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UvA Lecture 1:

One-Parameter Bifurcations of Fixed Points of Maps

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1. Critical cases

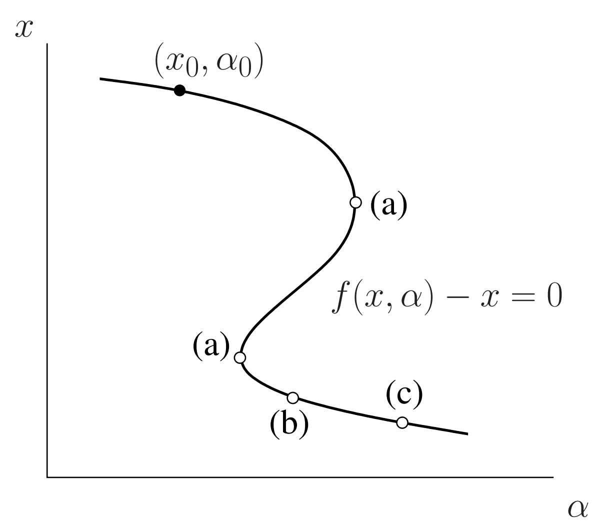
Consider a one-parameter system

$$x \mapsto \tilde{x} = f(x, \alpha), \quad x, \tilde{x} \in \mathbf{R}^n, \ \alpha \in \mathbf{R}^1.$$

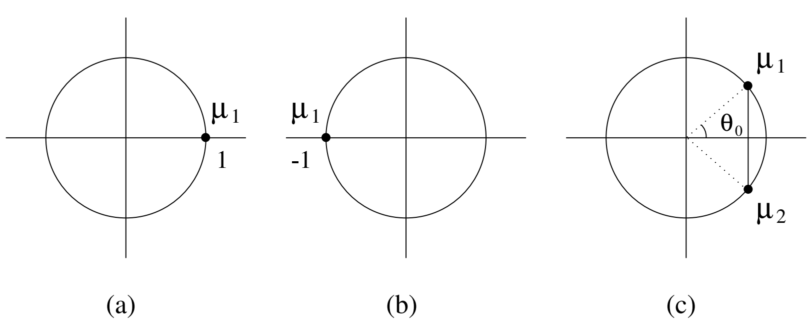
The **fixed-point manifold**

$$f(x, \alpha) - x = 0$$

is a smooth curve in $\mathbf{R}^{n+1}$, provided rank $J = n$, where $J = [f_x - I_n | f_\alpha]$.



Let $\{\mu_1, \mu_2, \ldots, \mu_n\}$ be eigenvalues of f_x. Critical cases: (a) LP; (b) PD; (c) NS

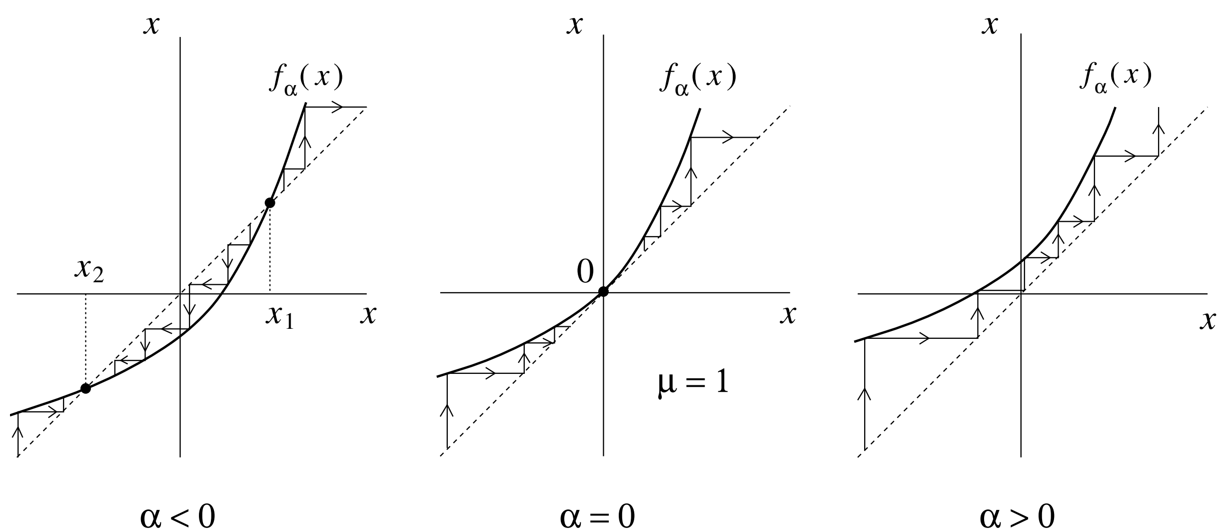


2. Fold (limit point) bifurcation

Example 1 (Fold normal form)
Consider

$$x \mapsto \alpha + x + x^2 \equiv f(x, \alpha) \equiv f_\alpha(x), \quad x, \alpha \in \mathbf{R}^1.$$

At $\alpha = 0$ this system has a nonhyperbolic fixed point $x_0 = 0$ with $\mu = f_x(0, 0) = 1$.



For $\alpha < 0$ there are two hyperbolic fixed points

$$x_{1,2}(\alpha) = \pm\sqrt{-\alpha}.$$

For $\alpha > 0$ there are no fixed points in the system.

Lemma 1 *The system*

$$x \mapsto \alpha + x + x^2 + O(x^3)$$

is locally topologically equivalent near the origin to the system

$$x \mapsto \alpha + x + x^2. \; \square$$

Th. 1 *Suppose that a one-dimensional system*

$$x \mapsto f(x, \alpha), \qquad x \in \mathbf{R}^1, \alpha \in \mathbf{R}^1,$$

with smooth f, has at $\alpha = 0$ the fixed point $x = 0$, and let $\mu = f_x(0, 0) = 1$. Assume that the following conditions are satisfied:

$(A.1)$ $\quad f_{xx}(0, 0) \neq 0 \quad$ (*nondegeneracy*)
$(A.2)$ $\quad f_{\alpha}(0, 0) \neq 0 \quad$ (*transversality*)

Then there are invertible smooth coordinate and parameter changes transforming the system into

$$\eta \mapsto \beta + \eta \pm \eta^2 + O(\eta^3).$$

Proof: Expand

$$f(x, \alpha) = f_0(\alpha) + f_1(\alpha)x + f_2(\alpha)x^2 + O(x^3).$$

Here $f_0(0) = f(0, 0) = 0$, $f_1(0) = f_x(0, 0) = 1$. Thus $f_1(\alpha) = 1 + g(\alpha)$, where $g(0) = 0$. Let

$$\xi = x + \delta,$$

where $\delta = \delta(\alpha)$ is to be defined later.

$$\tilde{\xi} = \tilde{x} + \delta = f(x, \alpha) + \delta = f(\xi - \delta, \alpha) + \delta.$$

Therefore,

$$\begin{aligned} \tilde{\xi} &= [f_0(\alpha) - g(\alpha)\delta + f_2(\alpha)\delta^2 + O(\delta^3)] \\ &\quad + \xi + [g(\alpha) - 2f_2(\alpha)\delta + O(\delta^2)]\xi \\ &\quad + [f_2(\alpha) + O(\delta)]\xi^2 + O(\xi^3). \end{aligned}$$

Write the coefficient in front of ξ as

$$F(\alpha, \delta) = g(\alpha) - 2f_2(\alpha)\delta + \delta^2\varphi(\alpha, \delta),$$

where φ is some smooth function. We have

$$F(0, 0) = 0, \ \left.\frac{\partial F}{\partial \delta}\right|_{(0,0)} = -2f_2(0) = f_{xx}(0, 0) \neq 0$$

by $(A.1)$.

The Implicit Function Theorem gives (local) existence and uniqueness of a smooth function $\delta = \delta(\alpha)$ such that $\delta(0) = 0$ and $F(\alpha, \delta(\alpha)) \equiv 0$. Now

$$\begin{aligned}\tilde{\xi} &= [f_0'(0)\alpha + O(\alpha^2)] + \xi \\ &\quad + [f_2(0) + O(\alpha)]\,\xi^2 + O(\xi^3).\end{aligned}$$

Consider the ξ-independent term as a new parameter

$$\mu(\alpha) = f_0'(0)\alpha + \alpha^2\varphi(\alpha).$$

We have $\mu(0) = 0$, $\mu'(0) = f_0'(0) = f_\alpha(0,0) \neq 0$, by $(A.2)$. Therefore, the Inverse Function Theorem implies local existence and uniqueness of a smooth function $\alpha = \alpha(\mu)$ with $\alpha(0) = 0$. Now the map reads

$$\tilde{\xi} = \mu + \xi + a(\mu)\xi^2 + O(\xi^3),$$

where $a(\mu)$ is a smooth function, $a(0) = f_2(0) \neq 0$. Let $\eta = |a(\mu)|\xi$ and $\beta = |a(\mu)|\mu$. Then we get

$$\tilde{\eta} = \beta + \eta + s\eta^2 + O(\eta^3),$$

where $s =$ sign $a(0) = \pm 1$.$\square$

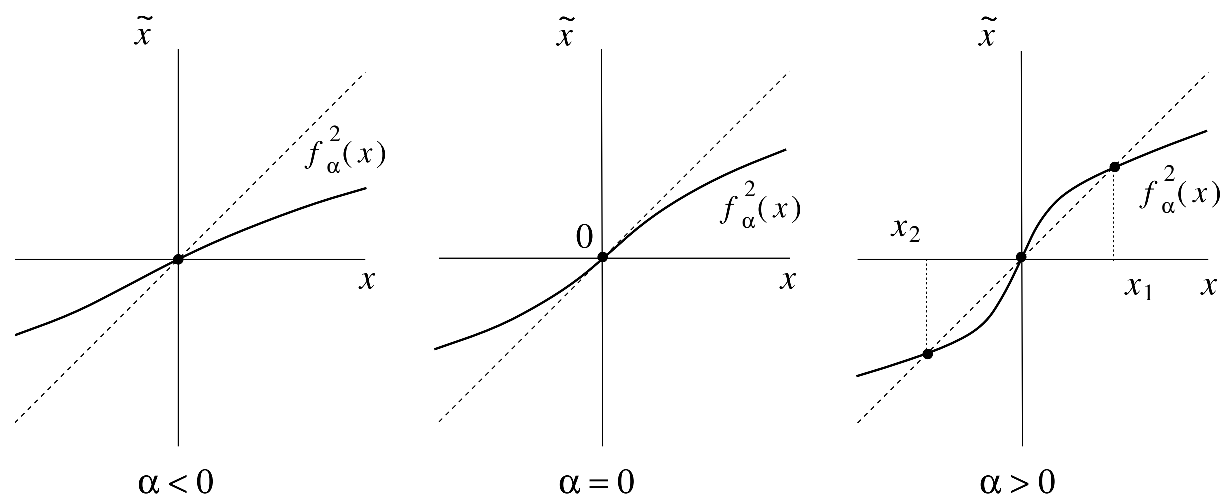
3. Flip (period doubling) bifurcation

Example 2 (Flip normal form) Consider

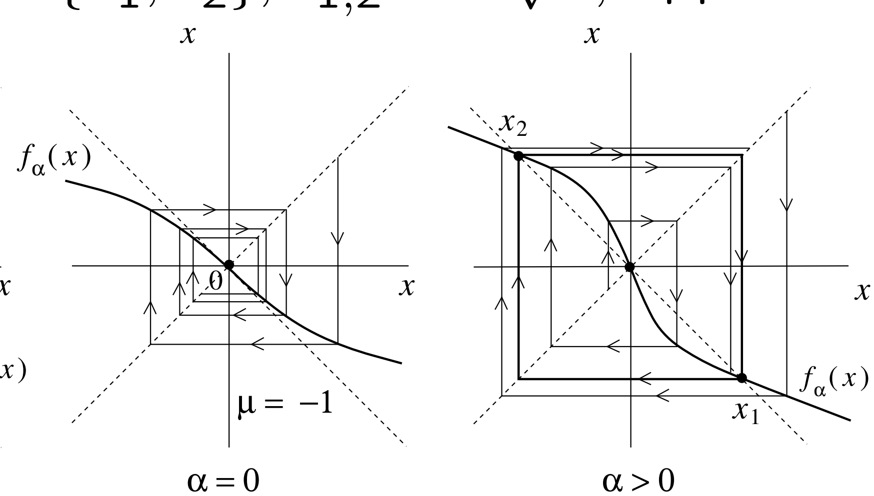
$$x \mapsto -(1+\alpha)x + x^3 \equiv f(x,\alpha) \equiv f_\alpha(x), \quad x, \alpha \in \mathbf{R}^1.$$

At $\alpha = 0$ this system has a nonhyperbolic fixed point $x_0 = 0$ with $\mu = f_x(0,0) = -1$. The **second iterate**

$$f_\alpha^2(x) = (1+\alpha)^2 x - [(1+\alpha)(2+2\alpha+\alpha^2)]x^3 + O(x^3).$$



Period-2 cycle $\{x_1, x_2\}, x_{1,2} = \pm\sqrt{\alpha}$, appears



Lemma 2 *The system*

$$x \mapsto -(1+\alpha)x + x^3 + O(x^4)$$

is locally topologically equivalent near the origin to the system

$$x \mapsto -(1+\alpha)x + x^3. \ \square$$

Th. 2 *Suppose that a one-dimensional system*

$$x \mapsto f(x,\alpha), \qquad x \in \mathbf{R}^1, \alpha \in \mathbf{R}^1,$$

with smooth f*, has the fixed point* $x_0 = 0$ *for all sufficiently small* $|\alpha|$*, and let* $\mu = f_x(0,0) = -1$*. Assume that the following conditions are satisfied:*

$(B.1)$ $\quad \frac{1}{2}(f_{xx}(0,0))^2 + \frac{1}{3}f_{xxx}(0,0) \neq 0$;
$(B.2)$ $\quad f_{x\alpha}(0,0) \neq 0$.

Then there are smooth invertible coordinate and parameter changes transforming the system into

$$\eta \mapsto -(1+\beta)\eta \pm \eta^3 + O(\eta^4).$$

Remark: If the map $x \mapsto f(\alpha)$ has a fixed point x_0 that depends on α, the condition $(B.2)$ has to be modified.

Proof:

Write the Taylor expansion of the function $f(x, \alpha)$ near $x = 0$ as

$$\tilde{x} = f(x, \alpha) = f_1(\alpha)x + f_2(\alpha)x^2 + f_3(\alpha)x^3 + O(x^4),$$

where $f_1(\alpha) = -[1 + g(\alpha)]$ for some smooth function g. Since $g(0) = 0$ and

$$g'(0) = f_{x\alpha}(0, 0) \neq 0,$$

according to $(B.2)$, the function g is locally invertible and can be used to introduce a new parameter: $\beta = g(\alpha)$. Thus

$$\tilde{x} = \mu(\beta)x + a(\beta)x^2 + b(\beta)x^3 + O(x^4),$$

where $\mu(\beta) = -(1 + \beta)$, and the functions $a(\beta)$ and $b(\beta)$ are smooth,

$$a(0) = \frac{1}{2} f_{xx}(0, 0), \quad b(0) = \frac{1}{6} f_{xxx}(0, 0).$$

Perform a smooth change of coordinate:

$$x = y + \delta y^2,$$

where $\delta = \delta(\beta)$ is a smooth function to be defined. The transformation is invertible in some neighborhood of the origin, and its inverse

$$y = x - \delta x^2 + 2\delta^2 x^3 + O(x^4).$$

Therefore

$$\begin{aligned}\tilde{y} &= \mu y + (a + \delta\mu - \delta\mu^2)y^2 \\ &+ (b + 2\delta a - 2\delta\mu(\delta\mu + a) + 2\delta^2\mu^3)y^3 \\ &+ O(y^4).\end{aligned}$$

Thus, the y^2-term can be "killed" for all sufficiently small $|\beta|$ by setting

$$\delta(\beta) = \frac{a(\beta)}{\mu^2(\beta) - \mu(\beta)}.$$

This is valid since $\mu^2(0) - \mu(0) = 2 \neq 0$ and, thus, the denominator is nonzero for all $|\beta|$ small.

The map takes the form

$$\begin{aligned}\tilde{y} &= \mu y + \left(b + \frac{2a^2}{\mu^2 - \mu}\right) y^3 + O(y^4) \\ &= -(1+\beta)y + c(\beta)y^3 + O(y^4)\end{aligned}$$

for some smooth function $c(\beta)$,

$$c(0) = a^2(0) + b(0) = \frac{1}{4}(f_{xx}(0,0))^2 + \frac{1}{6} f_{xxx}(0,0).$$

Notice that $c(0) \neq 0$ by $(B.1)$. The rescaling

$$y = \frac{\eta}{\sqrt{|c(\beta)|}}$$

brings the system into the desired form:

$$\tilde{\eta} = -(1+\beta)\eta + s\eta^3 + O(\eta^4),$$

where $s =$ sign $c(0) = \pm 1$. $\square$

Notice that

$$c(0) = -\frac{1}{12} \frac{\partial^3}{\partial x^3} \left. f_\alpha^2(x) \right|_{(x,\alpha)=(0,0)},$$

where $f_\alpha(x) = f(x, \alpha)$

Example 3 (Ricker's equation) The map

$$x \mapsto f(x, \alpha) = \alpha x e^{-x}$$

has at $\alpha_1 = e^2 = 7.38907\ldots$ the fixed point $x_1 = 2$ with the multiplier $\mu_1 = -1$ and

$$c(\alpha_1) = \frac{1}{6} > 0.$$

Thus, a stable period-2 cycle bifurcates from x_1. It exists for $\alpha > \alpha_1$ but is stable only for $|\alpha - \alpha_1|$ sufficiently small.