Lecture 3: Branching points

3.1 Definition and properties

A point $p \in M$ is called **singular** for ALCP

$$F(x) = 0, \quad F : \mathbb{R}^{N+1} \to \mathbb{R}^N, \tag{3.1}$$

if rank $F_x(p) < N$. Let $p = 0$ be a singular point and write the Taylor expansion

$$F(x) = Jx + \frac{1}{2}B(x, x) + O(\|x\|^3), \tag{3.2}$$

where $J = F_x(0)$ and $B(x, y) = F_{xx}(0)[x, y]$. Introduce two linear spaces:

$$\begin{aligned} N(J) &:= \{v \in \mathbb{R}^{N+1} : Jv = 0\}, \\ N(J^{\mathrm{T}}) &:= \{w \in \mathbb{R}^N : J^{\mathrm{T}}w = 0\}. \end{aligned}$$

Assume that rank $J = N - 1$. This implies that

$$\dim N(J) = 2 \quad \text{and} \quad \dim N(J^{\mathrm{T}}) = 1,$$

so that

$$\begin{aligned} N(J) &= \mathrm{span}\{q^{(1)}, q^{(2)}\}, \quad q^{(j)} \in \mathbb{R}^{N+1}, j = 1, 2, \\ N(J^{\mathrm{T}}) &= \mathrm{span}\{\varphi\}, \quad \varphi \in \mathbb{R}^N. \end{aligned}$$

This means that any $v \in N(J)$ can be written as

$$v = \beta_1 q^{(1)} + \beta_2 q^{(2)}$$

for some $\beta_j \in \mathbb{R}$, and any $w \in N(J^{\mathrm{T}})$ has the form

$$w = \alpha\varphi$$

for some $\alpha \in \mathbb{R}$. We can assume that

$$\|q^{(1)}\| = \|q^{(2)}\| = 1, \quad \langle q^{(1)}, q^{(2)} \rangle = 0,$$

and

$$\|\varphi\| = 1.$$

Lemma 8 *Any tangent vector $v \in \mathbb{R}^{N+1}$ to M at the singular point $p = 0$ satisfies the equation*

$$\langle \varphi, B(v,v) \rangle = 0. \tag{3.3}$$

Proof:

Consider a solution curve in M parametrized by $x = x(s)$ such that $x(0) = 0$ and $\dot{x}(0) = v$. Then the repeated differentiation w.r.t. s gives

$$\begin{aligned} F(x(s)) &= 0, \\ F_x(x(s))\dot{x}(s) &= 0, \\ F_{xx}(x(s))[\dot{x}(s), \dot{x}(s)] + F_x(x(s))\ddot{x}(s) &= 0. \end{aligned}$$

At $s = 0$, these equations are reduced to

$$\begin{aligned} F(0) &= 0, \\ Jv &= 0, \\ B(v,v) + J\ddot{x}(0) &= 0. \end{aligned}$$

Taking the scalar product of the last equation with φ, we obtain

$$0 = \langle \varphi, B(v,v) + J\ddot{x}(0) \rangle = \langle \varphi, B(v,v) \rangle + \langle J^{\mathrm{T}}\varphi, \ddot{x}(0) \rangle.$$

Since $J^{\mathrm{T}}\varphi = 0$, we get (3.3). □

Substituting $v = \beta_1 q^{(1)} + \beta_2 q^{(2)}$, we obtain the **Algebraic Branching Equation**

$$Q(\beta) := b_{11}\beta_1^2 + 2b_{12}\beta_1\beta_2 + b_{22}\beta_2^2 = 0, \tag{3.4}$$

where $b_{ij} := \langle \varphi, B(q^{(i)}, q^{(j)}) \rangle$ for $i, j = 1, 2$.

A singular point $p = 0 \in M$ is called a **simple branching point** for ALCP (3.1) if

(*i*) rank $J = N - 1$;
(*ii*) $b_{12}^2 - b_{11}b_{22} > 0$.

Theorem 2 *Near a simple branching point $p = 0$ of ALCP* (3.1)*, the solution manifold M consists of two smooth curves, Γ_1 and Γ_2, intersecting transversally at p.*

A vector tangent to Γ_2 is given by the formula

$$v^{(2)} = -\frac{b_{22}}{2b_{12}} q^{(1)} + q^{(2)},$$

where $q^{(1)} = v^{(1)}$ is the tangent vector to Γ_1 at $p = 0$, and a nonzero vector $q^{(2)} \in N(J)$ satisfies $\langle q^{(1)}, q^{(2)} \rangle = 0$.

Proof: Write

$$x = \beta_1 q^{(1)} + \beta_2 q^{(2)} + y, \quad y \in \mathbb{R}^{N+1},$$

and consider the equation

$$H(y, \mu, \beta) := \begin{pmatrix} \mu\varphi + F(\beta_1 q^{(1)} + \beta_2 q^{(2)} + y) \\ \langle q^{(1)}, y \rangle \\ \langle q^{(2)}, y \rangle \end{pmatrix} = 0 \tag{3.5}$$

with $H : \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^{N+2}$. Clearly, $H(0,0,0) = 0$. Moreover, the square $(N+2) \times (N+2)$ matrix

$$(H_y(0,0,0) \;\; H_\mu(0,0)) = \begin{pmatrix} J & \varphi \\ [q^{(1)}]^{\mathrm{T}} & 0 \\ [q^{(2)}]^{\mathrm{T}} & 0 \end{pmatrix}$$

is nonsingular. Indeed, if for some $w \in \mathbb{R}^{N+1}$ and $u \in \mathbb{R}$ holds

$$\begin{pmatrix} J & \varphi \\ [q^{(1)}]^{\mathrm{T}} & 0 \\ [q^{(2)}]^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then $Jw + u\varphi = 0$ implying $u = 0$, since $\varphi^{\mathrm{T}} J = 0$ but $\|\varphi\| = 1$. Thus $Jw = 0$ and $w = c_1 q^{(1)} + c_2 q^{(2)}$ with some $c_{1,2} \in \mathbb{R}$. The conditions $\langle q^{(j)}, w\rangle = 0$ lead to $c_1 = c_2 = 0$ and thus $w = 0$.

Therefore, the Implicit Function Theorem guarantees existence and uniqueness of smooth functions $y = Y(\beta)$ and $\mu = m(\beta)$ with $Y(0) = 0, m(0) = 0$, and such that

$$H(Y(\beta), m(\beta), \beta) = 0$$

for all $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ with sufficiently small $\|\beta\|$. The solutions of the original problem $F(x) = 0$ near $x = 0$ correspond to the *level curves*

$$m(\beta) = 0.$$

This gives relations between β_1 and β_2 to be used in

$$x = \beta_1 q^{(1)} + \beta_2 q^{(2)} + Y(\beta)$$

to parametrize different branches of the ALCP $F(x) = 0$ near the origin.

One can show that

$$m(\beta) = -\frac{1}{2} Q(\beta) + O(\|\beta\|^2), \tag{3.6}$$

where $Q(\beta)$ is defined in (3.4). Since condition (ii) in the definition of the simple branching point guarantees that the quadratic form $Q(\beta)$ has a *saddle point* at $\beta_1 = \beta_2 = 0$, there are locally two zero-level curves of $m(\beta)$ intersecting at a nonzero angle at the origin. Hence, we will prove the first part of the theorem if we verify (3.6).

Notice that the first equation in (3.5) implies the identity

$$m(\beta)\varphi + F(\beta_1 q^{(1)} + \beta_2 q^{(2)} + Y(\beta)) = 0 \tag{3.7}$$

for small $\|\beta\|$. Using the expansion (3.2), we see that this is equivalent to

$$\begin{aligned} 0 \;=\;& m(\beta)\varphi + JY(\beta) + \\ & \frac{1}{2}\left[\beta_1^2 B(q^{(1)}, q^{(1)}) + 2\beta_1\beta_2 B(q^{(1)}, q^{(2)}) + \beta_2^2 B(q^{(2)}, q^{(2)})\right] + \\ & \beta_1 B(q^{(1)}, Y(\beta)) + \beta_2 B(q^{(2)}, Y(\beta)) + \frac{1}{2} B(Y(\beta), Y(\beta)) + \dots, \end{aligned}$$

where dots stand for all cubic and higher-order terms in (Y, β).

Differentiating (3.7) w.r.t. β_j, we therefore obtain at $\beta = 0$

$$\frac{\partial m(0)}{\partial \beta_j}\varphi + J\frac{\partial Y(0)}{\partial \beta_j} = 0 \tag{3.8}$$

for $j = 1, 2$. Computing the product with φ^{T} from the left and taking into account $\varphi^{\mathrm{T}} J = 0$ with $\|\varphi\| = 1$, we see that

$$\frac{\partial m(0)}{\partial \beta_1} = \frac{\partial m(0)}{\partial \beta_2} = 0.$$

Then (3.8) implies that

$$\frac{\partial Y(0)}{\partial \beta_j} \in N(J)$$

so that

$$\frac{\partial Y(0)}{\partial \beta_j} = c_{j1}q^{(1)} + c_{j2}q^{(2)}$$

with some $c_{jk} \in \mathbb{R}$ for $j, k = 1, 2$. Then from the last two equations in (3.5) it follows that

$$\langle q^{(1)}, \frac{\partial Y(0)}{\partial \beta_j}\rangle = \langle q^{(2)}, \frac{\partial Y(0)}{\partial \beta_j}\rangle = 0,$$

so that $c_{11} = c_{12} = c_{21} = c_{22} = 0$, ensuring

$$\frac{\partial Y(0)}{\partial \beta_1} = \frac{\partial Y(0)}{\partial \beta_2} = 0.$$

Thus, the functions $m(\beta)$ and $Y(\beta)$ not only vanish at $\beta = 0$ but contain no linear terms:

$$m(\beta) = O(\|\beta\|^2), \quad Y(\beta) = O(\|\beta\|^2).$$

Differentiating now (3.7) w.r.t. β_1 and β_2 twice at $\beta = 0$ and multiplying with φ^{T} from the left, we see in the same manner that

$$\frac{\partial^2 m(0)}{\partial \beta_j \partial \beta_k} = -\langle \varphi, B(q^{(j)}, q^{(k)})\rangle = -b_{jk}, \quad j, k = 1, 2,$$

which proves (3.6).

To complete the proof, notice that by construction the nonzero vector $q^{(2)}$ is orthogonal in $N(J)$ to $q^{(1)}$ and is therefore linearly independent of $q^{(1)}$. Since $v^{(1)} = q^{(1)}$, the equality

$$v^{(1)} = \beta_1^{(1)} q^{(1)} + \beta_2^{(1)} q^{(2)}$$

implies $\beta_1^{(1)} = 1$ and $\beta_2^{(1)} = 0$. Thus, because $Q(\beta^{(1)}) = 0$, we must have $b_{11} = 0$. Therefore, the coordinates $\beta_j^{(2)}$ in the decomposition

$$v^{(2)} = \beta_1^{(2)} q^{(1)} + \beta_2^{(2)} q^{(2)}$$

should satisfy

$$2b_{12}\beta_1^{(2)} + b_{22}\beta_2^{(2)} = 0$$

or

$$\beta_1^{(2)} = -\frac{b_{22}}{2b_{12}}\beta_2^{(2)} .$$

Here $b_{12} \neq 0$, since $b_{12}^2 - b_{11}b_{22} = b_{12}^2 > 0$ due to the simplicity of the braching point. □

The above theorem solves the problem of switching to the **secondary branch** Γ_2 at a simple branching point, since (an approximation of) $v^{(1)}$ is known from the continuation of the **primary branch** Γ_1.

3.2 Detection of branching points

Suppose that $s = 0$ corresponds to a branching point of ALCP (3.1) in the solution branch Γ_1 parametrized by $x^{(1)}(s)$ such that

$$x^{(1)}(0) = 0,\ \|v^{(1)}(0)\| = \|\dot{x}^{(1)}(0)\| = 1.$$

Theorem 3 *Define the* $(N+1)\times(N+1)$ *matrix*

$$D(s) = \begin{pmatrix} F_x(x^{(1)}(s) \\ [\dot{x}^{(1)}(s)]^{\mathrm{T}} \end{pmatrix}$$

and introduce

$$\psi_{\mathrm{BP}}(s) := \det(D(s)).$$

At a simple branching point holds

$$\psi_{\mathrm{BP}}(0) = 0 \quad \text{and} \quad \dot{\psi}_{\mathrm{BP}}(0) \neq 0.$$

Proof (under an extra genericity assumption):

The matrix $D(0)$ is singular. Indeed, $D(0)q^{(2)} = 0$ where $q^{(2)} \in N(J)$ is a nonzero vector satisfying $\|q^{(2)}\| = 1$ and orthogonal to $q^{(1)} = v^{(1)}$, the normalized tangent vector to Γ_1 at $x^{(1)}(0)$. The vectors $q^{(1)}$ and $q^{(2)}$ together span $N(J)$. Thus $\psi_{\mathrm{BP}}(0) = 0$.

The null-space $N(D(0))$ is one-dimensional. Indeed, any vector $q \in \mathbb{R}^{N+1}$ that satisfies $D(0)q = 0$ satisfies

$$\begin{cases} Jq &= 0, \\ \langle q, q^{(1)}\rangle &= 0, \end{cases}$$

and is unique (modulo scaling), since there is only one direction in the two-dimensional space $N(J)$ that is orthogonal to $q^{(1)}$. This implies that the null-space $N(D^{\mathrm{T}}(0)$ is also one-dimensional, and there is a unique (modulo scaling) $P \in \mathbb{R}^{N+1}$ such that $D^{\mathrm{T}}(0)P = 0$. Clearly,

$$P = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \tag{3.9}$$

where $J^{\mathrm{T}}\varphi = 0$ and we have assumed that $\|\varphi\| = 1$ implying $\|P\| = 1$.

Assume that $\lambda = 0$ is algebraically simple eigenvalue of $D(0)$, which is generic. Consider a smooth continuation $\lambda(s)$ of this eigenvalue and its corresponding eigenvector $u(s)$ satisfying

$$D(s)u(s) = \lambda(s)u(s),$$

where $\lambda(0) = 0, u(0) = q^{(2)}$. By differentiating w.r.t. s, we get

$$\dot{D}(s)u(s) + D(s)\dot{u}(s) = \dot{\lambda}(s)u(s) + \lambda(s)\dot{u}(s)$$

or

$$\begin{pmatrix} F_{xx}(x^{(1)}(s))[\dot{x}^{(1)}(s), u(s)] \\ [\ddot{x}^{(1)}(s)]^{\mathrm{T}} u(s) \end{pmatrix} + \begin{pmatrix} F_x(x^{(1)}(s) \\ [\dot{x}^{(1)}(s)]^{\mathrm{T}} \end{pmatrix} \dot{u}(s) = \dot{\lambda}(s)u(s) + \lambda(s)\dot{u}(s).$$

At $s = 0$ this gives

$$\begin{pmatrix} B(q^{(1)}, q^{(2)}) \\ [\ddot{x}^{(1)}(0)]^{\mathrm{T}} q^{(2)} \end{pmatrix} + \begin{pmatrix} J \\ [\dot{q}^{(1)}]^{\mathrm{T}} \end{pmatrix} \dot{u}(0) = \dot{\lambda}(0) q^{(2)}.$$

Multiplying the last equation from the left with P^{T} defined by (3.9), we obtain

$$\langle \varphi, B(q^{(1)}, q^{(2)}) \rangle + \varphi^{\mathrm{T}} J \ \dot{u}(0) = \dot{\lambda}(0) \langle P, q^{(2)} \rangle,$$

from which it follows that

$$\dot{\lambda}(0) = \frac{\langle \varphi, B(q^{(1)}, q^{(2)}) \rangle}{\langle P, q^{(2)} \rangle} = \frac{b_{12}}{\langle P, q^{(2)} \rangle} \neq 0.$$

Here, we have taken into account that the branching point is simple and that $\langle P, q^{(2)} \rangle \neq 0$ due to simplicity of $\lambda(0) = 0$.

Thus, $\lambda(s)$ has a regular zero at $s = 0$. Therefore, $\psi_{\mathrm{BP}}(s)$ also has a regular zero at $s = 0$, since it is the product of all eigenvalues of $D(s)$. □

3.3 Location of branching points

Theorem 4 *Let $x = 0$ be a simple branching point of ALCP* (3.1) *and $\varphi \in \mathbb{R}^N$ be such that $\varphi^{\mathrm{T}} J = 0, \|\varphi\| = 1$.*

Then $(x, y, z) = (0, \varphi, 0) \in \mathbb{R}^{N+1} \times \mathbb{R}^N \times \mathbb{R}$ is a regular solution of the system

$$\begin{cases} F(x) + zy &= 0, \\ y^{\mathrm{T}} F_x(x) &= 0, \\ y^{\mathrm{T}} y - 1 &= 0. \end{cases} \tag{3.10}$$

Proof:

Denote the LHS of (3.10) by $G(x, y, z)$. Then $(0, \varphi, 0)$ is obviously a solution to $G(x, y, z) = 0$. The Jacobian matrix of G at $(0, \varphi, 0)$ is

$$N = \begin{pmatrix} J & 0 & \varphi \\ \varphi^{\mathrm{T}} F_{xx}(0) & J^{\mathrm{T}} & 0 \\ 0 & 2\varphi^{\mathrm{T}} & 0 \end{pmatrix},$$

where the elements of the $(N+1) \times (N+1)$ matrix $\varphi^{\mathrm{T}} F_{xx}(0)$ are given by

$$(\varphi^{\mathrm{T}} F_{xx}(0))_{jk} = \sum_{i=1}^{N} \varphi_i \frac{\partial^2 F_i(0)}{\partial x_j \partial x_k}, \quad j, k = 1, 2, \ldots, N+1.$$

Suppose that N has a nontrivial null-vector

$$N\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0, \quad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \neq 0.$$

Then

$$\begin{aligned} JX + Z\varphi &= 0, & (3.11)\\ \varphi^{\mathrm{T}} F_{xx}(0)X + J^{\mathrm{T}}Y &= 0, & (3.12)\\ 2\varphi^{\mathrm{T}}Y &= 0. & (3.13) \end{aligned}$$

Equation (3.11) implies

$$\varphi^{\mathrm{T}} JX + Z\varphi^{\mathrm{T}}\varphi = 0,$$

i.e. $Z = 0$. Thus, (3.11) actually has the form $JX = 0$ so that $X \in N(J)$ and can be written as

$$X = \beta_1 q^{(1)} + \beta_2 q^{(2)} \tag{3.14}$$

for some $\beta_i \in \mathbb{R}, i = 1, 2$. Substituting this expression in (3.12), we get

$$\beta_1\varphi^{\mathrm{T}} F_{xx}(0)q^{(1)} + \beta_2\varphi^{\mathrm{T}} F_{xx}(0)q^{(2)} + J^{\mathrm{T}}Y = 0.$$

Now multiply the last equation with $[q^{(i)}]^{\mathrm{T}}$ from the left to get

$$\left\{ \begin{aligned} \beta_1\langle\varphi, B(q^{(1)}, q^{(1)})\rangle + \beta_2\langle\varphi, B(q^{(1)}, q^{(2)})\rangle + \langle q^{(1)}, J^{\mathrm{T}}Y\rangle &= 0,\\ \beta_1\langle\varphi, B(q^{(1)}, q^{(2)})\rangle + \beta_2\langle\varphi, B(q^{(2)}, q^{(2)})\rangle + \langle q^{(2)}, J^{\mathrm{T}}Y\rangle &= 0. \end{aligned} \right.$$

But $\langle q^{(i)}, J^{\mathrm{T}}Y\rangle = \langle Jq^{(i)}, Y\rangle = 0$ for $i = 1, 2$. Thus, taking into account the definition of b_{ij},

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the 2×2 matrix is nonsingular since $x = 0$ is the simple branching point. We see that $\beta_1 = \beta_2 = 0$ and $X = 0$ due to (3.14).

The equation (3.12) now reads: $J^{\mathrm{T}}Y = 0$, i.e. $Y \in N(J^{\mathrm{T}})$. Thus $Y = c\varphi$ for some $c \in \mathbb{R}$. Substituting this expression into (3.13) we get

$$2c\varphi^{\mathrm{T}}\varphi = 0.$$

This implies $c = 0$ and thus $Y = 0$.

We see that

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0,$$

a contradiction. Therefore $(0, \varphi, 0)$ is a regular solution of (3.10). □