4. Computation of normal forms for LP and Hopf bifurcations

4.1. Normal forms on center manifolds

4.2. Fredholm's Alternative

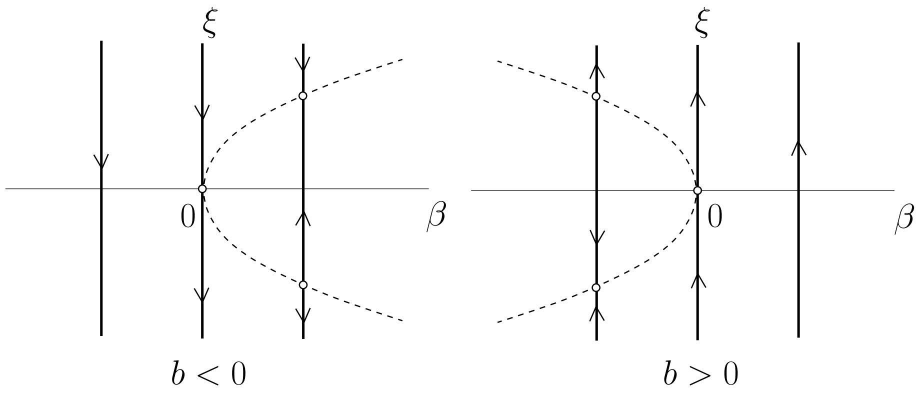
4.3. Critical LP-coefficient

4.4. Critical H-coefficient

4.5. Approximation of multilinear forms by finite differences

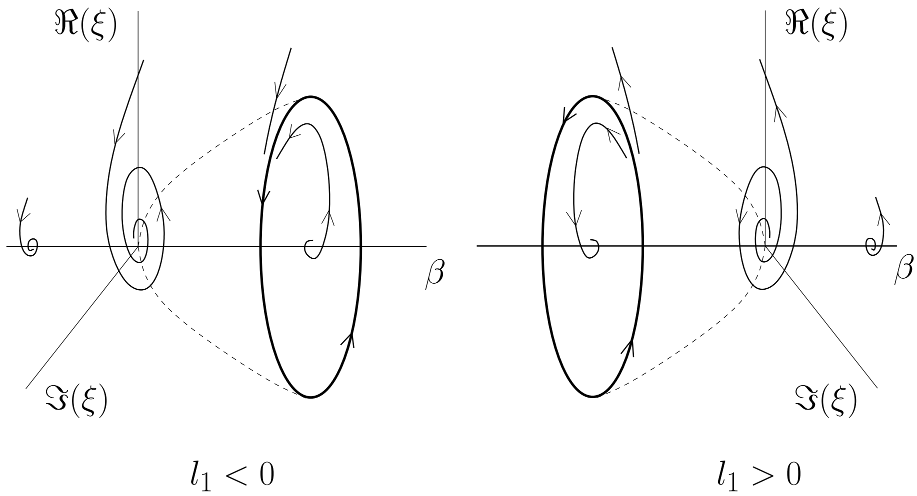
4.1. Normal forms on center manifolds

- LP: $\dot{\xi} = \beta + b\xi^2, \; b \neq 0$



Equilibria: $\beta + b\xi^2 = 0 \Rightarrow \xi_{1,2} = \pm\sqrt{-\frac{\beta}{b}}$

- H: $\dot{\xi} = (\beta + i\omega)\xi + c\xi|\xi|^2, \; l_1 = \frac{1}{\omega}\Re(c) \neq 0$



Limit cycle:

$$\begin{cases} \dot{\rho} &= \rho(\beta + \Re(c)\rho^2), \\ \dot{\varphi} &= \omega + \Im(c)\rho^2, \end{cases} \Rightarrow \rho_0 = \sqrt{-\frac{\beta}{\Re(c)}}$$

4.2. Fredholm's Alternative

- **Lemma 1** *The linear system* $Ax = b$ *with* $b \in \mathbb{R}^n$ *and a singular* $n \times n$ *real matrix* A *is solvable if and only if* $\langle p, b \rangle = 0$ *for all* p *satisfying* $A^{\top} p = 0$.

  Indeed, $\mathbb{R}^n = L \oplus R$ with $L \perp R$, where

  $$L = \mathcal{N}(A^{\top}) = \{p \in \mathbb{R}^n : A^T p = 0\}$$

  and

  $$R = \{x \in \mathbb{R}^n : x = Ay \quad \text{for some} \quad y \in \mathbb{R}^n\}.$$

  The proof is completed by showing that the orthogonal complement $L^{\perp}$ to L coincides with R.

- In the complex case:

$$\begin{array}{rcl} \mathbb{R}^n & \Rightarrow & \mathbb{C}^n \\ \langle p, b \rangle & = & \bar{p}^{\top} b \\ A^{\top} & \Rightarrow & A^* = \bar{A}^{\top} \end{array}$$

4.3. Critical LP-coefficient b

- Let $Aq = A^T p = 0$ with $\langle q, q\rangle = \langle p, q\rangle = 1$.

- Write the RHS at the bifurcation as
$$F(u) = Au + \frac{1}{2}B(u,u) + O(\|u\|^3),$$
and locally represent the center manifold W_0^c as the graph of a function $H : \mathbb{R} \to \mathbb{R}^n$,
$$u = H(\xi) = \xi q + \frac{1}{2}h_2\xi^2 + O(\xi^3),\ \ \xi \in \mathbb{R},\ h_2 \in \mathbb{R}^n.$$
The restriction of $\dot{u} = F(u)$ to W_0^c is
$$\dot{\xi} = G(\xi) = b\xi^2 + O(\xi^3).$$

- The invariance of the center manifold $H_\xi(\xi)\dot{\xi} = F(H(\xi))$ implies
$$H_\xi(\xi)G(\xi) = F(H(\xi))$$
Substitute all expansions into this **homological equation** and collect the coefficients of the ξ^j-terms.

We have

$$A(\xi q + \frac{1}{2}h_2\xi^2) \;\; + \;\; \frac{1}{2}B(\xi q, \xi q) + O(|\xi|^3)$$
$$= b\xi^2 q \;\; + \;\; b\xi^3 h_2 \;\; + \;\; O(|\xi|^4)$$

- The ξ-terms give the identity: $Aq = 0$.

- The ξ^2-terms give the equation for h_2:

$$Ah_2 = -B(q,q) + 2bq.$$

  It is singular and its **Fredholm solvability**

$$\langle p, -B(q,q) + 2bq \rangle = 0$$

  implies

$$b = \frac{1}{2}\langle p, B(q,q)\rangle$$

4.4. Critical H-coefficient c

- $Aq = i\omega_0 q, A^{\mathsf{T}} p = -i\omega_0 p, \langle q, q\rangle = \langle p, q\rangle = 1.$

- Write

  $$F(u) = Au + \frac{1}{2}B(u,u) + \frac{1}{3!}C(u,u,u) + O(\|u\|^4)$$

  and locally represent the center manifold W_0^c as the graph of a function $H : \mathbb{C} \to \mathbb{R}^n$,

  $$u = H(\xi, \bar{\xi}) = \xi q + \bar{\xi}\,\bar{q} + \sum_{2 \le j+k \le 3} \frac{1}{j!k!} h_{jk} \xi^j \bar{\xi}^k + O(|\xi|^4).$$

  The restriction of $\dot{u} = F(u)$ to W_0^c is

  $$\dot{\xi} = G(\xi, \bar{\xi}) = i\omega_0 \xi + c\xi|\xi|^2 + O(|\xi|^4).$$

- The invariance of W_0^c

  $$H_\xi(\xi, \bar{\xi})\dot{\xi} + H_{\bar{\xi}}(\xi, \bar{\xi})\dot{\bar{\xi}} = F(H(\xi, \bar{\xi}))$$

  implies

  $$H_\xi(\xi, \bar{\xi})G(\xi, \bar{\xi}) + H_{\bar{\xi}}(\xi, \bar{\xi})\overline{G}(\xi, \bar{\xi}) = F(H(\xi, \bar{\xi})).$$

- Quadratic ξ^2- and $|\xi|^2$-terms give

$$\begin{aligned} h_{20} &= (2i\omega_0 I_n - A)^{-1} B(q,q), \\ h_{11} &= -A^{-1} B(q,\overline{q}). \end{aligned}$$

- Cubic $w^2\overline{w}$-terms give the singular system

$$\begin{aligned} (i\omega_0 I_n - A) h_{21} &= C(q,q,\overline{q}) \\ &+ B(\overline{q}, h_{20}) + 2B(q, h_{11}) \\ &- 2cq. \end{aligned}$$

The solvability of this system implies

$$\begin{aligned} c &= \frac{1}{2}\langle p, C(q,q,\overline{q}) \\ &+ B(\overline{q}, (2i\omega_0 I_n - A)^{-1} B(q,q)) \\ &- 2B(q, A^{-1} B(q,\overline{q}))\rangle \end{aligned}$$

- The **first Lyapunov coefficient**

$$l_1 = \frac{1}{\omega_0} \Re(c).$$

4.5. Approximation of multilinear forms by finite differences

- Finite-difference approximation of directional derivatives:

$$
\begin{aligned}
B(q,q) &= \frac{1}{h^2}\left[f(u_0+hq,\alpha_0)+f(u_0-hq,\alpha_0)\right] \\
&+ O(h^2) \\
C(r,r,r) &= \frac{1}{8h^3}\left[f(u_0+3hr,\alpha_0)-3f(u_0+hr,\alpha_0)\right. \\
&+ \left.3f(u_0-hr,\alpha_0)-f(u_0-3hr,\alpha_0)\right] \\
&+ O(h^2).
\end{aligned}
$$

- Polarization identities:

$$
B(q,r)=\frac{1}{4}\left[B(q+r,q+r)-B(q-r,q-r)\right],
$$

$$
\begin{aligned}
C(q,q,r) &= \frac{1}{6}[C(q+r,q+r,q+r) \\
&- C(q-r,q-r,q-r)] \\
&- \frac{1}{3}C(r,r,r).
\end{aligned}
$$

5. Detection of codim 2 bifurcations

- codim 2 cases along the LP-curve:
  - **Bogdanov-Takens (BT)**: $\lambda_{1,2} = 0$
    ($\psi_{BT} = \langle p, q \rangle$ with $\langle q, q \rangle = \langle p, p \rangle = 1$)
  - **fold-Hopf (ZH)**: $\lambda_1 = 0, \lambda_{2,3} = \pm i\omega_0$
    ($\psi_{ZH} = \det(2A \odot I_n)$)
  - **cusp (CP)**: $\lambda_1 = 0, b = 0$ ($\psi_{CP} = b$)

- Critical cases along the H-curve:
  - **Bogdanov-Takens (BT)**: $\lambda_{1,2} = 0$
    ($\psi_{BT} = \kappa$)
  - **fold-Hopf (ZH)**: $\lambda_{1,2} = \pm i\omega_0, \lambda_3 = 0$
    ($\psi_{ZH} = \det A$)
  - **double Hopf (HH)**: $\lambda_{1,2} = \pm i\omega_0, \lambda_{3,4} = \pm i\omega_1$
    ($\psi_{HH} = \det(2A^{\perp} \odot I_{n-2})$)
  - **Bautin (GH)**: $\lambda_{1,2} = \pm i\omega_0, l_1 = 0$
    ($\psi_{GH} = l_1$)