

# Codimension 2 bifurcations of periodic orbits in MatCont

Fabio Della Rossa, Virginie De Witte, Willy Govaerts and Yuri A. Kuznetsov

In this tutorial we show how to construct the bifurcation diagram in Figure 1, originally proposed in [1]. First recall the system:

$$\begin{cases} \dot{x} = r \left(1 - \frac{x}{K}\right) x - p(x, t)y, \\ \dot{y} = ep(x, t)y - dy, \end{cases} \quad (1)$$

where  $x$  and  $y$  are the numbers of individuals, respectively of prey and predator populations or suitable (but equivalent) measures of density or biomass. The parameters present in system (1) are the intrinsic growth rate  $r$ , the carrying capacity  $K$ , the efficiency  $e$  and the death rate  $d$  of the predator. The function  $p(x, t)$  is a functional response, for which the Holling type II is chosen, with constant attack rate  $a$  and half saturation  $b(t)$  that varies periodically with period one (year), i.e.

$$p(x, t) = \frac{ax}{b(t) + x}, \quad b(t) = b_0(1 + \varepsilon \cos 2\pi t).$$

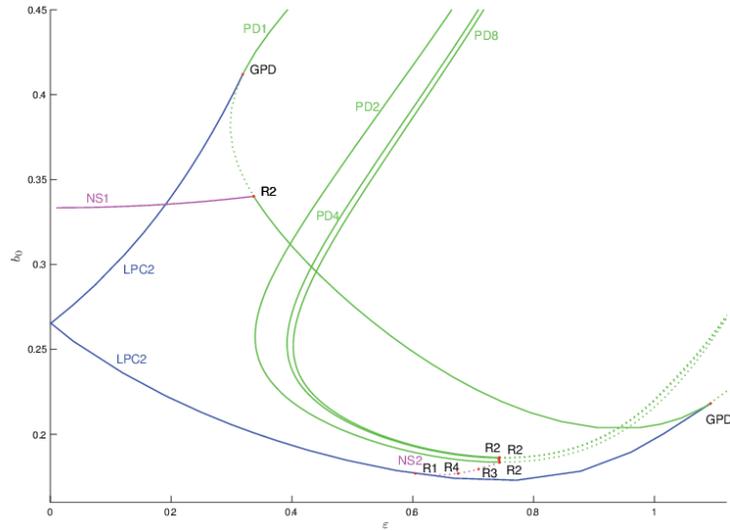


Figure 1: Bifurcation diagram of limit cycles in model (2). Blue are limit point of cycles bifurcations, green period doubling bifurcations and magenta Neimark-Sacker bifurcations. Continuous/dotted curves correspond to supercritical/subcritical bifurcations, respectively.

Instead of system (1), we consider the extended autonomous system

$$\begin{cases} \dot{x} = r \left(1 - \frac{x}{K}\right) x - \frac{axy}{b_0(1 + \epsilon u) + x}, \\ \dot{y} = e \frac{axy}{b_0(1 + \epsilon u) + x} - dy, \\ \dot{u} = u - 2\pi v - (u^2 + v^2)u, \\ \dot{v} = 2\pi u + v - (u^2 + v^2)v, \end{cases} \quad (2)$$

where the last two equations are decoupled from the first two and have a stable limit cycle with  $u(t) = \cos(2\pi t + \varphi)$ ,  $v(t) = \sin(2\pi t + \varphi)$ , and a phase shift  $\varphi$  depending on the initial conditions.

As a preliminary step, install Matlab on your computer. Then download MatCont 6.10 or higher version (see <http://sourceforge.net/projects/matcont/files/matcont>), unzip it and move the current Matlab folder to the one which contains the file `matcont.m`. Now start MatCont by typing `matcont` in the Matlab Command Window. (warning: depending on the operating system it can be necessary to first build the MEX-files from the c-files in the directory `Limitcycle`; see the documentation and readme - files).

## 1 Introduce the system in MatCont

First of all we have to create a new system. This can be done through the menu `Select -> System -> New`. We insert the system in the `System` window, and obtain the result shown in Figure 2. Note that the semicolons are optional. When pressing the `OK` button, the model is pre-computed (the selected derivatives

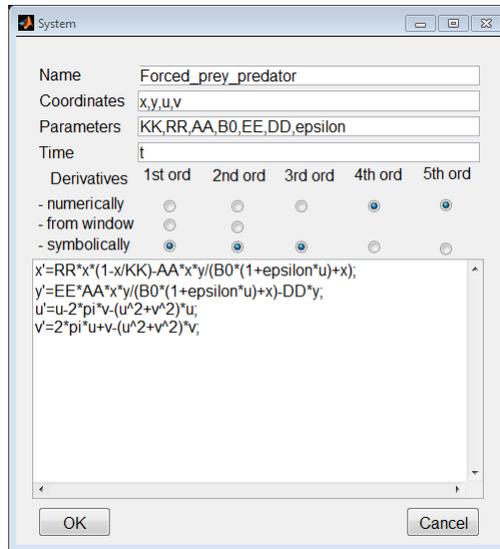


Figure 2: The `System` window for (2).

are computed symbolically), and we return to the MatCont principal window.

## 2 Find an initial solution

As a first step we have to find a periodic solution to which our continuation analysis can be applied. Therefore, we start a trajectory from a random point of our system, hoping to converge to a stable limit cycle.

When clicking the menu `Type -> Initial point -> Point` and `Type -> Curve -> Orbit`, two windows are opened, the `Starter` and the `Integrator` window. Put all the initial state and parameter values in the `Starter` window (fix the value of the state variables to  $x = y = 0.1$ , while  $u = 1$  and  $v = 0$  in order to have  $\varphi = 0$ , and the parameter values to  $K = 1, r = 6.28, a = 12.56, b_0 = 0.25, e = 1, d = 6.28, \varepsilon = 0.2$ ). Put 100 as `Interval` in the `Integrator` window. Figure 3 is obtained as a result.

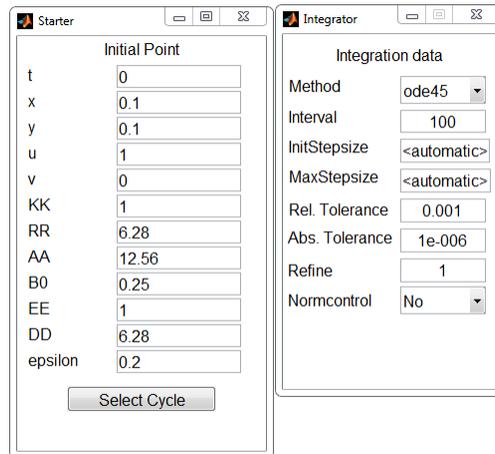


Figure 3: `Starter` and `Integrator` window.

Open a plot window, through the menu `Window -> Graphic -> 2Dplot` which (by default) visualizes the  $(x, x)$  plane. Change this into visualising the  $(x, y)$ – plane by the command `Layout -> Variables on axes`. By computing the trajectory making use of time-integration, through the menu `Compute -> Forward`, we obtain the result shown in Figure 4. The figure shows that the system has converged to a stable limit

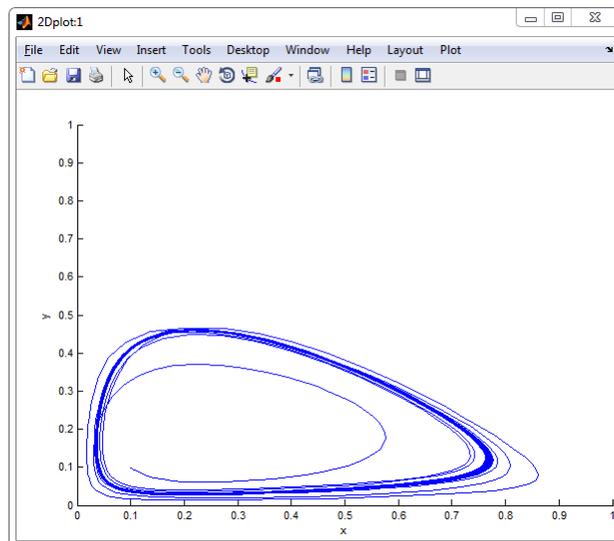


Figure 4: A trajectory of system (2).

cycle. We now want to select that invariant cycle in order to start a continuation. First we select a point on the limitcycle, through the `MatCont` menu `Select -> Initial Point` and then selecting the last point of the computed trajectory. Then we clear the plot, through the `Plot -> Clear` menu in the `2Dplot:1`

window. We want to do an integration over a small interval, say 1, by changing the **Interval** field in the **Integrator** window, and compute a new trajectory through the **Compute -> Forward** menu in the **MatCont** window. In order to obtain the limit cycle, the trajectory should be extended once, through the **Compute -> Extend** menu in the **MatCont** window. We can then use the **Select Cycle** button in the **Starter** window, in order to select the obtained periodic solution. After having done that, using the default tolerance and the number of discretization intervals, the **Curve** field in the **MatCont** window automatically changes in a limit cycle continuation (namely **LC.LC(1)**), and a new **Starter** window and the **Continuer** windows are opened; it is now possible to do a limit cycle continuation.

### 3 Continue a periodic solution

Select the parameter  $b_0$  and the period of the limit cycle as free in the **Starter** window (note **Period** is approximately equal to 2). Turn on the detection of all singularities except **BPC (Branching)** and change the **MaxStepsize** in the **Continuer** window to 0.5 and **MaxNumPoints** to 30. The obtained windows are displayed in Figure 5. Start the continuation as usual with **Compute -> Forward** in the **MatCont** window.

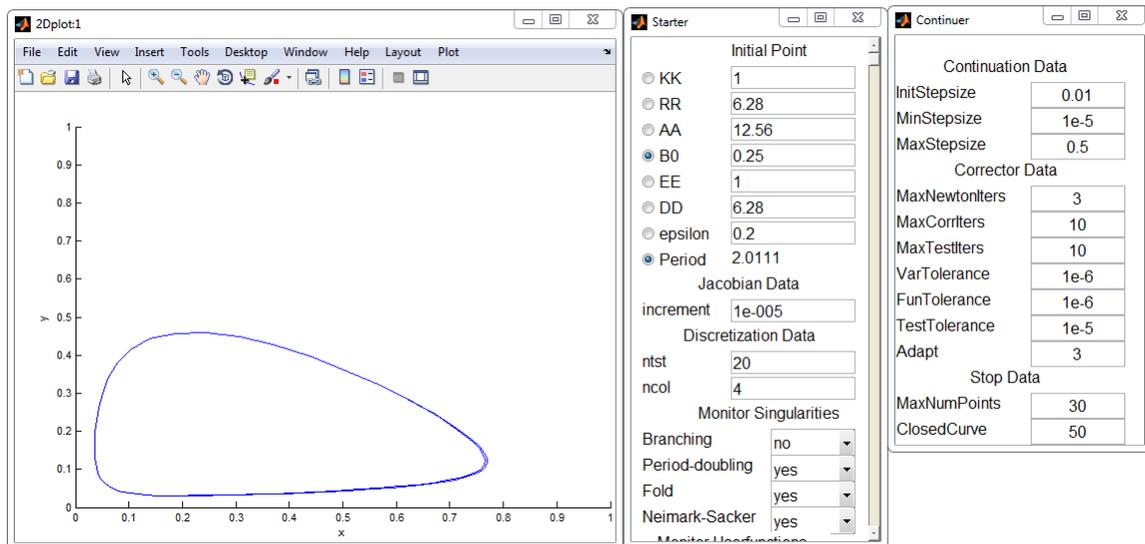


Figure 5: Ready for limit cycle continuation.

We detect the following bifurcation

```
Limit point cycle (period = 2.000000e+00, parameter = 3.399917e-01)
Normal form coefficient = -1.453699e+09
```

This is the first point that we will later use to compute the bifurcation diagram. We note that (in general) the normal form coefficients depend very much on the details of the computations and can be reliably used only if the bifurcation points are computed to high accuracy. Resume the computation; we detect another bifurcation

```
Limit point cycle (period = 2.000000e+00, parameter = 2.213010e-01)
Normal form coefficient = -5.354929e+08
```

Stop the computations and save this curve as **LC1with2LPC** through the **Select -> Curve** menu in the **MatCont** window and the **Actions -> Rename** menu in the opened **Curves** window.

Change the variables on the axes of the 2Dplot:1 window through the menu **Layout -> Variables on axes** and select the two system parameters  $\varepsilon$  and  $b_0$  as plotting variables. Change the scale of the axes through the menu **Layout -> Plotting region** into  $\varepsilon \in [0, 1.1]$  and  $b_0 \in [0.16, 0.45]$ . Select parameter  $\varepsilon$  as free in the **Starter** window, while keeping the period active. Turn on the detection of all singularities as shown in Figure 6, and start the continuation with **Compute -> Backward** in the MatCont window. After some continuation steps we detect another bifurcation

Period Doubling (period = 2.000000e+00, parameter = 3.408389e-01)  
 Normal form coefficient = -2.786829e-01

Depending on details of the computation, it can be necessary to compute backward instead of forward to find this point. Stop and save this curve as **LC2with1PD** through the **Select -> Curve** menu in the MatCont window and the **Actions -> Rename** menu in the opened **Curves** window.

If we now redraw the bifurcation diagram (**Plot -> Redraw diagram** in the 2Dplot:1 window), we obtain the lines showed in Figure 6.

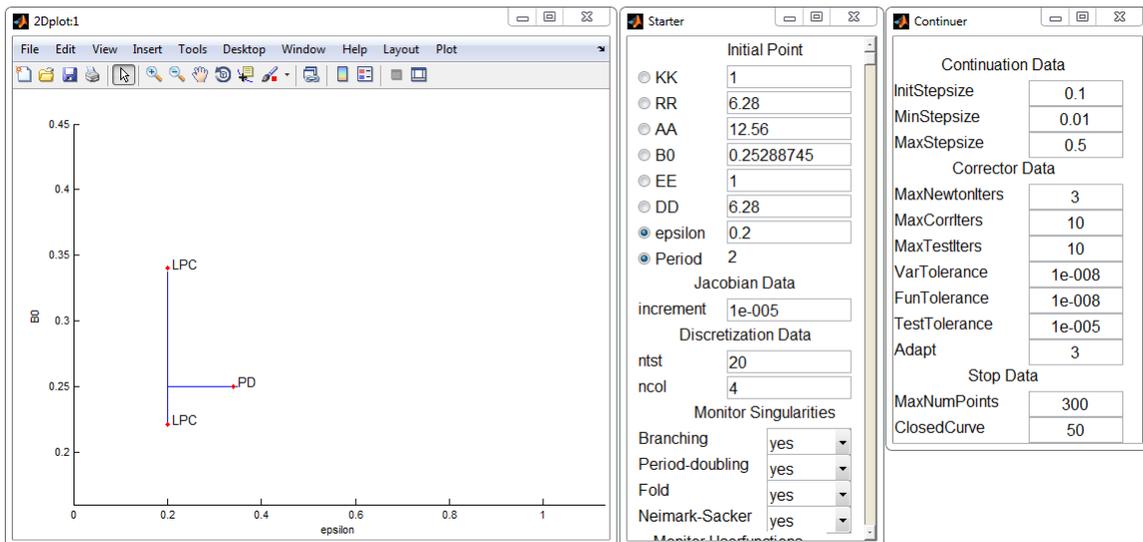


Figure 6: Ready for bifurcation continuation.

## 4 Compute the bifurcation diagram

Select the first Limit point cycle (LPC) bifurcation point in the limit cycle curve **LC1with2LPC** as a starting point of the continuation, through the **Select -> Initial point** menu in the MatCont window. The **Curve** field in the MatCont window automatically changes in a LPC continuation (curve named **LPC\_LPC(1)**). Select the parameters  $b_0$  and  $\varepsilon$  as free (active) in the **Starter** window, turn the monitoring of all singularities on and compute the bifurcation curve both forward and backward until it leaves the 2DPlot window or the computations stop. One of these continuations may lead to a "stepsize too small" error message. Save the two obtained **LPC\_LPC** curves as **LPC1for** and **LPC1back**, respectively (**Select -> Curve** and then **Actions -> Rename**).

Select the second Limit point cycle bifurcation point of the limit cycle curve **LC1with2LPC** as a starting point of our continuation, through the **Select -> Initial point** menu in the MatCont window. Select parameters  $b_0$  and  $\varepsilon$  as free (active) in the **Starter** window, turn the monitoring of all singularities on, and compute the bifurcation curve forward and backward until it leaves the 2DPlot window or the computations

stop. One of these computations may lead to a "Current step size too small" error message. Save the two obtained curves as LPC2for and LPC2back, respectively (Select -> Curve and then Actions -> Rename). Figure 7 shows the obtained LPC curves.

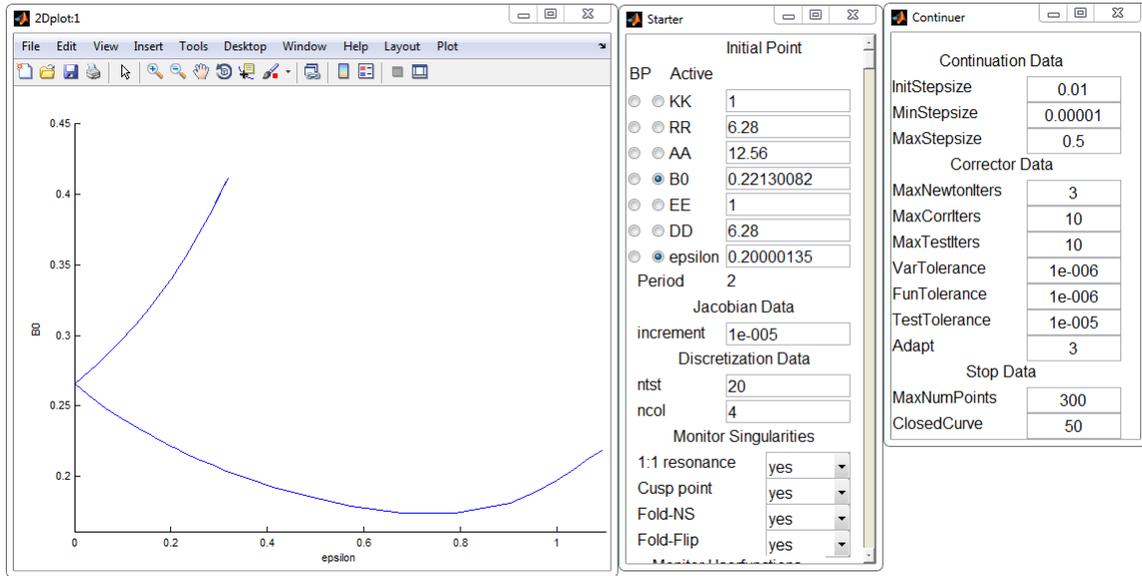


Figure 7: The limit point of cycle curves.

Note that

1. An error message "stepsize too small" pops up during a limit point of cycles continuation, which in this case indicates that a codimension 2 point is encountered. We will see later that there are Generalized Period Doubling (GPD) points (for a limit cycle with period one).
2. The two LPC bifurcation curves meet at  $\varepsilon = 0$ . This is typical for periodically forced systems: that point is the resonant point in which the non-forced system has a Hopf bifurcation.
3. By analyzing the resonance 1:1 test function along the lower Limit point of cycles curve, a change of sign can be seen around  $\varepsilon = 0.6$ . We will return to this later.

A robust way to find the period doubling bifurcation curve PD1 in Figure 8 is to start a P\_0 computation for  $b_0 = 0.4$ ,  $\varepsilon = 0.2$  with initial values  $x = 0.1$ ,  $y = 0.1$ ,  $u = 1$ , and  $v = 0$ . Using **Interval** equal to 100 for integration, we see (eventually in a separate 2Dplot window) that the orbit rapidly converges to a stable limit cycle with period 1. We select a point on the limit cycle, through the MatCont menu **Select -> Initial Point** and then selecting the last point of the computed orbit. Then we do an integration over time interval 1.1, by changing the **Interval** field in the **Integrator** window, and compute a new orbit through the **Compute -> Forward** menu in the MatCont window. We can then use the **Select Cycle** button in the **Starter** window to select the obtained period-1 solution with the default discretization data. After having done that, the **Curve** field in the MatCont window automatically changes in a limit cycle continuation (namely LC\_LC(1)), a new **Starter** window and the **Continuer** window are opened; it is now possible to do a limit cycle continuation.

Select the period of the limit cycle (that now is equal to 1) and  $\varepsilon$  as free in the **Starter** window, turn on the detection of all the singularities except BPC, and continue the limit cycle backward. We then find a period doubling bifurcation of the period-1 cycle for  $\varepsilon \approx 0.3050916$  :

```
Period Doubling (period = 1.000000e+00, parameter = 3.050916e-01)
Normal form coefficient = 6.269269e-01
```

After that the computations can be stopped.

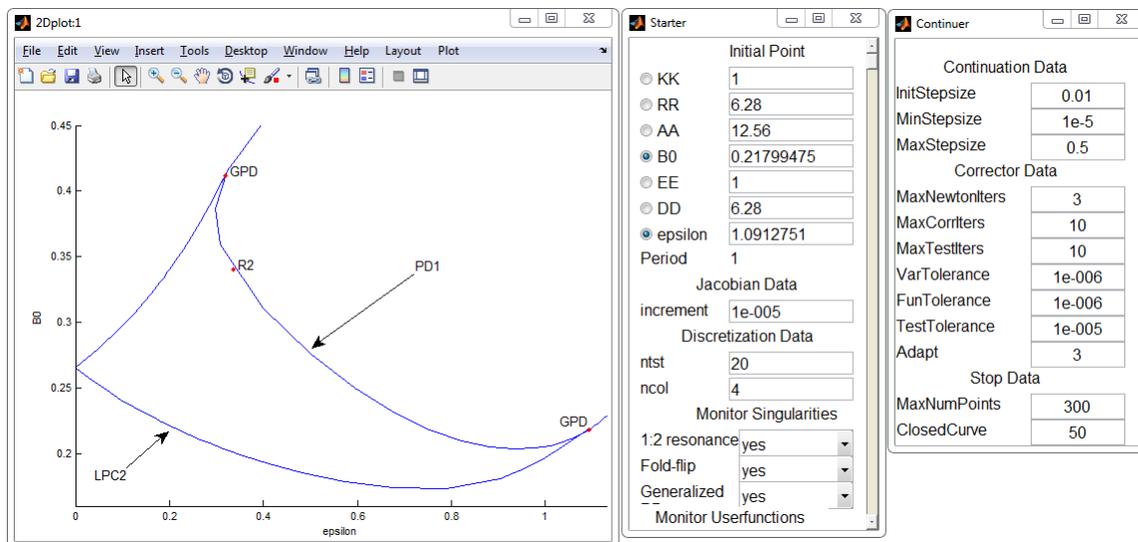


Figure 8: The period-doubling bifurcation curve PD1.

Select (Select -> Initial point) this PD point, and prepare MatCont to continue the period-doubling bifurcation curve (Type -> Curve -> Period-doubling) in two parameters. Set MaxStepsize equal to 0.5 and MaxNumPoints 30 in the Continuer window. Select  $b_0$  and  $\varepsilon$  as free parameters in the Starter window, turn the detection of the singularities on and compute the PD bifurcation curve forward and backward. During the computations we obtain the following output

```
Generalized period doubling(period = 1.000000e+00, parameters = 4.117418e-01, 3.182303e-01)
e = -5.859323e+01
Resonance 1:2 (period = 1.000000e+00, parameters = 3.401299e-01, 3.362219e-01)
(a,b)=(3.400450e+00, -1.292207e+01)
Generalized period doubling(period = 1.000000e+00, parameters = 2.183706e-01, 1.092945e+00)
e = 4.155492e+01
```

The result of the redrawing of the bifurcation curves is presented in Figure 8. Notice that we found the two Generalized Period Doubling (GPD) bifurcation points. Save the two obtained curves (by Select -> Curve and then Actions -> Rename) as PD1with1GPD and PD1with1R21GPD.

Select (with Select -> Initial point) the Resonance 1:2 point on the PD1with1R21GPD curve. Normal form analysis learns us that this point can be used as starting point for a Neimark-Sacker curve of the period-1 limit cycle (Type -> Initial point -> Neimark-Sacker). The Curve type automatically switches to an NS continuation. Select  $b_0$  and  $\varepsilon$  as free parameters, and manually swap their numerical values<sup>1</sup>. Turn on the detection of all singularities and compute (Compute -> Forward) the Neimark-Sacker bifurcation curve until it leaves the window. Save the obtained curve as NS1, the result is shown in Figure 9.

We now come back to the result obtained in the previous section in order to compute a missing part of the bifurcation diagram. Select the PD bifurcation point of the second limit cycle curve, i.e. LC2with1PD, and use it as a starting point for a period doubling bifurcation curve (Type -> Curve -> Period-doubling). Select  $b_0$  and  $\varepsilon$  parameters as free (in the Starter window), turn on the detection of all codimension 2

<sup>1</sup>This is a known bug in MatCont that will be fixed later. You can find the correct numerical values in the MATLAB Command Window.

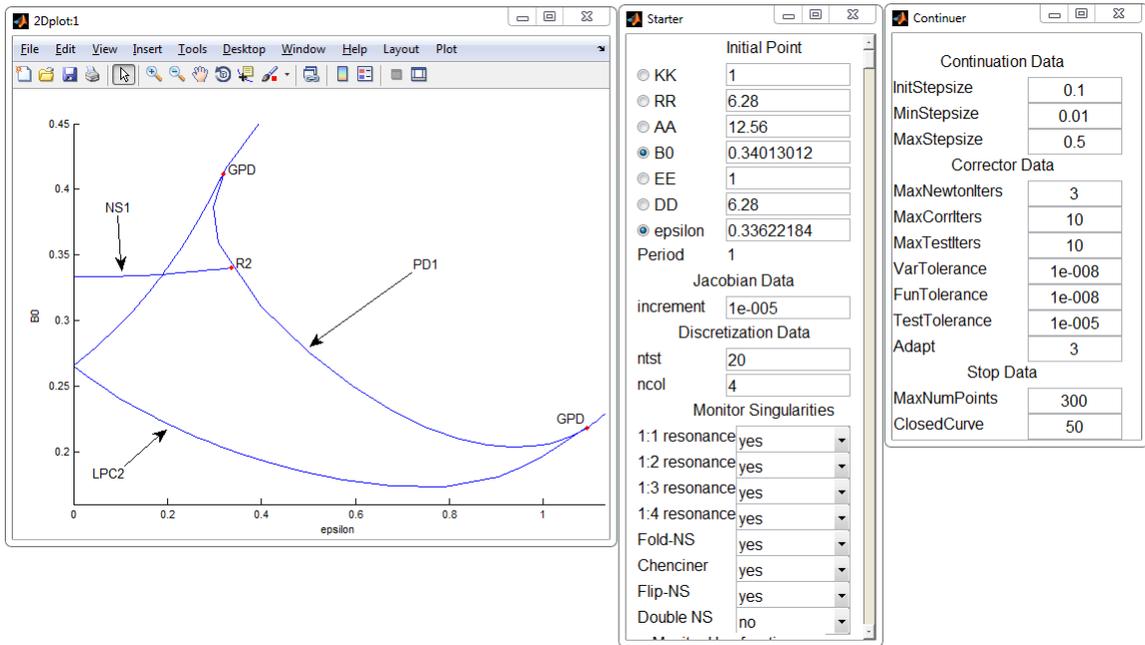


Figure 9: The Neimark-Sacker bifurcation curve NS1 is added.

bifurcations, compute the bifurcation curve forward and backward, and save the obtained curves as PD2for and PD2back (via Select -> Curve and Actions -> Rename). You can now delete all earlier computed

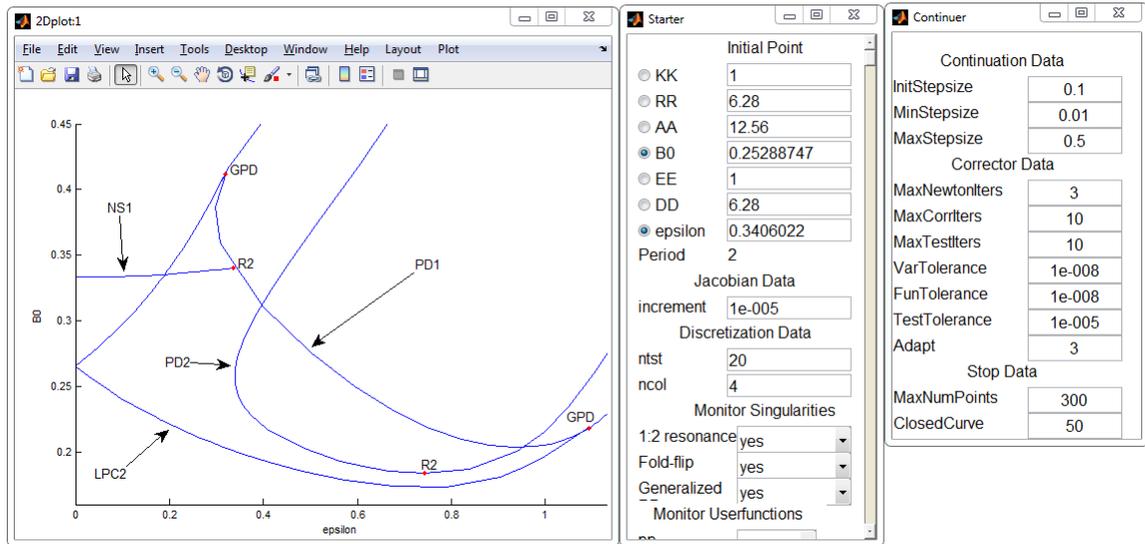


Figure 10: The period doubling curve PD2 of period-2 limit cycles is added.

orbits (P<sub>0</sub>) and limit cycle curves (LC<sub>LC</sub>, LC<sub>1with2LPC</sub>, and LC<sub>2with1PD</sub>) and redraw the diagram to see the result presented in Figure 10.

During the backward continuation, a Resonance 1:2 point is found

Resonance 1:2 (period = 2.000000e+00, parameters = 1.838264e-01, 7.431613e-01)

(a,b)=(-6.555022e+01, 1.623705e+01)

Normal form analysis of this point learns us that it can be used as a starting point of a Neimark-Sacker bifurcation curve of the period 2 limit cycle and of a Neimark-Sacker bifurcation curve of a period 4 limit cycle. We now proceed in the same way as for the last resonance 1:2 point, i.e. select this point as the starting point (Select -> Initial point). Select it to be on the Neimark-Sacker bifurcation curve (Type -> Initial point -> Neimark-Sacker), turn on all singularity detections (in the Starter window), activate the parameters  $b_0$  and  $\varepsilon$  (and swap their numerical values). Perform a forward continuation (Compute -> Forward). You will get the following output in the MATLAB Command window:

```
Resonance 1:3 (period = 2.000000e+00, parameters = 1.798043e-01, 7.084330e-01)
(b,Re(c))=(5.612848e+00 + (2.993580e+00) i,9.133880e+00)
Resonance 1:4 (period = 2.000000e+00, parameters = 1.772744e-01, 6.740365e-01)
(A,d)=(1.032146e-01+ (-7.519306e-01) i, 2.352604e+01+ (-1.098091e+02) i)
Resonance 1:1 (period = 2.000000e+00, parameters = 1.772440e-01, 6.020674e-01)
ab=1.283421e+03
```

Stop after the 1:1 point R1 is located and save the computed NS curve as NS2.

In order to obtain the second Neimark-Sacker curve we have to pass to the period-doubled solution and find the corresponding Neimark-Sacker bifurcation. So, after having selected the Resonance 1:2 point on the period doubling bifurcation curve PD2back (Select -> Initial point), we change the point and curve type (Type -> Initial point -> Period doubling, Type -> Curve -> Limit Cycle). We swap the parameter values of  $b_0$  and  $\varepsilon$ , leave only the parameter  $b_0$  free, and turn on the detection of Neimark-Sacker bifurcations. We start the computation (Compute -> Backward) and after 3 continuation steps we find a point of the Neimark-Sacker curve

```
Neimark-Sacker (period = 4.000002e+00, parameter = 1.838116e-01)
Normal form coefficient = -2.037732e-04
```

That point can be used (Select -> Initial point) as the starting point for the secondary Neimark-Sacker curve rooted at the resonance 1:2 point. Since we want to compute a longer trajectory, we increase the number of mesh intervals, changing the ntest field in the Starter window to 30. Denote  $b_0$  and  $\varepsilon$  as the free parameters, turn the detection of the singularities on and do the continuation (Compute -> Forward). We obtain

```
Resonance 1:4 (period = 4.000000e+000, parameters = 1.850697e-001, 7.426656e-001)
(A,d)=(-1.339790e-002+ (-6.458158e-001) i, -4.166417e+002+ (-4.891725e+002) i)
Resonance 1:3 (period = 4.000000e+000, parameters = 1.857095e-001, 7.424365e-001)
(b,Re(c))=(4.043754e-001 + (1.143509e+001) i,-8.836545e+000)
Resonance 1:2 (period = 4.000000e+000, parameters = 1.863586e-001, 7.422040e-001)
(a,b)=(-2.684811e+002, -1.818726e+001)
```

and therefore find the bifurcation diagram presented in Figure 11.

After having saved the obtained bifurcation curve, we can use the last found resonance 1:2 point as a starting point of the period 4 period doubling bifurcation curve and of the Neimark-Sacker of period 8 limit cycles bifurcation curve. In this tutorial we only discuss the first curve; the procedure to obtain all the other curves is similar to the one described in the previous cases. Select the resonance 1:2 point on the Neimark-Sacker of period 4 limit cycle bifurcation curve (Select -> Initial point). This point is a period-doubling bifurcation point (Type -> Initial point -> Period doubling) and we want to continue a period doubling bifurcation curve (Type -> Curve -> Period-doubling). Select  $b_0$  and  $\varepsilon$  as free parameters in the Starter window and turn the singularities detection on. Compute the bifurcation curve, we obtain

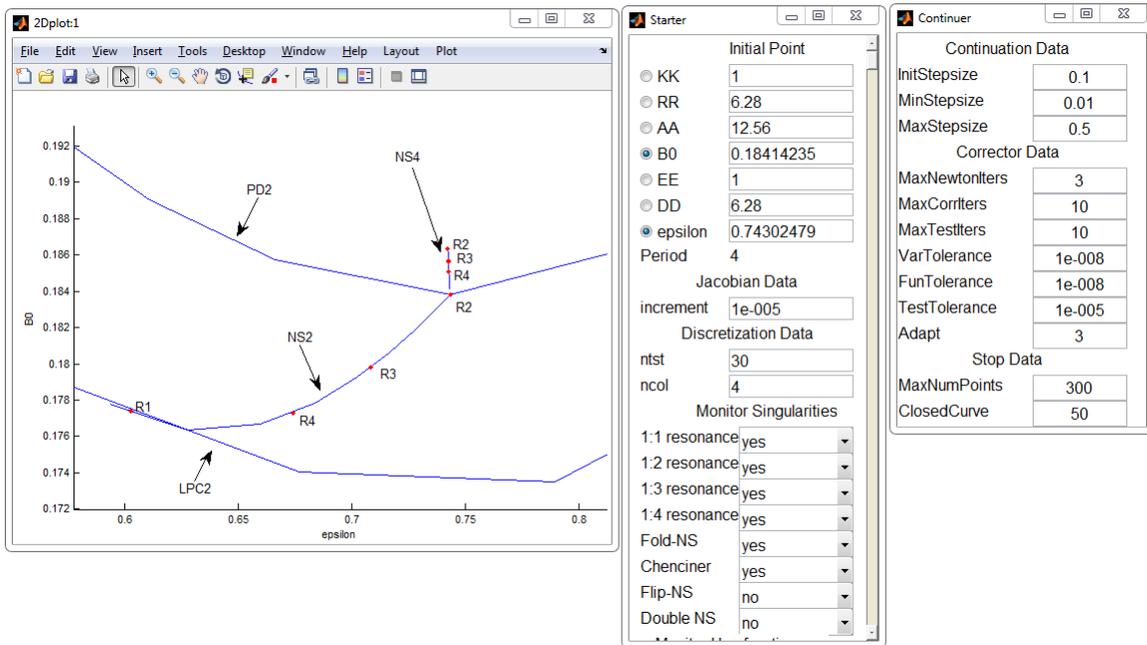


Figure 11: The secondary Neimark-Sacker bifurcation curve NS4 rooted at the resonance 1:2 point on the period doubling bifurcation curve of the period 2 limit cycle.

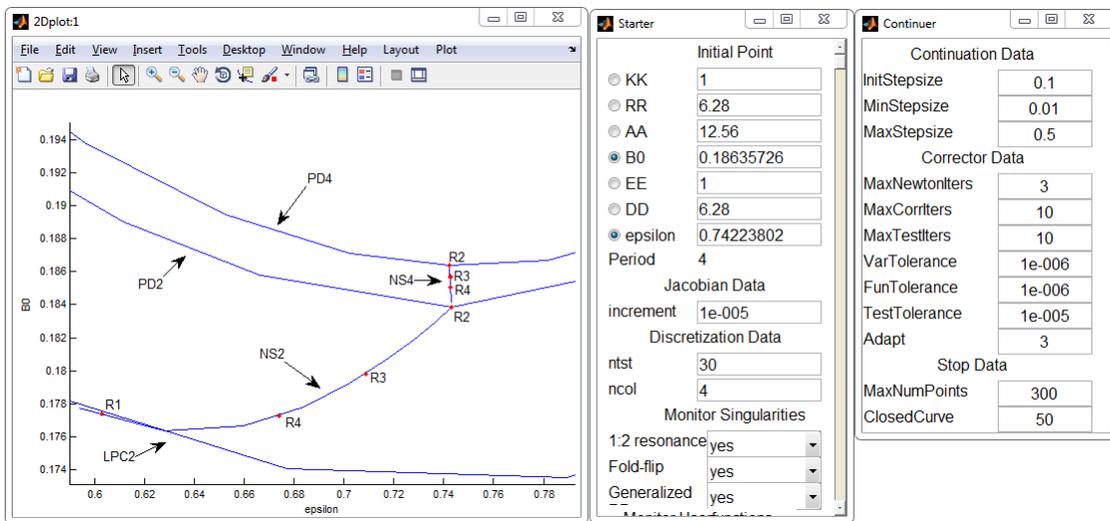


Figure 12: The period doubling bifurcation curve PD4 of the period 4 limit cycles.

1:2 resonance (period = 4.000000e+000, parameters = 1.863585e-001, 7.422109e-001)  
(a,b)=(-9.217016e+002, 1.658100e+001)

and thus the bifurcation diagram shown in Figure 12.

Notice that in order to obtain the bifurcation diagram shown in Figure 1 the bifurcation curves should be computed with higher precision with respect to the ones used in this tutorial: here it has not been done in order to make this tutorial as easy as possible, but it is possible to achieve more accurate bifurcation curves

by simply reducing the `MaxStepsize` parameter in the `Continuer` window.

## References

- [1] YU.A. KUZNETSOV, S. MURATORI, AND S. RINALDI, *Bifurcations and chaos in a periodic predator-prey model*, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 2 (1992), pp. 117–128.