Lecture 10

Numerical periodic normalization for codim 1 bifurcations of limit cycles

Yu.A. Kuznetsov (Utrecht University, NL)

April 9, 2014

Contents

1. Codim 1 bifurcations of limit cycles.

2. Periodic normal forms.

3. Critical normal form coefficients.

4. Remarks

1. Codim 1 bifurcations of limit cycles

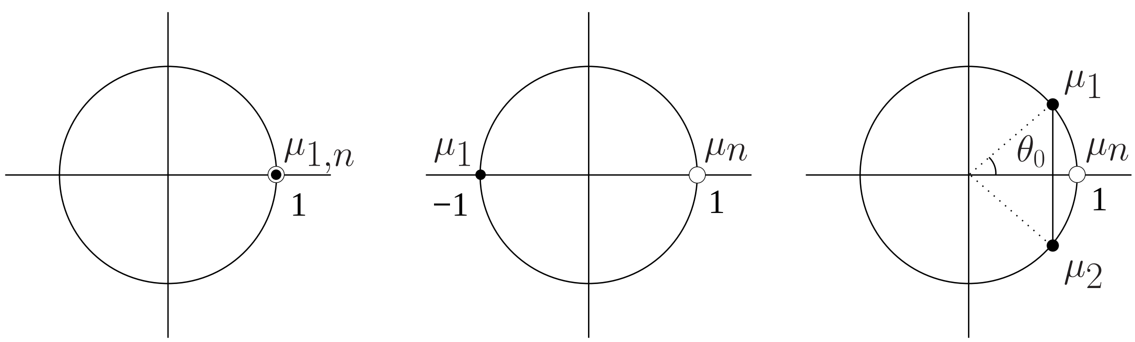
- Consider

$$\dot{x} = f(x, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

A **limit cycle** C_0 corresponds to a periodic solution $x_0(t+T_0) = x_0(t)$ and has **Floquet multipliers** $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 1$, the eigenvalues of $M(T_0)$:

$$\dot{M}(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \quad M(0) = I_n.$$

- Critical cases:



  - **Fold (LPC)**: $\mu_1 = 1$;

  - **Flip (PD)**: $\mu_1 = -1$;

  - **Torus (NS)**: $\mu_{1,2} = e^{\pm i\theta_0}, 0 < \theta_0 < \pi,$

$$\theta_0 \neq \frac{\pi}{2} \quad \text{and} \quad \theta_0 \neq \frac{2\pi}{3};$$

2. Periodic normal forms

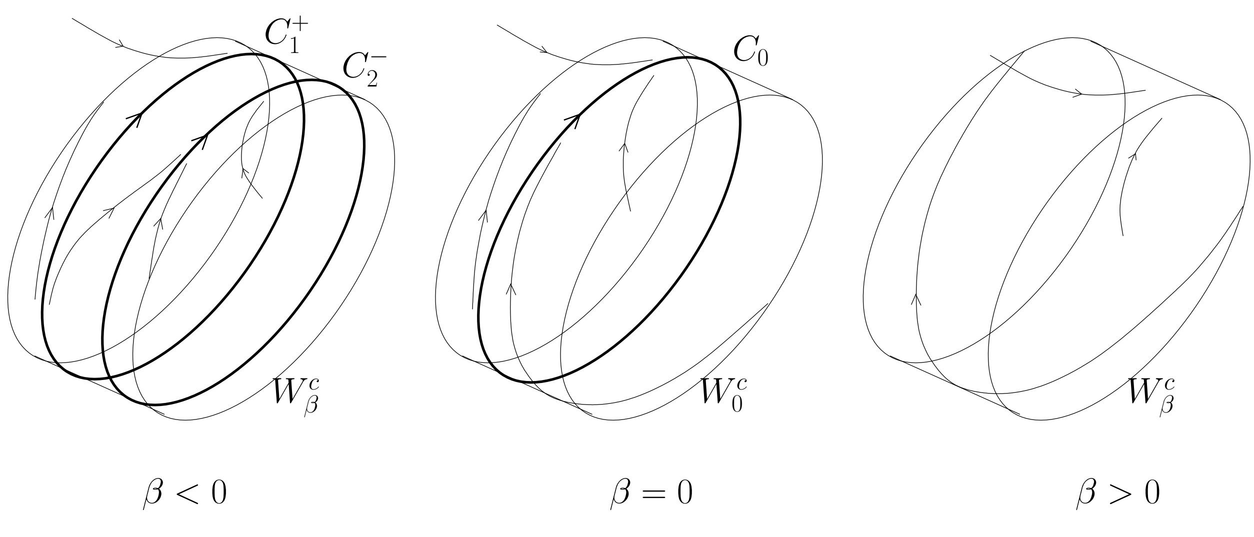
LPC bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1 + \nu(\beta,\xi) - \xi + a(\beta)\xi^2 + \mathcal{O}(\xi^3), \\ \dfrac{d\xi}{dt} &= \beta + b(\beta)\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

where $\nu(\beta,\xi) = \nu_0(\beta) + \nu_1(\beta)\xi,\ \nu_{0,1}(0) = 0$, $a, b \in \mathbb{R}$ with $b(0) \neq 0$, and the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

- Phase portraits ($b(0) > 0$):



Collision and disappearance of two limit cycles: $C_1^- + C_2^+ \to \emptyset$

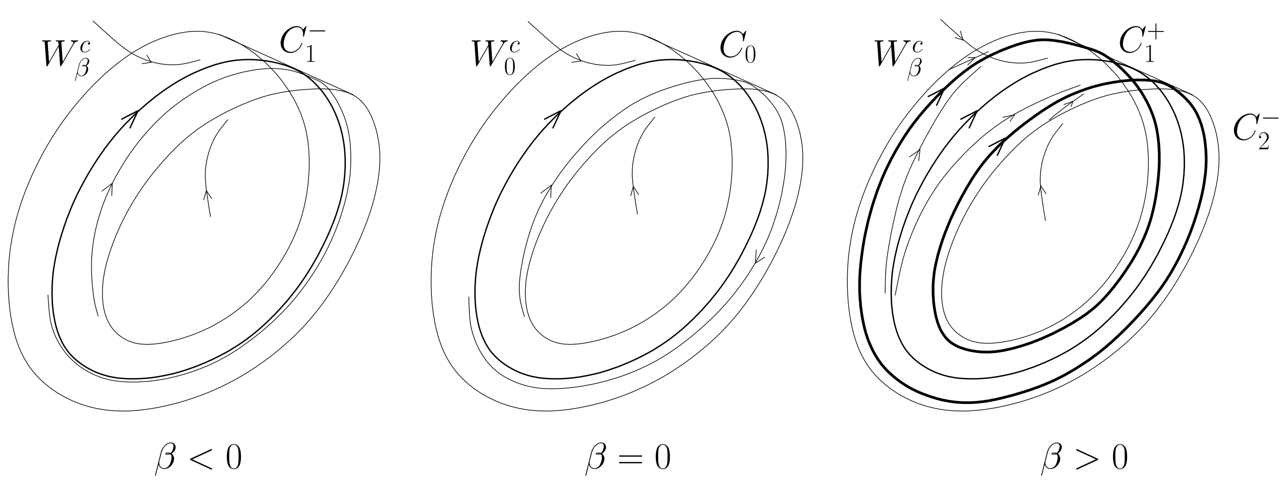
PD bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\beta,\xi) + a(\beta)\xi^2 + \mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} = \beta\xi + c(\beta)\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $\nu(\beta,\xi) = \nu_0(\beta) + \nu_1(\beta)\xi,\ \nu_{0,1}(0) = 0$, $a, c \in \mathbb{R}$, and the $\mathcal{O}(\xi^3)$-terms are $2T_0$-periodic in τ.

- Phase portraits ($c(0) < 0$):



Period-doubling: $C^-_1 \to C^+_1 + C^-_2$

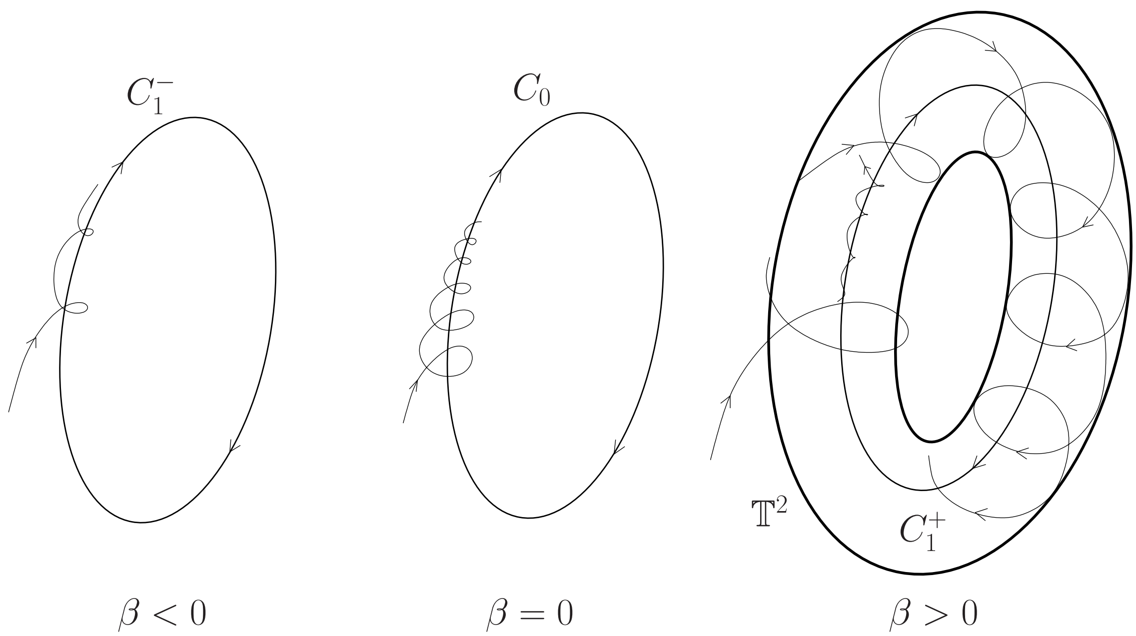
NS bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1 + \nu(\beta,\xi,\bar{\xi}) + a(\beta)|\xi|^2 + \mathcal{O}(|\xi|^3), \\ \dfrac{d\xi}{dt} &= \left(\beta + \dfrac{i\theta(\beta)}{T(\beta)}\right)\xi + d(\beta)\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{cases}$$

where $\nu(\beta,\xi,\bar{\xi}) = \nu_0(\beta) + \nu_1(\beta)\xi + \overline{\nu}_1(\beta)\bar{\xi} + \nu_2(\beta)\xi^2 + \overline{\nu}_2(\beta)\bar{\xi}^2$ with $\nu_j(0) = 0$, $a \in \mathbb{R}$, $d \in \mathbb{C}$, and the $\mathcal{O}(|\xi|^4)$-terms are T_0-periodic in τ.

- Phase portraits ($\Re(d(0)) < 0$):



Torus generation: $C_1^- \to C_1^+ + \mathbb{T}^2$

3. Critical normal form coefficients

- **Fredholm technique for BVPs**

  Assume $\varphi, \varphi^* \in \mathcal{C}^1([0,T_0],\mathbb{R}^n)$ satisfy

$$\left\{ \begin{array}{rcl} \dot{\varphi}(\tau) - A(\tau)\varphi(\tau) & = & 0,\ \tau \in [0,T_0], \\ \varphi(0) - \varphi(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi(\tau), \varphi(\tau) \rangle d\tau - 1 & = & 0, \end{array} \right.$$

  and

$$\left\{ \begin{array}{rcl} \dot{\varphi}^*(\tau) + A^{\mathsf{T}}(\tau)\varphi^*(\tau) & = & 0,\ \tau \in [0,T_0], \\ \varphi^*(0) - \varphi^*(T_0) & = & 0. \end{array} \right.$$

  If $h \in \mathcal{C}^1([0,T_0],\mathbb{R}^n)$ is a solution to

$$\left\{ \begin{array}{rcl} \dot{h}(\tau) - A(\tau)h(\tau) & = & g(\tau),\ \tau \in [0,T_0], \\ h(0) - h(T_0) & = & 0, \end{array} \right.$$

  with $g \in \mathcal{C}^0([0,T_0],\mathbb{R}^n)$, then

$$\int_0^{T_0} \langle \varphi^*(\tau), g(\tau) \rangle \ d\tau = 0$$

  (**Fredholm solvability condition**). When it holds, there is a unique solution h satisfying

$$\int_0^{T_0} \langle \varphi^*(\tau), h(\tau) \rangle \ d\tau = 0.$$

- **Multilinear forms**

At a codimension-one point write

$$\begin{aligned} f(x_0(t)+v,\alpha_0) &= f(x_0(t),\alpha_0) \\ &+ A(t)v + \frac{1}{2}B(t;v,v) \\ &+ \frac{1}{6}C(t;v,v,v) + \mathcal{O}(\|v\|^4), \end{aligned}$$

where $A(t) = f_x(x_0(t),\alpha_0)$ and the components of the multilinear functions B and C are given by

$$B_i(t;u,v) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 f_i(x,\alpha_0)}{\partial x_j \partial x_k}\right|_{x=x_0(t)} u_j v_k$$

and

$$C_i(t;u,v,w) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 f_i(x,\alpha_0)}{\partial x_j \partial x_k \partial x_l}\right|_{x=x_0(t)} u_j v_k w_l,$$

for $i = 1,2,\ldots,n$.

These are T_0-periodic in t.

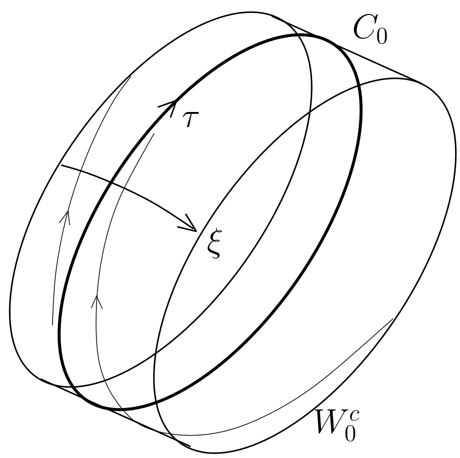
Fold (LPC): $\mu_1 = \mu_n = 1$ (double non-semisimple)

- Critical center manifold W_0^c:

$$x = x_0(\tau) + \xi v(\tau) + H(\tau, \xi),$$

where $\tau \in [0, T_0], \xi \in \mathbb{R}, H(T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2} h_2(\tau)\xi^2 + \mathcal{O}(\xi^3)$$



- Critical periodic normal form on W_0^c:

$$\begin{cases} \frac{d\tau}{dt} = 1 - \xi + a\xi^2 + \mathcal{O}(\xi^3), \\ \frac{d\xi}{dt} = b\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$, while the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

LPC: Generalized and adjoint eigenfunctions

$$\left\{ \begin{array}{rcl} \dot{v}(\tau) - A(\tau)v(\tau) - f(x_0(\tau), \alpha_0) & = & 0, \ \ \tau \in [0, T_0], \\ v(0) - v(T_0) & = & 0, \\ \int_0^{T_0} \langle v(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau & = & 0, \end{array} \right.$$

implying

$$\int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0) \rangle \, d\tau = 0,$$

where φ^* satisfies

$$\left\{ \begin{array}{rcl} \dot{\varphi}^*(\tau) + A^{\mathsf{T}}(\tau)\varphi^*(\tau) & = & 0, \ \ \tau \in [0, T_0], \\ \varphi^*(0) - \varphi^*(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), v(\tau) \rangle d\tau - 1 & = & 0. \end{array} \right.$$

LPC: Computation of b

- Substitute into

$$\frac{dx}{dt} = \frac{\partial x}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial x}{\partial \tau}\frac{d\tau}{dt}$$

- Collect

$$\begin{aligned}
\xi^0 &: \ \dot{x}_0 = f(x_0, \alpha_0), \\
\xi^1 &: \ \dot{v} - A(\tau)v = \dot{x}_0, \\
\xi^2 &: \ \dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) - 2af(x_0, \alpha_0) + \\
&\qquad\qquad\qquad\qquad 2\dot{v} - 2bv.
\end{aligned}$$

- Fredholm solvability condition

$$b = \frac{1}{2}\int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) + 2A(\tau)v(\tau) \rangle d\tau$$

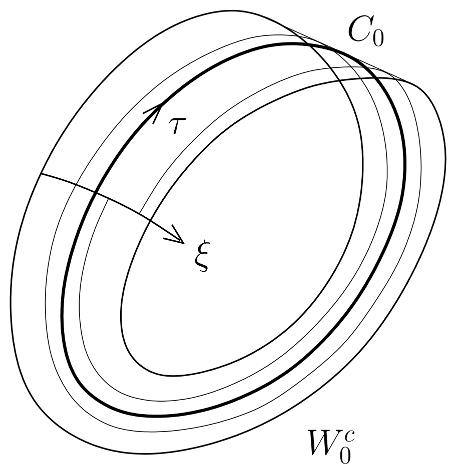
Flip (PD): $\mu_1 = -1, \mu_n = 1$ (both simple)

- Critical center manifold W_0^c:

$$x = x_0(\tau) + \xi w(\tau) + H(\tau, \xi),$$

where $\tau \in [0, 2T_0], \xi \in \mathbb{R}, H(2T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2} h_2(\tau)\xi^2 + \frac{1}{6} h_3(\tau)\xi^3 + \mathcal{O}(\xi^4)$$



- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1 + a\xi^2 + \mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} &= c\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$, while the $\mathcal{O}(\xi^4)$-terms are $2T_0$-periodic in τ.

PD: Eigenfunctions

$$
\begin{aligned}
w(\tau) &= \begin{cases} v(\tau), & \tau \in [0, T_0], \\ -v(\tau - T_0), & \tau \in [T_0, 2T_0], \end{cases}, \\
w^*(\tau) &= \begin{cases} v^*(\tau), & \tau \in [0, T_0], \\ -v^*(\tau - T_0), & \tau \in [T_0, 2T_0], \end{cases}
\end{aligned}
$$

with

$$
\begin{cases}
\dot{v}(\tau) - A(\tau)v(\tau) = 0, & \tau \in [0, T_0], \\
v(0) + v(T_0) = 0, & \\
\int_0^{T_0} \langle v(\tau), v(\tau) \rangle d\tau - 1 = 0, &
\end{cases}
$$

$$
\begin{cases}
\dot{v}^*(\tau) + A^{\mathsf{T}}(\tau)v^*(\tau) = 0, & \tau \in [0, T_0], \\
v^*(0) + v^*(T_0) = 0, & \\
\int_0^{T_0} \langle v^*(\tau), v(\tau) \rangle d\tau - 1/2 = 0. &
\end{cases}
$$

PD: Quadratic terms

$$\xi^2 \; : \; \dot{h}_2 - A(\tau)h_2 = B(\tau; w, w) - 2a\dot{x}_0, \quad \tau \in [0, 2T_0].$$

Since $\mathcal{N}\left(\frac{d}{d\tau} - A(\tau)\right) = \text{span}\{w, \psi = \dot{x}_0\}$, we must have

$$\begin{cases} \int_0^{2T_0} \langle w^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau)\rangle \, d\tau &= 0, \\ \int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau)\rangle \, d\tau &= 0, \end{cases}$$

where ψ^* satisfies

$$\begin{cases} \dot{\psi}^*(\tau) + A^{\mathsf{T}}(\tau)\psi^*(\tau) &= 0, \; \tau \in [0, T_0], \\ \psi^*(0) - \psi^*(T_0) &= 0, \\ \int_0^{T_0} \langle \psi^*(\tau), f(x_0(\tau), \alpha_0)\rangle \, d\tau - 1/2 &= 0, \end{cases}$$

and is extended to $[T_0, 2T_0]$ by periodicity.

PD: Computation of a and h_2

- The first Fredholm condition holds identically for all a, while the second gives

$$\begin{aligned} a &= \frac{1}{2}\int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)\rangle d\tau \\ &= \int_0^{T_0} \langle \psi^*(\tau), B(\tau; v(\tau), v(\tau)\rangle d\tau. \end{aligned}$$

- Define h_2 on $[0, T_0]$ as the unique solution to

$$\left\{ \begin{array}{rcl} \dot{h}_2(\tau) - A(\tau)h_2(\tau) & & \\ -B(\tau; v(\tau), v(\tau)) + 2af(x_0(\tau), \alpha_0) & = & 0, \\ h_2(0) - h_2(T_0) & = & 0, \\ \int_0^{T_0} \langle \psi^*(\tau), h_2(\tau) \rangle \, d\tau & = & 0, \end{array} \right.$$

and extend it by periodicity to $[T_0, 2T_0]$.

PD: Computation of c

Cubic terms: ξ^3

$$\dot{h}_3 - A(\tau)h_3 = C(\tau; w, w, w) + 3B(\tau; w, h_2) - 6a\dot{w} - 6cw$$

The Fredholm solvability condition implies

$$\begin{aligned} 6c &= \int_0^{2T_0} \langle w^*(\tau), C(\tau; w(\tau), w(\tau), w(\tau)) + \\ &\qquad 3B(\tau; w(\tau), h_2(\tau))\rangle d\tau \\ &- \int_0^{2T_0} \langle w^*(\tau), 6aA(\tau)w(\tau)\rangle d\tau \end{aligned}$$

or

$$\begin{aligned} c &= \frac{1}{3}\int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), v(\tau)) + \\ &\qquad 3B(\tau; v(\tau), h_2(\tau)) - 6aA(\tau)v(\tau)\rangle d\tau \end{aligned}$$

Torus (NS): $\mu_{1,2} = \mathrm{e}^{\pm i\theta_0}, \mu_n = 1$ (all simple)

No **strong resonances**: $\mathrm{e}^{i\nu\theta_0} \neq 1, \nu = 1,2,3,4.$

- Critical center manifold $W_0^c : \tau \in [0, T_0],\ \xi \in \mathbb{C}$

$$x = x_0(\tau) + \xi v(\tau) + \bar{\xi}\bar{v}(\tau) + H(\tau, \xi, \bar{\xi}),$$

where $H(T_0, \xi, \bar{\xi}) = H(0, \xi, \bar{\xi})$,

$$\begin{aligned} H(\tau, \xi, \bar{\xi}) &= \frac{1}{2}h_{20}(\tau)\xi^2 + h_{11}(\tau)\xi\bar{\xi} + \frac{1}{2}h_{02}(\tau)\bar{\xi}^2 \\ &+ \frac{1}{6}h_{30}(\tau)\xi^3 + \frac{1}{2}h_{21}(\tau)\xi^2\bar{\xi} \\ &+ \frac{1}{2}h_{12}(\tau)\xi\bar{\xi}^2 + \frac{1}{6}h_{03}(\tau)\bar{\xi}^3 \\ &+ \mathcal{O}(|\xi|^4). \end{aligned}$$

- Critical periodic normal form on W_0^c:

$$\left\{ \begin{aligned} \frac{d\tau}{dt} &= 1 + a|\xi|^2 + \mathcal{O}(|\xi|^4), \\ \frac{d\xi}{dt} &= \frac{i\theta_0}{T_0}\xi + d\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{aligned} \right.$$

where $a \in \mathbb{R}, d \in \mathbb{C}$, and the $\mathcal{O}(|\xi|^4)$-terms are T_0-periodic in τ.

NS: Complex eigenfunctions

$$\left\{\begin{array}{rcl} \dot{v}(\tau) - A(\tau)v(\tau) + \dfrac{i\theta_0}{T_0}v(\tau) & = & 0, \ \ \tau \in [0, T_0], \\ v(0) - v(T_0) & = & 0, \\ \int_0^{T_0} \langle v(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

and

$$\left\{\begin{array}{rcl} \dot{v}^*(\tau) + A^{\mathsf{T}}(\tau)v^*(\tau) + \dfrac{i\theta_0}{T_0}v^*(\tau) & = & 0, \ \ \tau \in [0, T_0], \\ v^*(0) - v^*(T_0) & = & 0, \\ \int_0^{T_0} \langle v^*(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

NS: Quadratic terms

- $\xi^2\bar{\xi}^0$:

$$\dot{h}_{20} - A(\tau)h_{20} + \frac{2i\theta_0}{T_0}h_{20} = B(\tau; v, v)$$

Since $e^{2i\theta_0}$ is not a multiplier of the critical cycle, the BVP

$$\left\{ \begin{array}{rcl} \dot{h}_{20} - A(\tau)h_{20} & & \\ +\frac{2i\theta_0}{T_0}h_{20} - B(\tau; v(\tau), v(\tau)) & = & 0, \\ h_{20}(0) - h_{20}(T_0) & = & 0. \end{array} \right.$$

has a unique solution on $[0, T_0]$.

- $|\xi|^2$:

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - a\dot{x}_0$$

Here

$$\mathcal{N}\left(\frac{d}{d\tau} - A(\tau)\right) = \text{span}(\varphi = \dot{x}_0).$$

NS: Computation of a and h_{11}

- Define φ^* as the unique solution of

$$\begin{cases} \dot{\varphi}^*(\tau) + A^{\mathsf{T}}(\tau)\varphi^*(\tau) & = & 0, \\ \varphi^*(0) - \varphi^*(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau - 1 & = & 0. \end{cases}$$

- Fredholm solvability:

$$a = \int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), \bar{v}(\tau)) \rangle \ d\tau$$

- Then find h_{11} on $[0, T_0]$ from the BVP

$$\begin{cases} \dot{h}_{11}(\tau) - A(\tau)h_{11}(\tau) & & \\ -B(\tau; v(\tau), \bar{v}(\tau)) + af(x_0(\tau), \alpha_0) & = & 0, \\ h_{11}(0) - h_{11}(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), h_{11}(\tau) \rangle \ d\tau & = & 0. \end{cases}$$

NS: Computation of d

- Cubic terms: $\xi^2\bar{\xi}$

$$\begin{aligned} \dot{h}_{21} - Ah_{21} + \frac{i\theta_0}{T_0}h_{21} &= 2B(\tau; h_{11}, v) \\ &+ B(\tau; h_{20}, \bar{v}) \\ &+ C(\tau; v, v, \bar{v}) \\ &- 2a\dot{v} - 2dv. \end{aligned}$$

- Fredholm solvability condition:

$$\begin{aligned} d &= \frac{1}{2}\int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), \bar{v}(\tau))\rangle \, d\tau \\ &+ \frac{1}{2}\int_0^{T_0} \langle v^*(\tau), 2B(\tau; h_{11}(\tau), v(\tau)) + \\ &\qquad B(\tau; h_{20}(\tau), \bar{v}(\tau))\rangle \, d\tau \\ &- a\int_0^{T_0} \langle v^*(\tau), A(\tau)v(\tau)\rangle \, d\tau + \frac{ia\theta_0}{T_0}. \end{aligned}$$

4. Remarks

- Only the derivatives of $f(x, \alpha_0)$ are used, not those of the Poincaré map.

- Detection of codim 2 points is easy.

- After discretization via orthogonal collocation, all linear BVPs involved have the standard sparsity structure.

- One can re-use solutions to linear BVPs appearing in the continuation to compute the normal form coefficients.

- Actually implemented in MATCONT by scaling to the interval $[0, 1]$.