Lecture 7: Computation of codim 1 bifurcations of equilibria

In this lecture, we will present regular defining systems to compute fold and (Andronov-)Hopf bifurcations of equilibria in

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}. \tag{7.14}$$

These systems will have the form

$$F(X) = 0, \quad X \in \mathbb{R}^N,$$

where $X = (u, \ldots, \alpha)^{\mathrm{T}}$. A solution $X_0 = (u_0, \ldots, \alpha_0)^{\mathrm{T}}$ will give the critical equilibrium u_0 at the bifurcation parameter value α_0. We assume for simplicity that $u_0 = 0$ and $\alpha_0 = 0$, and write

$$f(u, 0) = Au + \frac{1}{2}B(u, u) + O(3). \tag{7.15}$$

The regularity of the defining system at X_0, i.e. the non-singularity of its Jacobian matrix $F_X(X_0)$, will guarantee that this solution can be continued w.r.t. any other system parameter, say $\beta \in \mathbb{R}$. The corresponding solution curve will (locally) define a bifurcation boundary in the (α, β)-plane.

7.1 Generic bifurcation points

7.1.1 Simple fold points

Assume that $(u, \alpha) = (0, 0)$ corresponds to a limit point (see Lecture 2) of the equilibrium manifold of (7.14),

$$f(u, \alpha) = 0.$$

We know that $A = f_u(0, 0)$ has the one-dimensional null-space spanned by $q_0 \in \mathbb{R}^n$ such that

$$Aq_0 = 0, \quad \langle q_0, q_0 \rangle = 1,$$

while A^{T} also has the one-dimensional null-space spanned by $p_0 \in \mathbb{R}^n$ such that

$$A^{\mathrm{T}} p_0 = 0 \quad \text{or} \quad p_0^{\mathrm{T}} A = 0.$$

The matrix

$$J = (A \ \ f_\alpha^0), \quad f_\alpha^0 = f_\alpha(0, 0),$$

has rank n, which implies that $f_\alpha^0 \notin R(A)$ (otherwise rank $J < n$). This condition can be expressed more explicitly using the **Fredholm Decomposition**

$$\mathbb{R}^n = R(A) \oplus N(A^{\mathrm{T}}), \tag{7.16}$$

where $\oplus$ denotes the direct orthogonal sum of two linear subspaces. Since $p_0 \in N(A^{\mathrm{T}})$, the condition $f_\alpha^0 \notin R(A)$ is equivalent to

$$\langle p_0, f_\alpha^0 \rangle \neq 0.$$

Generically, the critical eigenvalue $\lambda_1 = 0$ of A is algebraically simple, implying $\langle p_0, q_0 \rangle \neq 0$. Indeed, in this case, $N(A)$ and $R(A)$ are the complementary invariant subspaces for A with $\dim N(A) = 1$ and $\dim R(A) = n-1$. Since $q_0 \notin R(A)$ (because q_0 spans $N(A)$), (7.16) implies that q_0 is not orthogonal to p_0. Thus we can assume

$$\langle q_0, q_0 \rangle = \langle p_0, q_0 \rangle = 1.$$

If a limit point is quadratic, we also have

$$a = \frac{1}{2} \langle p_0, B(q_0, q_0) \rangle \neq 0.$$

By definition, a **simple limit point** (or **simple fold**) is characterized by the following conditions:

(*i*) $\lambda_1 = 0$ is an algebraically simple eigenvalue of A and is the only eigenvalue with $\Re(\lambda) = 0$;

(*ii*) $\langle p_0, f^0_\alpha \rangle \neq 0$;

(*iii*) $\langle p_0, B(q_0, q_0) \rangle \neq 0$.

Choose a parametrization of the equilibrium manifold near the simple fold point

$$u = u(s), \quad \alpha = \alpha(s),$$

such that $u(0) = 0$, $\alpha(0) = 0$, $u'(0) = q_0$, $\alpha'(0) = 0$. Since $\lambda_1 = 0$ is algebraically simple, there exists a smooth continuation of the critical eigenvector, i.e. a smooth vector-function $q(s)$ and a smooth function $\lambda(s)$ satisfying for all sufficiently small $|s|$

$$f_u(u(s), \alpha(s))q(s) = \lambda(s)q(s)$$

and such that $q(0) = q_0$, $\lambda(0) = 0$. Differentiating the last equation w.r.t. s, we obtain

$$\begin{aligned} f_{uu}(u(s), \alpha(s))[u'(s), q(s)] &+ f_{u\alpha}(u(s), \alpha(s))[q(s), \alpha'(s)] \\ &+ f_u(u(s), \alpha(s))q'(s) = \lambda'(s)q(s) + \lambda(s)q'(s), \end{aligned}$$

which at $s = 0$ gives

$$f^0_{uu}[q_0, q_0] + f^0_u q'(0) = \lambda'(0)q_0 \qquad \text{or} \qquad B(q_0, q_0) + Aq'(0) = \lambda'(0)q_0.$$

Computing the scalar product of the last equation with p_0, we see that

$$\langle p_0, B(q_0, q_0) \rangle + \langle p_0, Aq'(0) \rangle = \lambda'(0) \langle p_0, q_0 \rangle.$$

Since $\langle p_0, q_0 \rangle = 1$ and $A^{\mathrm{T}} p_0 = 0$ implies $\langle p_0, Aq'(0) \rangle = \langle A^{\mathrm{T}} p_0, q'(0) \rangle = 0$, we can conclude that

$$\lambda'(0) = \langle p_0, B(q_0, q_0) \rangle \neq 0$$

at a simple fold point.

7.2 Simple Hopf points

The point $(u, \alpha) = (0, 0)$ is a Hopf point if $A = f_u(0, 0)$ has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$. Generically, these eigenvalues are algebraically simple, which allows one to introduce the corresponding eigenvectors $q_0, p_0 \in \mathbb{C}^n$,

$$Aq_0 = i\omega_0 q_0, \quad A^{\mathrm{T}} p_0 = -i\omega_0 p_0,$$

and assume that

$$\langle q_0, q_0\rangle = \langle p_0, q_0\rangle = 1,$$

where $\langle p_0, q_0\rangle := \bar{p}_0^{\mathrm{T}} q_0$.

Since A is nonsingular, the Implicit Function Theorem guarantees the existence of the unique local smooth continuation $u_e(\alpha)$ of the critical equilibrium $u_e(0) = 0$ that satisfies

$$f(u_e(\alpha), \alpha) = 0$$

for all sufficiently small parameter values. Differentiating this equation w.r.t. α we obtain

$$A(\alpha)u_e'(\alpha) + f_\alpha(u_e(\alpha), \alpha) = 0,$$

where $A(\alpha) := f_u(u_e(\alpha), \alpha)$. Substituting $\alpha = 0$ yields

$$Au_e'(0) + f_\alpha^0 = 0$$

or

$$u_e'(0) = -A^{-1} f_\alpha^0. \tag{7.17}$$

The Jacobian matrix $A(\alpha)$ has a smooth pair of complex-conjugate eigenvalues $\lambda(\alpha), \bar{\lambda}(\alpha)$, where

$$\lambda(\alpha) = \mu(\alpha) + i\omega(\alpha)$$

with $\mu(0) = 0$ and $\omega(0) = \omega_0$.

Lemma 12 *It holds that*

$$\mu'(0) = \Re\langle p_0, A_\alpha(0) q_0\rangle. \tag{7.18}$$

Proof:

Since the critical eigenvalues are algebraically simple, there exists a smooth complex vector-function $q(\alpha)$ with $q(0) = q_0$ and a smooth complex function $\lambda(\alpha)$ with $\lambda(0) = i\omega_0$, such that

$$A(\alpha)q(\alpha) = \lambda(\alpha)q(\alpha)$$

for all sufficiently small $|\alpha|$. Differentiating this equation w.r.t. α we obtain

$$A_\alpha(\alpha)q(\alpha) + A(\alpha)q'(\alpha) = \lambda'(\alpha)q(\alpha) + \lambda(\alpha)q'(\alpha).$$

Evaluation at $\alpha = 0$ gives

$$A_\alpha(0)q_0 + Aq'(0) = \lambda'(0)q_0 + i\omega_0 q'(0)$$

implying $\langle p_0, A_\alpha(0)q_0\rangle = \lambda'(0)$. Indeed, $\langle p_0, q_0\rangle = 1$ and

$$\langle p_0, Aq'(0)\rangle = \langle A^{\mathrm{T}} p_0, q'(0)\rangle = -\langle i\omega_0 p_0, q'(0)\rangle = i\omega_0\langle p_0, q'(0)\rangle.$$

Since $\mu'(0) = \Re(\lambda'(0))$, (7.18) follows. □

Taking into account

$$A_\alpha(\alpha)q(\alpha) = f_{uu}(u_e(\alpha),\alpha)[u_e'(\alpha), q(\alpha)] + f_{u\alpha}(u_e(\alpha),\alpha)q(\alpha),$$

we get

$$A_\alpha(0)q_0 = B(u_e'(0), q_0) + f^0_{u\alpha}q_0,$$

that leads to

$$\mu'(0) = \Re\langle p_0, -B(A^{-1}f^0_\alpha, q_0) + f^0_{u\alpha}q_0\rangle.$$

By definition, a **simple Hopf point** satisfies the following conditions:

(i) $\lambda_{1,2} = \pm i\omega_0$ are algebraically simple eigenvalues of A and are the only eigenvalues with $\Re(\lambda) = 0$;

(ii) $\mu'(0) = \Re\langle p_0, -B(A^{-1}f^0_\alpha, q_0) + f^0_{u\alpha}q_0\rangle \neq 0$.

The second condition is called the **Hopf transversality**.

Write $q_0 = q_1 + iq_2$ and $p_0 = p_1 + ip_2$ with $q_{1,2}, p_{1,2} \in \mathbb{R}^n$. In the simple Hopf case one can select these real vectors to satisfy

$$\langle q_j, q_k\rangle = \langle p_j, q_k\rangle = \frac{1}{2}\delta_{jk}, \tag{7.19}$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

We can now write (7.18) in the real form

$$\mu'(0) = p_1^{\mathrm{T}} A_\alpha(0)q_1 + p_2^{\mathrm{T}} A_\alpha(0)q_2, \tag{7.20}$$

where by linearity

$$\begin{aligned} A_\alpha(0)q_1 &= B(u_e'(0), q_1) + f^0_{u\alpha}q_1, \\ A_\alpha(0)q_2 &= B(u_e'(0), q_2) + f^0_{u\alpha}q_2. \end{aligned} \tag{7.21}$$

7.3 Bordering thechnique II

We need the following generalization of (7.16) to rectangular complex matrices.

Theorem 8 (General Fredholm's Decomposition) *Let $C \in \mathbb{C}^{n\times m}$ be a complex $n \times m$ matrix. Then*

$$\mathbb{C}^n = R(C) \oplus N(C^*),$$

where $\oplus$ denotes the direct orthogonal sum of two complex-linear subspaces of $\mathbb{C}^n$, and $C^* := \overline{C}^{\mathrm{T}}$. □

Notice that in the theorem the orthogonality w.r.t. the scalar product $\langle u, v\rangle := u^* v = \bar{u}^{\mathrm{T}} v$ is used for $u, v \in \mathbb{C}^n$. If C is real, we have

$$\mathbb{R}^n = R(C) \oplus N(C^{\mathrm{T}}),$$

where $\oplus$ denotes the direct orthogonal sum of two linear subspaces of $\mathbb{R}^n$.

Theorem 9 (Construction of Nonsingular Bordered Matrices) *Consider a real $(n+m)\times(n+m)$-matrix*

$$M = \begin{pmatrix} A & B \\ C^{\mathrm{T}} & D \end{pmatrix},$$

where $A \in \mathbb{R}^{n\times n}$, $B, C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times m}$, and assume that $r = \mathrm{rank}\ A = n - m$, so that m is the rank defect of A.

If $R(B)$ is a complement to $R(A)$ and $R(C)$ is a complement to $R(A^{\mathrm{T}})$, then M is nonsingular.

Remark:

Theorem 8 implies that it is sufficient to take B such that its columns span $N(A^{\mathrm{T}})$, and C such that its columns span $N(A)$. By continuity, all sufficiently small perturbations of M also remain nonsingular.

Proof of Theorem 9:

Suppose that M is singular, i.e.

$$\begin{pmatrix} A & B \\ C^{\mathrm{T}} & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for some $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equiavalent to the system

$$\left\{ \begin{array}{rcl} Ax + By & = & 0, \\ C^{\mathrm{T}}x + Dy & = & 0. \end{array} \right.$$

In its first equation, $Ax \in R(A)$ and $By \in R(B)$, so that $Ax = 0$ and $By = 0$, since $R(A)$ and $R(B)$ are complementary. Since $\dim R(A) = r$ then $\dim R(B) = n - r = m$ and B has full column rank (equal to m). This implies $y = 0$ and the system reduces to

$$\left\{ \begin{array}{rcl} Ax & = & 0, \\ C^{\mathrm{T}}x & = & 0. \end{array} \right.$$

This means that $x \in N(C^{\mathrm{T}})$ and $x \in N(A)$.

By Theorem 8, $N(C^{\mathrm{T}})$ is the orthogonal complement to $R(C)$, while $N(A)$ is the orthogonal complement to $R(A^{\mathrm{T}})$. Since $R(C)$ is complementary to $R(A^{\mathrm{T}})$, we conclude that $N(C^{\mathrm{T}})$ is a complement to $N(A)$. Thus, $x = 0$.

We have $x = 0$ and $y = 0$, a contradiction. Hence, M is nonsingular. □

Theorem 10 *Let*

$$M = \begin{pmatrix} A & B \\ C^{\mathrm{T}} & D \end{pmatrix}$$

be a nonsingular $(n+m)\times(n+m)$ *block-matrix with* $A \in \mathbb{R}^{n\times n}$, $B, C \in \mathbb{R}^{n\times m}$, *and* $D \in \mathbb{R}^{m\times m}$. *Let its inverse be decomposed as*

$$M^{-1} = \begin{pmatrix} P & Q \\ R^{\mathrm{T}} & S \end{pmatrix}$$

with $P \in \mathbb{R}^{n\times n}$, $Q, R \in \mathbb{R}^{n\times m}$, *and* $S \in \mathbb{R}^{m\times m}$.

If $\nu \leq \min(m, n)$ *then* A *has rank defect* ν *if and only if* S *has rank defect* ν.

Proof:

$$MM^{-1} = I_{n+m} \quad \Leftrightarrow \quad \begin{pmatrix} A & B \\ C^{\mathrm{T}} & D \end{pmatrix} \begin{pmatrix} P & Q \\ R^{\mathrm{T}} & S \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}.$$

Thus, in particularly,

$$AQ + BS = 0.$$

If A has a left singular vector p, then $p^{\mathrm{T}} A = 0$ and

$$p^{\mathrm{T}} AQ + p^{\mathrm{T}} BS = (p^{\mathrm{T}} B)S = (B^{\mathrm{T}} p)^{\mathrm{T}} S = 0.$$

Thus, $q = B^{\mathrm{T}} p$ is a left singular vector of S. Notice that $B^{\mathrm{T}} p$ is a linear combination of the rows of B and must be nonzero, since M has full rank. Therefore $q \neq 0$ and the dimension of the left null-space of S is at least that of the left null-space of A.

Similarly,

$$M^{-1} M = I_{n+m} \quad \Leftrightarrow \quad \begin{pmatrix} P & Q \\ R^{\mathrm{T}} & S \end{pmatrix} \begin{pmatrix} A & B \\ C^{\mathrm{T}} & D \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}.$$

Thus, in particularly,

$$R^{\mathrm{T}} A + SC^{\mathrm{T}} = 0.$$

If S has a left singular vector q, then $q^{\mathrm{T}} S = 0$ and

$$q^{\mathrm{T}} R^{\mathrm{T}} A + q^{\mathrm{T}} SC^{\mathrm{T}} = (q^{\mathrm{T}} R^{\mathrm{T}})A = (R^{\mathrm{T}} q)^{\mathrm{T}} A = 0.$$

As above, we conclude that $p = R^{\mathrm{T}} q \neq 0$ is a left singular vector of A. This implies that the dimension of the left null-space of A is at least that of the left null-space of S.

Therefore, the left null-spaces of A and S have equal dimensions. In the same manner, one establishes the equality of the dimensions of the right null-spaces of A and S, which proves the result. □

Suppose that matrix A depends smoothly on parameter $\beta \in \mathbb{R}$, i.e. we have

$$M(\beta) = \begin{pmatrix} A(\beta) & B \\ C^{\mathrm{T}} & D \end{pmatrix},$$

where constant B, C, and D are selected as before to make $M(0)$ nonsingular. Then $S = S(\beta)$ and there are two obvious ways to compute $S(\beta)$, namely, either by solving the bordered system

$$M(\beta) \begin{pmatrix} V(\beta) \\ S(\beta) \end{pmatrix} = \begin{pmatrix} 0 \\ I_m \end{pmatrix} \tag{7.22}$$

or

$$(W^{\mathrm{T}}(\beta) \quad S(\beta)) M(\beta) = (\, 0 \quad I_m) \tag{7.23}$$

that is equivalent to

$$M^{\mathrm{T}}(\beta)\begin{pmatrix} W(\beta) \\ S(\beta) \end{pmatrix} = \begin{pmatrix} 0 \\ I_m \end{pmatrix}.$$

There is an effifcient method to compute the derivative $S_\alpha(\alpha)$ using equations (7.22) and (7.23). Differentiating (7.22) w.r.t. β we obtain

$$M(\beta)\begin{pmatrix} V_\beta(\beta) \\ S_\beta(\beta) \end{pmatrix} + \begin{pmatrix} A_\beta(\beta) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V(\beta) \\ S(\beta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Multiplying this equation from the left by $(W^{\mathrm{T}}(\beta) \;\; S(\beta))$ and using (7.23) we find

$$S_\beta(\beta) = -W^{\mathrm{T}}(\beta)A_\beta(\beta)V(\beta). \tag{7.24}$$

7.4 Minimally augmeneted defining systems

7.4.1 Fold

Using the bordering technique, we can introduce the system

$$\begin{cases} f(u,\alpha) &= 0, \\ g(u,\alpha) &= 0, \end{cases} \tag{7.25}$$

where $g(u,\alpha)$ is defined by solving the linear system

$$\begin{pmatrix} f_u(u,\alpha) & p_0 \\ q_0^{\mathrm{T}} & 0 \end{pmatrix}\begin{pmatrix} w(u,\alpha) \\ g(u,\alpha) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.26}$$

with $q_0, p_0 \in \mathbb{R}^n$ satisfying

$$Aq_0 = A^{\mathrm{T}}p_0 = 0, \quad \langle q_0, q_0\rangle = \langle p_0, q_0\rangle = 1,$$

where $A = f_u^0 = f_u(0,0)$. The system (7.26) is a particular instance of the general bordered system (7.22) with $m = 1$.

Theorem 11 *Let $(u,\alpha) = (0,0)$ be a simple quadratic fold point. Then the Jacobian matrix of* (7.25) *at this point*

$$J = \begin{pmatrix} f_u^0 & f_\alpha^0 \\ g_u^0 & g_\alpha^0 \end{pmatrix}$$

is nonsingular.

Proof:

Theorem 9 (or Lemma 9 from Lecture 3) guarantees that

$$\begin{pmatrix} A & p_0 \\ q_0^{\mathrm{T}} & 0 \end{pmatrix}$$

is nonsingular. This implies that the matrix of the bordered system (7.26) is nonsingular for all sufficiently small $\|u\|$ and $|\alpha|$. Thus, $g(u, \alpha)$ is locally well defined. Furthermore, it follows from (7.24) (or just from Lemma 10 in Lecture 3) that

$$g_u(u, \alpha) = -p_0^{\mathrm{T}} f_{uu}(u, \alpha) q_0, \quad g_\alpha(u, \alpha) = -p_0^{\mathrm{T}} f_{u\alpha}(u, \alpha) q_0.$$

Here we treat the gradient g_u as the one-row matrix.

Theorem 9 ensures that matrix J is nonsingular if

$$f_\alpha^0 \notin R(f_u^0) = R(A) \quad \text{and} \quad [g_u^0]^{\mathrm{T}} \notin R([f_u^0]^{\mathrm{T}}) = R(A^{\mathrm{T}}).$$

By Fredholm's Decomposition these conditions are equivalent to the following inequalities:

$$p_0^{\mathrm{T}} f_\alpha^0 = \langle p_0, f_\alpha^0 \rangle \neq 0 \quad \text{and} \quad [g_u^0]^{\mathrm{T}} q_0 = -\langle p_0, B(q_0, q_0) \rangle \neq 0,$$

which hold since $(u, \alpha) = (0, 0)$ is a simple quadratic fold. □

7.4.2 Hopf

At a simple Hopf point, $A = f_u^0$ has a simple eigenvalue $\lambda_1 = i\omega_0$ with $\omega_0 > 0$. Its corresponding complex eigenvector $q_0 = q_1 + iq_2$ satisfies $Aq_0 = i\omega_0 q_0$. Moreover, there is a complex eigenvector $p_0 = p_1 + ip_2$ of the transposed matrix satisfying $A^{\mathrm{T}} p_0 = -i\omega_0 p_0$. Thus

$$\left\{ \begin{array}{rcl} Aq_1 + \omega_0 q_2 & = & 0, \\ Aq_2 - \omega_0 q_1 & = & 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} A^{\mathrm{T}} p_1 - \omega_0 p_2 & = & 0, \\ A^{\mathrm{T}} p_2 + \omega_0 p_1 & = & 0, \end{array} \right.$$

and the normalization conditions (7.19) are assumed to hold. These systems imply that

$$(A^2 + \omega_0^2 I_n) q_{1,2} = 0 \quad \text{and} \quad ([A^{\mathrm{T}}]^2 + \omega_0^2 I_n) p_{1,2} = 0$$

so that the matrix $(A^2 + \omega_0^2 I_n)$ has rank defect $\nu = 2$.

According to Lemma 9 the matrix

$$M(u,\alpha,\kappa) = \begin{pmatrix} f_u^2(u,\alpha) + \kappa I_n & B \\ C^{\mathrm{T}} & 0 \end{pmatrix},$$

where the columns of $B = (b_1\ b_2)$ span a space that is not orthogonal to $N([A^{\mathrm{T}}]^2 + \omega_0^2 I_n)$ and the columns of $C = (c_1\ c_2)$ span a space that is not orthogonal to $N(A^2 + \omega_0^2 I_n)$, is nonsingular at the simple Hopf point

$$(u,\alpha,\kappa) = (0,0,\omega_0^2).$$

Consider now the following bordered system

$$M(u,\alpha,\kappa)\begin{pmatrix} V \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ I_2 \end{pmatrix} \Leftrightarrow M(u,\alpha,\kappa)\begin{pmatrix} v_1 & v_2 \\ g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and its solution

$$v_j = v_j(u,\alpha,\kappa), \quad g_{jk} = g_{jk}(u,\alpha,\kappa), \qquad j,k = 1,2.$$

According to Theorem 10 in the case $m = 2$, the matrix $f_u^2(u,\alpha) + \kappa I_n$ has rank defect $\nu = 2$ if and only if $G \equiv 0$, i.e. $g_{11} = g_{12} = g_{21} = g_{22} = 0$. Thus $g_{jk}(0,0,\omega_0^2) = 0$ for all $j,k = 1,2$ at a Hopf point $(u,\alpha) = (0,0)$. This indicates that the system

$$\left\{ \begin{array}{rcl} f(u,\alpha) & = & 0, \\ g_{i_1 j_1}(u,\alpha,\kappa) & = & 0, \\ g_{i_2 j_2}(u,\alpha,\kappa) & = & 0, \end{array} \right. \tag{7.27}$$

where $(i_1 j_1)$ and $(i_2 j_2)$ are different index pairs, can be considered as a defining system for Hopf bifurcation.

In practice, the following modification is used. Consider the bordered system

$$M(u,\alpha,\kappa)\begin{pmatrix} v \\ h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{7.28}$$

and its solution

$$v = v(u,\alpha,\kappa), \quad h_j = h_j(u,\alpha,\kappa), \qquad j = 1,2.$$

Since $g_{jk}(0,0,\omega_0^2) = 0$ for all $j,k = 1,2$, we have

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

at a Hopf point. Thus as a defining system for the Hopf bifurcation one can use

$$\begin{cases} f(u,\alpha) &= 0, \\ h_1(u,k,\alpha) &= 0, \\ h_2(u,k,\alpha) &= 0. \end{cases} \tag{7.29}$$

Indeed, the following theorem holds.

Theorem 12 *Let* $(u,\alpha,\kappa) = (0,0,\omega_0^2)$ *correspond to a simple Hopf point. Then the Jacobian matrix of* (7.29) *at this point*

$$J = \begin{pmatrix} f_u^0 & f_\alpha^0 & 0 \\ h_{1u}^0 & h_{1\alpha}^0 & h_{1\kappa}^0 \\ h_{2u}^0 & h_{2\alpha}^0 & h_{2\kappa}^0 \end{pmatrix}$$

is nonsingular.

Proof:

First notice that the derivatives of $h_j(u,\alpha,\kappa)$ w.r.t. any component of (u,α,κ) can be efficiently computed using the bordering technique. Introduce $w_{1,2} = w_{1,2}(u,\alpha,k) \in \mathbb{R}^n$ as the solutions of the nonsingular system

$$\begin{pmatrix} w_1^{\mathrm{T}} & h_{11} & h_{12} \\ w_1^{\mathrm{T}} & h_{11} & h_{12} \end{pmatrix} M(u,k,\alpha) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} h_{1u} &= -w_1^{\mathrm{T}}(f_u^2)_u v, & h_{2u} &= -\,w_2^{\mathrm{T}}(f_u^2)_u v, \\ h_{1\alpha} &= -w_1^{\mathrm{T}}(f_u^2)_\alpha v, & h_{2\alpha} &= -\,w_2^{\mathrm{T}}(f_u^2)_\alpha v \end{aligned}$$

and

$$h_{1k} = -w_1^{\mathrm{T}} v, \quad h_{2k} = -w_2^{\mathrm{T}} v,$$

where $v = v(u,\alpha,\kappa)$ is defined by solving (7.28).

Suppose that at the simple Hopf point $(0,0,\omega_0^2)$ there is a vector $(U,\beta,K) \in \mathbb{R}^{n+2}$ such that

$$\begin{pmatrix} f_u^0 & f_\alpha^0 & 0 \\ h_{1u}^0 & h_{1\alpha}^0 & h_{1\kappa}^0 \\ h_{2u}^0 & h_{2\alpha}^0 & h_{2\kappa}^0 \end{pmatrix} \begin{pmatrix} U \\ \beta \\ K \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7.30}$$

The first equation in (7.30) is

$$f_u^0 U + f_\alpha^0 \beta = 0,$$

Since $A = f_u^0$ is invertible at the simple Hopf point and $f_\alpha^0 = -f_u^0 u_e'(0)$ (see (7.17)), we have

$$U = \beta u_e'(0).$$

Using the expressions for the derivatives, we have at $(u, \alpha) = (0, 0)$:

$$(f_u^2)_u vU = B(f_u^0 v, U) + f_u^0 B(v, U) = \beta[B(Av, u_e'(0)) + AB(v, u_e'(0))]$$

and

$$(f_u^2)_\alpha v\beta = \beta(f_{u\alpha}^0 f_u^0 + f_u^0 f_{u\alpha}^0)v = \beta(f_{u\alpha}^0 Av + Af_{u\alpha}^0 v).$$

At the simple Hopf point $v \in N(M)$ and $\{w_1, w_2\}$ form a basis in $N(M^{\mathrm{T}})$. Therefore (7.21) implies

$$\begin{aligned} A_\alpha(0)v &= B(v, u_e'(0)) + f_{u\alpha}^0 v, \\ A_\alpha(0)Av &= B(Av, u_e'(0)) + f_{u\alpha}^0 Av, \end{aligned}$$

so that and the second and third equations in (7.30) can now be written as

$$\begin{aligned} -\beta w_1^{\mathrm{T}}[AA_\alpha(0) + A_\alpha(0)A]v - (w_1^{\mathrm{T}} v)K &= 0, \\ -\beta w_2^{\mathrm{T}}[AA_\alpha(0) + A_\alpha(0)A]v - (w_2^{\mathrm{T}} v)K &= 0. \end{aligned}$$

Making a linear combination of the last two equations, we have

$$\begin{aligned} -\beta p_1^{\mathrm{T}}[AA_\alpha(0) + A_\alpha(0)A]q_1 - (p_1^{\mathrm{T}} q_1)K &= 0, \\ -\beta p_2^{\mathrm{T}}[AA_\alpha(0) + A_\alpha(0)A]q_1 - (p_2^{\mathrm{T}} q_1)K &= 0. \end{aligned}$$

Using the normalization conditions (7.19), we see that these equations are equivalent to

$$\begin{cases} 2\beta p_1^{\mathrm{T}}[A\,A_\alpha(0) + A_\alpha(0)A]q_1 + K = 0, \\ \beta p_2^{\mathrm{T}}[A\,A_\alpha(0) + A_\alpha(0)A]q_1 = 0. \end{cases} \tag{7.31}$$

However, $Aq_1 = -\omega_0 q_2$ and $p_2^{\mathrm{T}} A = -\omega_0 p_1^{\mathrm{T}}$, so that the second equation in (7.31) reads

$$\beta\omega_0[p_1^{\mathrm{T}} A_\alpha(0)q_1 + p_2^{\mathrm{T}} A_\alpha(0)q_2] = 0$$

or, taking into account (7.20),

$$\beta\omega_0\mu'(0) = 0.$$

Since $\omega_0\mu'(0) \neq 0$ at a simple Hopf point, we mast have $\beta = 0$. Then $U = 0$ and the first equation in (7.31) implies $K = 0$. Thus $(U, \beta, K) = 0$ is the only solution to (7.30). Therefore, the Jacobian maitrix J is nonsingular. □

7.5 Standard augmented defining systems

Here we present without proof some alternative defining systems for the fold and Hopf bifurcations.

7.5.1 Fold

Consider the system

$$\left\{\begin{array}{rcl} f(u,\alpha) & = & 0, \\ f_u(u,\alpha)q & = & 0, \\ \langle q_0, q\rangle - 1 & = & 0, \end{array}\right. \tag{7.32}$$

where $u, q, q_0 \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. System (7.32) has the form

$$F(X) = 0, \quad X \in \mathbb{R}^N,$$

where $N = 2n + 1$ and

$$X = \begin{pmatrix} u \\ q \\ \alpha \end{pmatrix}, \quad F(X) = \begin{pmatrix} f(u,\alpha) \\ f_u(u,\alpha)q \\ \langle q_0, q\rangle - 1 \end{pmatrix}.$$

Theorem 13 *Let $(u,\alpha) = (0,0)$ be a simple fold point and let q_0 denote a normalized null-vector of $A = f_u^0 = f_u(0,0)$. Then the Jacobian matrix of (7.32) is nonsingular at $(u,q,\alpha) = (0,q_0,0)$.* □

7.5.2 Hopf

Consider the system

$$\left\{\begin{array}{rcl} f(u,\alpha) & = & 0, \\ f_u(u,\alpha)q - i\omega q & = & 0, \\ \langle q_0, q\rangle - 1 & = & 0, \end{array}\right. \tag{7.33}$$

where $u \in \mathbb{R}^n$, $q, q_0 \in \mathbb{C}^n$, $\alpha \in \mathbb{R}$, and $\langle q_0, q\rangle \equiv \bar{q}_0^{\mathrm{T}} q$. This system has the form

$$G(Z) = 0, \quad Z \in \mathbb{R}^n \times \mathbb{C}^n \times \mathbb{R}^2,$$

where

$$Z = \begin{pmatrix} u \\ q \\ \omega \\ \alpha \end{pmatrix}, \quad G(Z) = \begin{pmatrix} f(u,\alpha) \\ f_u(u,\alpha)q - i\omega q \\ \langle q_0, q\rangle - 1 \end{pmatrix}.$$

Introducing $q = v + iw$ and $q_0 = v_0 + iw_0$ with $v, w, v_0, w_0 \in \mathbb{R}^n$, we can re-write (7.33) in the real form

$$\left\{ \begin{array}{rcl} f(u,\alpha) & = & 0, \\ f_u(u,\alpha)v + \omega w & = & 0, \\ f_u(u,\alpha)w - \omega v & = & 0, \\ \langle v_0, v\rangle + \langle w_0, w\rangle - 1 & = & 0, \\ \langle w_0, v\rangle - \langle v_0, w\rangle & = & 0, \end{array} \right. \tag{7.34}$$

This system has the form

$$F(X) = 0, \quad X = \begin{pmatrix} u \\ v \\ w \\ \omega \\ \alpha \end{pmatrix} \in \mathbb{R}^{3n+2}.$$

Theorem 14 *Let* $(u,\alpha) = (0,0)$ *be a simple Hopf point and let* $q_0 \in \mathbb{C}^n$ *denote a normilized by* $\langle q_0, q_0\rangle = 1$ *eigenvector of* $A = f_u^0 = f_u(0,0)$ *corresponding to* $\lambda_1 = i\omega_0, \omega_0 > 0$. *Then the Jacobian matrix of* (7.33) *has the trivial null-space at* $(u,q,\omega,\alpha) = (0,q_0,\omega_0,0)$ *and the Jacobian matrix of* (7.34) *is nonsingular at* $(u,v,w,\omega,\alpha) = (0,v_0,w_0,\omega_0,0)$. □