0.716434\dots$ the message **Neimark-Sacker** indicates a *torus* bifurcation. In the MATLAB Command Window, the following message appears:

```
Neimark-Sacker (period = 1.091213e+01, parameter = 7.164336e-01)
Normal form coefficient = -4.912065e-08
```

Indeed, there are two complex multipliers with (approximately) $|\mu| = 1$ and one trivial multiplier (approximately) equal to 1. This can be seen in the **Numeric** window. The normal form coefficient is small but negative, indicating that a stable two-dimensional *invariant torus* bifurcates from the limit cycle.

Rename the computed curve into **cycle**.

2.3 Two-parameter analysis

2.3.1 Continuation of the Hopf bifurcation curve

Select the **H: Hopf** point in the computed **Equilibrium(+)** curve starting with the **Select|Curve** command. Then navigate in the **Data Browser** to the **Equilibrium(+)** curve, double-click it and then select the Hopf point.

Select **Type|Curve|Hopf**. Activate two parameters, namely k_7 and k_8 , in the **Starter** window.

Open a new **Plot2D** window with the parameters k_7 and k_8 as abscissa and ordinate, respectively, and the visibility limits:

Abscissa	0	7
Ordinate	0.3	1

Continue the Hopf bifurcation curve with **Compute|Forward** followed by **Compute|Extend**.
At

$$(k_7, k_8) = (6.336084\dots, 0.413039\dots)$$

a *generalized Hopf* bifurcation will be found, where the first Lyapunov coefficient vanishes (label GH). Resume the continuation and terminate it when the Hopf curve leaves the window. In the MATLAB Command Window, the following message has appeared:

```
label = GH, x = ( 50.40856 0.97900 0.01342 0.13184 6.33604 0.41303 0.59436 )
l2=3.325910e-03
```

from which it follows that this codim 2 bifurcation is nondegenerate (the second Lyapunov coefficient l2 is nonzero). Rename the computed curve to **Hopf(+)**.

The backward continuation of the Hopf bifurcation curve reveals one more generalized Hopf point at

$$(k_7, k_8) = (0.999480\dots, 0.645896\dots).$$

Resume and extend until the curve leaves the window. You should get Figure 7.

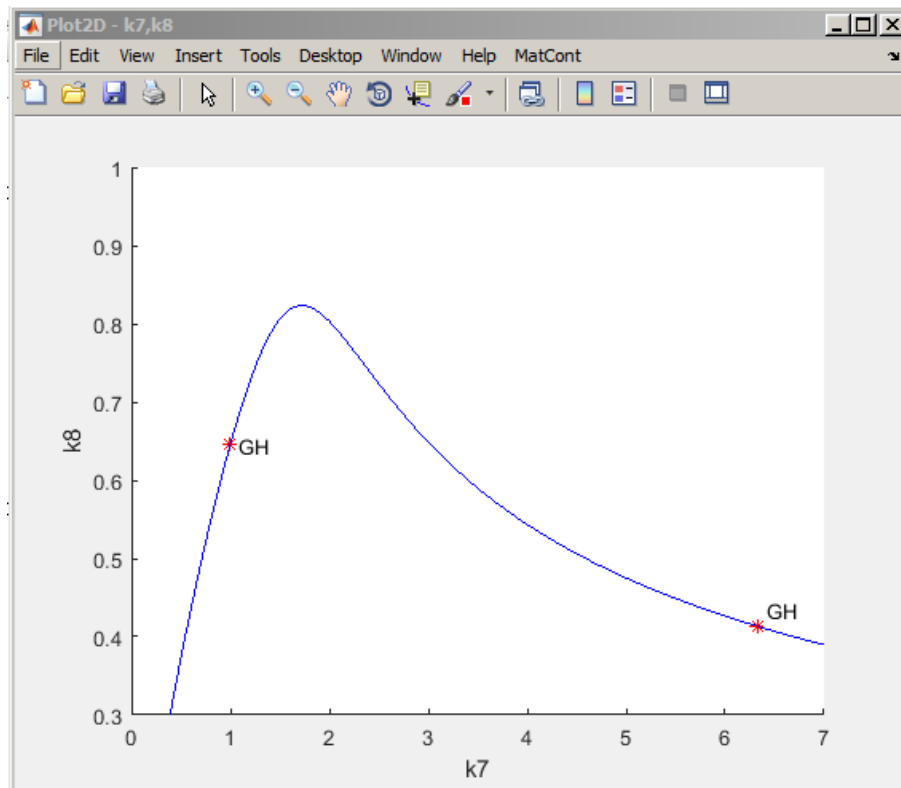


Figure 7: The Hopf bifurcation curve with two generalized Hopf points GH.

The MATLAB Command Window contains the message:

```
label = GH, x = ( 3.00921 14.18442 0.01797 0.26026 0.99947 0.64589 0.29229 )
l2=-4.044531e-03
```

indicating that this GH is also nondegenerate. Rename this curve to **Hopf(-)**.

2.3.2 Continuation of the LPC-bifurcation starting at the GH-point

Using **Select|Curve** and the **Data Browser** select **GH: Generalized Hopf** point in the **Hopf(+)** curve as initial point. Since it is known that an LPC curve originates there, select **Type|Curve|Limit point of cycles** to continue. The **Starter** and **Continuer** windows for the continuation of the fold bifurcation of cycles from the generalized Hopf point will appear.

In the **Starter** window, set **yes** to monitor for all singularities except the **Cusp** bifurcation of cycles (LPC) and increase the **amplitude** to 0.01. To increase speed set **ntst** to 20 and check that **ncol** is 4. In the **Continuer** window, set **MaxStepsize** equal to 2.0 and **MaxNumPoints** equal to 300. The **Starter** and **Continuer** windows should look like in Figure 8.

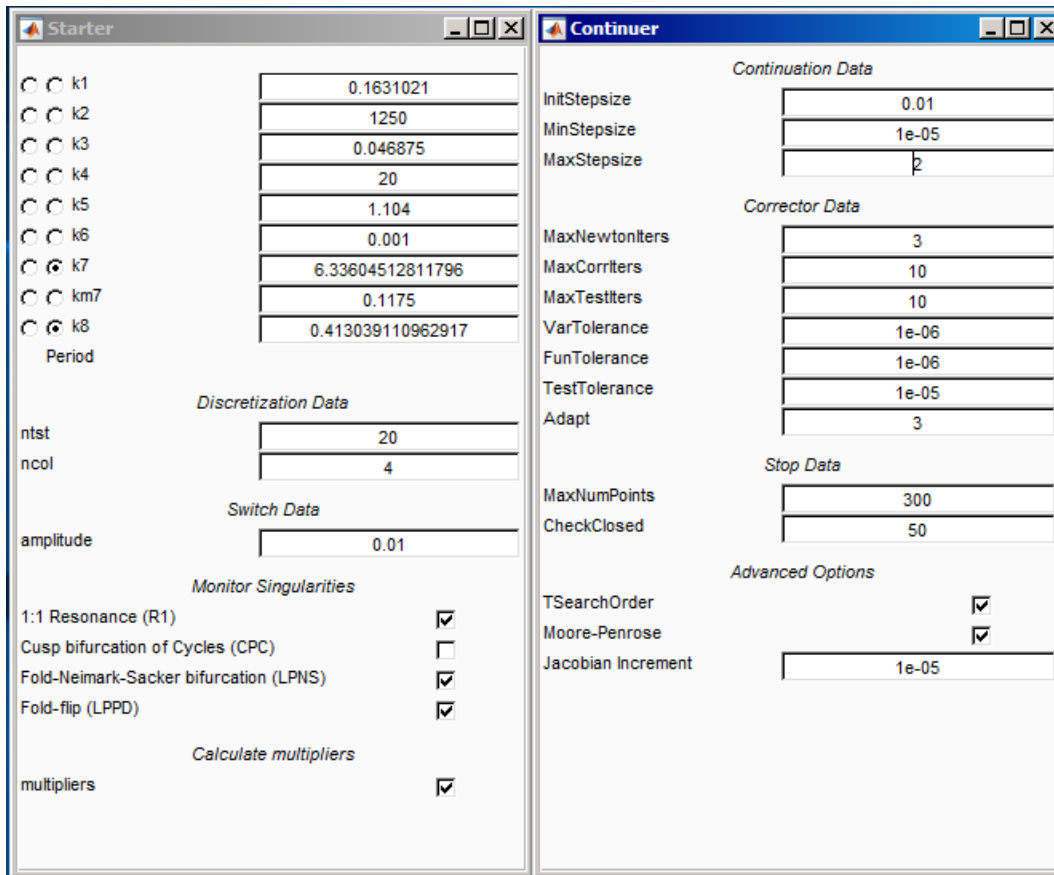


Figure 8: The **Starter** and **Continuer** windows for the LPC-continuation from the GH point.

Compute|Forward the LPC-bifurcation curve (with **Compute|Extend**) and observe that it actually connects the two found GH points in the Hopf curve, see Figure 9. The continuation (which takes some time) stops with the message **Current step size too small** near the left GH point.

The computed curve will also contain two extra codim 2 points labeled by R1. These are *strong resonance* 1:1 points, where the cycle has a triple multiplier 1 (counting the trivial multiplier). In the **Numeric** window, you can read the corresponding values of (k_7, k_8) . The MATLAB Command Window gives the following messages:

```
Resonance 1:1 (period = 1.40019e+01, parameters = 1.85767e+00, 9.30422e-01)
ab=1.432638e-01
Resonance 1:1 (period = 1.23945e+01, parameters = 1.17955e+00, 7.23957e-01)
ab=-2.686129e-03
```

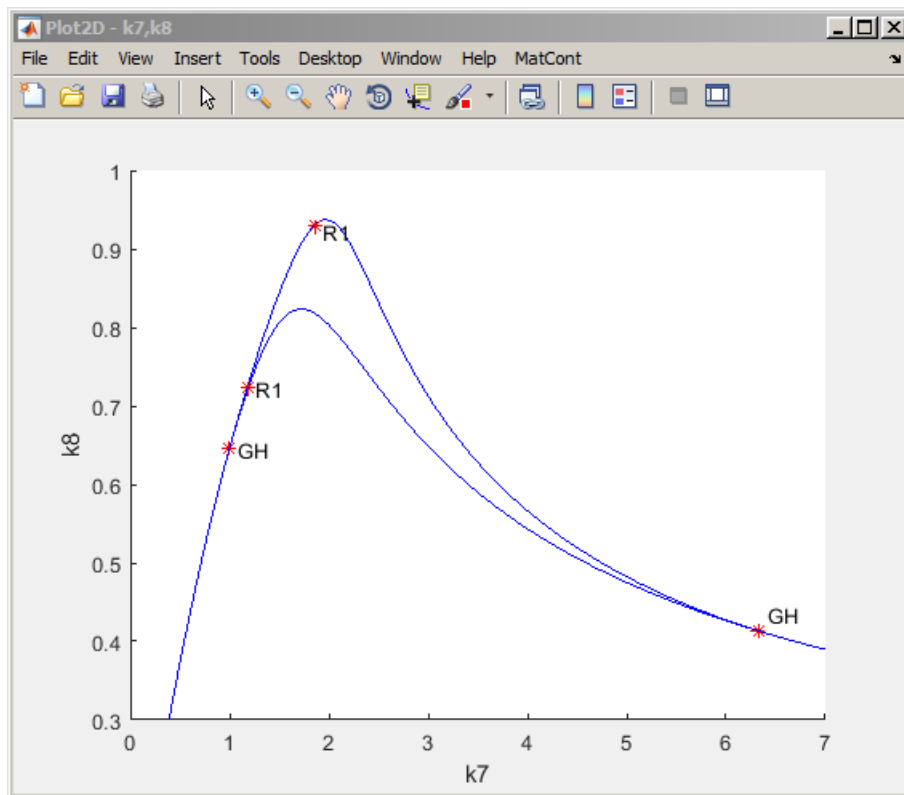


Figure 9: Hopf and limit point of cycles curves connecting the GH points.

where the critical cycle period and parameter values, as well as the product of the normal form coefficients for R1, are reported.

Rename the computed GH_LPC curve to `cyclefold`.

2.3.3 Continuation of the NS-curve in two parameters

Take the NS: Neimark-Sacker point in the limit cycle curve `cycle` as initial point. By default, the Curve Type is Neimark-Sacker. Activate k_7, k_8 as free parameters.

To speed up the continuation, untick the boxes for monitoring any singularity. Compute the multipliers, though. Set `MaxNumPoints` to 300.

Change the plotting region of the **Plot2D** window to

Abscissa	1	2.2
Ordinate	0.6	1

and redraw the diagram.

Click **Compute**|**Forward** and wait until the computed NS-curve enters the window (this takes some time). Extend the computation several times to obtain (after some editing) Figure 10.

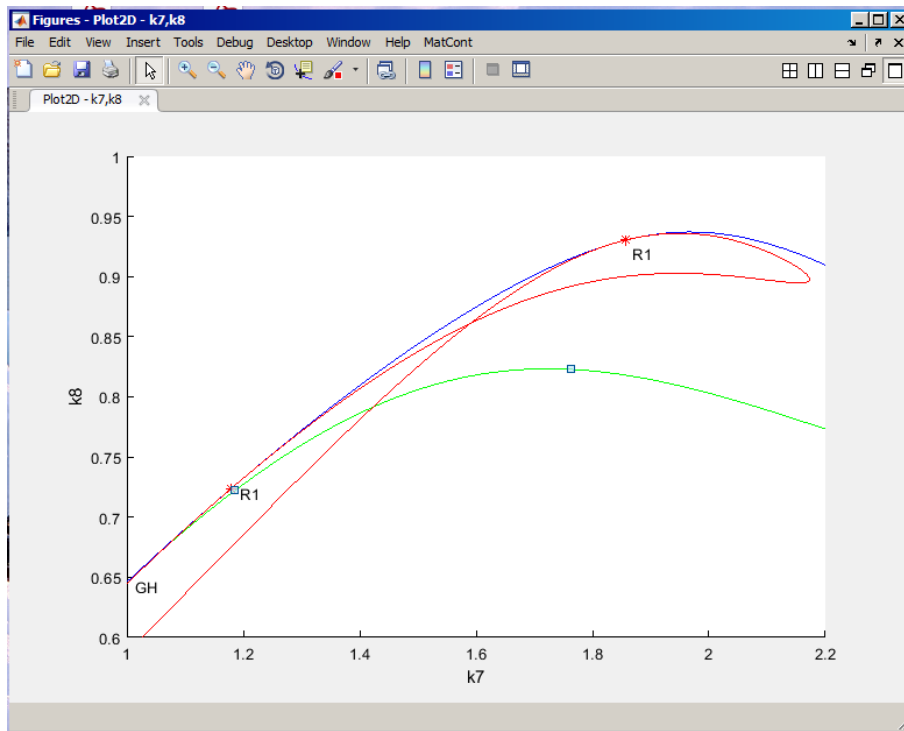


Figure 10: Hopf (green), limit point of cycles (blue) and Neimark-Sacker (red) curves: The NS-curve between two R1-points corresponds to neutral saddle cycles.

The computed NS-curve passes through the R1 points of strong resonance 1:1, where it tangentially meets the LPC-curve and turns from the Neimark-Sacker bifurcation curve into the non-bifurcation neutral saddle cycle curve (both are characterised by the presence of two multipliers with $\mu_1\mu_2 = 1$).

Warning: There are many other bifurcations in the model and Figure 10 shows only a few of them.

3 Additional Problems

A. Consider the following system by Lorenz [1984]:

$$\begin{cases} \dot{x} &= -y^2 - z^2 - ax + aF, \\ \dot{y} &= xy - bxz - y + G, \\ \dot{z} &= bxy + xz - z, \end{cases} \quad (1)$$

where (a, b, F, G) are parameters.

- Using MATCONT, compute fold and Hopf bifurcation curves for equilibria of (1) with $a = \frac{1}{4}, b = 4$, in the parameter domain

$$\{(G, F) : 0 \leq G \leq 3, 0 \leq F \leq 3\}.$$

Find numerical parameter values $(G_{\text{ZH}}, F_{\text{ZH}})$ at which (1) exhibits a *fold-Hopf bifurcation*, i.e. has an equilibrium with eigenvalues $\lambda_1 = 0, \lambda_{2,3} = \pm i\omega_0, \omega_0 > 0$. Find numerical parameter values $(G_{\text{CP}}, F_{\text{CP}})$ at which (1) exhibits a *cusplike bifurcation*, i.e. has an equilibrium with a triple zero.

- Derive analytical expressions for $(G_{\text{ZH}}, F_{\text{ZH}})$ and $(G_{\text{CP}}, F_{\text{CP}})$ as functions of (a, b) and verify the numerical results above.

Hints: At the fold-Hopf both the trace and the determinant of the Jacobian matrix of (1) vanish. To find a triple equilibrium, reduce the equilibrium system for (1) to one cubic equation for the x -coordinate of the equilibrium and look for its triple root.

B. Consider the *adaptive control system of Lur'e type*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\alpha z - \beta y - x + x^2. \end{cases} \quad (2)$$

- Continue the equilibrium $(x, y, z) = (0, 0, 0)$ of (2) with respect to parameter α starting at $\alpha = 2, \beta = 1$ and detect its Hopf bifurcation.
- Compute the Hopf bifurcation curve in the (α, β) -plane and plot it using the visibility limits

alpha	0	1.5
beta	0	2

- Find the period-doubling (PD) of the cycle bifurcating from the Hopf point and continue it in two parameters.
- Find the period-doubling of the *doubled* cycle and continue it in the (α, β) -plane.

C. Consider the following prey-predator system from mathematical ecology by Bazykin and Khibnik [1981]:

$$\begin{cases} \dot{x} &= \frac{x^2(1-x)}{n+x} - xy, \\ \dot{y} &= -y(m-x), \end{cases} \quad (3)$$

where $n > 0, 0 < m < 1$.

- Compare its bifurcation diagrams with respect to parameter m for $n_1 = \frac{1}{4}$ and $n_2 = \frac{1}{16}$ using MATCONT.
- Compute the two-parameter bifurcation diagram in the (m, n) -plane and sketch all qualitatively different phase portraits.

Hint: The diagram should include a Hopf bifurcation curve (H) and a cycle fold curve (LPC) that connects a *generalized Hopf point* (GH) on H with the point $(m, n) = (0, 0)$.

3. Derive an analytic expression for the Hopf bifurcation curve in (3). *Hint:* Consider the orbitally-equivalent polynomial system

$$\begin{cases} \dot{x} &= x^2(1-x) - xy(n+x), \\ \dot{y} &= -y(m-x)(n+x). \end{cases} \quad (4)$$

4. Verify the numerically found **GH**-point by proving that the first Lyapunov coefficient l_1 vanish at $(m_{\text{GH}}, n_{\text{GH}}) = (\frac{1}{4}, \frac{1}{8})$. *Hints:*

- (a) At Hopf parameter values, translate the origin in (4) to the equilibrium and scale the variables to obtain a system in the form

$$\begin{cases} \dot{\xi} &= -\omega\eta + P(\xi, \eta), \\ \dot{\eta} &= \omega\xi + Q(\xi, \eta), \end{cases}$$

where P, Q contain terms of order two and higher in (ξ, η) .

- (b) Introduce complex variables $z = \xi + i\eta$ and $\bar{z} = \xi - i\eta$, and derive the equation

$$\dot{z} = i\omega z + \sum_{2 \leq j+k \leq 3} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4).$$

- (c) Compute

$$l_1 = \frac{1}{2\omega^2} \operatorname{Re}(ig_{20}g_{11} + \omega g_{21}).$$