Recent progress in numerical bifurcation analysis of DDEs

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  Diekmann, O., Verduyn Lunel, S.M., van Gils, S.A. Walther, H.-O.
  Delay Equations: Functional-, Complex-, and Nonlinear Analysis
  Applied Mathematical Sciences 110, Springer (1995)

- Normal forms for codim 2 bifurcations of equilibria and branch switching to codim 1 bifurcation of cycles and (some) codim 1 homoclinic bifurcations [DDE-BIFTOOL]

  Bosschaert, M.M., Janssens, S, and Kuznetsov, Yu.A.
  Switching to nonhyperbolic cycles from codimension two bifurcations of equilibria of delay differential equations
  SIAM J. Appl. Dyn. Syst. **19** (2020), 252-303

  Bosschaert, M.M. and Kuznetsov, Yu.A.
  Bifurcation analysis of Bogdanov-Takens bifurcations in Delay Differential Equations
  SIAM J. Appl. Dyn. Syst. **23** (2024), 553-591

- Normal forms for codim 1 bifurcations of cycles
- Normal forms for codim 2 bifurcations of cycles and branch switching to codim 1 bifurcations of cycles

Local codim 1 bifurcations of cycles in ODEs

- Periodic normal forms for codim 1 bifurcations [MATCONT]

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Lentjes, B., Windmolders, M., and Kuznetsov, Yu.A.
Periodic center manifolds for nonhyperbolic limit cycles in ODEs
Int. J. Bifurcation & Chaos **33** (2023), 2350184 (29 pages)

Kuznetsov, Yu.A.
Elements of Applied Bifurcation Theory, 4th ed.
Applied Mathematical Sciences 113, Springer (2023)

Critical cases

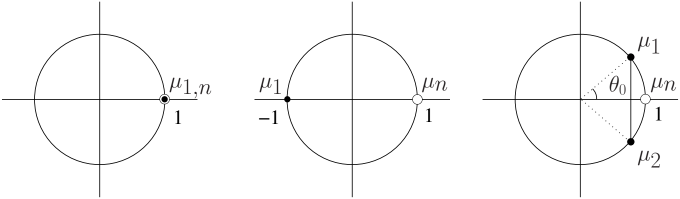
Consider a smooth ODE

$$\dot{x} = f(x), \;\; x \in \mathbb{R}^n.$$

A *limit cycle* C_0 corresponds to a periodic solution $x^0(t+T_0) = x^0(t)$ and has *Floquet multipliers* $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 1$, the eigenvalues of $U(s+T_0, s)$:

$$\dot{U}(t,s) - Df(x^0(t))\,U(t,s) = 0, \quad U(s,s) = I_n.$$

Critical cases:



- Fold (LPC): $\mu_1 = \mu_n = 1$;
- Flip (PD): $\mu_1 = -1, \; \mu_n = 1$;
- Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}, \;\; 0 < \theta_0 < \pi, \; \theta_0 \neq \frac{\pi}{2}$ and $\theta_0 \neq \frac{2\pi}{3}, \;\; \mu_n = 1$;

Center Manifold Theorem for ODEs

Let $X_0(s)$ denote the (n_0+1)-dimensional *center subspace* of $U(s+T_0,s)$ defined by the direct sum of all its generalized eigenspaces with a Floquet multiplier on the unit circle and let $X_0 := \{(s,y_0) \in \mathbb{R}\times\mathbb{R}^n : y_0 \in X_0(s)\}$ denote the *center fiber bundle.*

Theorem (Lentjes et al., 2023)

Consider a system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

with a C^{k+1}-smooth right-hand side $f:\mathbb{R}^n \to \mathbb{R}^n$ for some finite $k \geq 1$. Let x^0 be a T_0-periodic solution of this system such that the associated cycle C_0 is nonhyperbolic with $n_0+1 \geq 2$ multipliers satisfying $|\mu| = 1$. Then there exists a locally defined T_0-periodic C^k-smooth (n_0+1)-dimensional invariant manifold W_0^c defined around C_0 and tangent to the center fiber bundle X_0. □

Moreover, a sufficiently smooth system

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m,$$

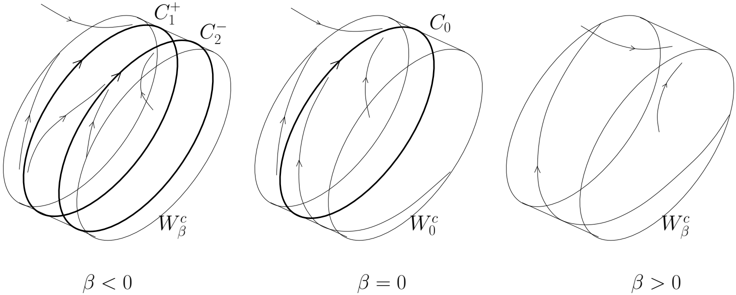
such that $f(x,0)$ coincides with $f(x)$, has a locally defined periodic smooth (n_0+1)-dimensional invariant manifold W_α^c that is a smooth continuation of W_0^c.

Generic LPC bifurcation: $\mu_1 = \mu_n = 1$

Periodic normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\beta) + \xi + a(\beta)\xi^2 + O(\xi^3), \\ \dfrac{d\xi}{dt} = \beta + b(\beta)\xi^2 + O(\xi^3), \end{cases}$$

where $\nu(0) = 0$ but $b(0) \neq 0$. The $O(\xi^3)$-terms are T_0-periodic in τ.

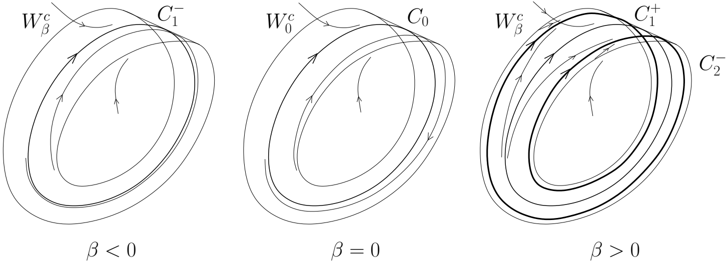


Generic PD bifurcation: $\mu_1 = -1$

Periodic normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\beta) + a(\beta)\xi^2 + O(\xi^4), \\ \dfrac{d\xi}{dt} = \beta\xi + c(\beta)\xi^3 + O(\xi^4), \end{cases}$$

where $\nu(0) = 0$ but $c(0) \neq 0$. The $O(\xi^4)$-terms are $2T_0$-periodic in τ.

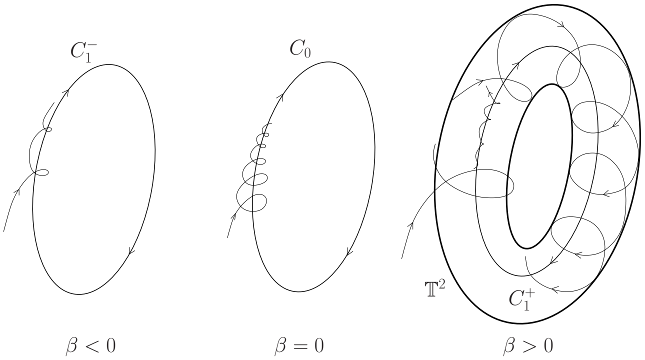


Generic NS bifurcation: $\mu_{1,2} = e^{\pm i\theta_0}$

Complex periodic normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\beta) + a(\beta)|\xi|^2 + O(|\xi|^4), \\ \dfrac{d\xi}{dt} = \left(\beta + \dfrac{i\theta(\beta)}{T(\beta)}\right)\xi + d(\beta)\xi|\xi|^2 + O(|\xi|^4), \end{cases}$$

where $\nu(0) = 0$ but $\mathrm{Re}(d(0)) \neq 0$. The $O(|\xi|^4)$-terms are T_0-periodic in τ.



Critical normal form coefficients for ODEs

- At a codimension-one point write the ODE as

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n.$$

- Taylor expansion with multilinear forms:

$$f(x^0(t) + v) = f(x^0(t)) + A(t)v + \frac{1}{2}B(t; v, v) + \frac{1}{6}C(t; v, v, v) + O(\|v\|^4),$$

where $A(t) = Df(x^0(t))$, while $B(t; u, v) = D^2 f(x^0(t))(u, v)$ and $C(t; u, v, w) = D^3 f(x^0(t))(u, v, w)$ with the components

$$B_i(t; u, v) = \sum_{j,k=1}^{n} \left. \frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} \right|_{x=x^0(t)} u_j v_k$$

and

$$C_i(t; u, v, w) = \sum_{j,k,l=1}^{n} \left. \frac{\partial^3 f_i(x)}{\partial x_j \partial x_k \partial x_l} \right|_{x=x^0(t)} u_j v_k w_l,$$

for $i = 1, 2, \ldots, n$. These are T_0-periodic in t.

- Fredholm technique for BVPs

Assume unique modulus scaling $\varphi, \varphi^* \in C^1([0,T_0],\mathbb{R}^n)$ satisfy

$$\begin{cases} \dot{\varphi}(\tau) - A(\tau)\varphi(\tau) &= 0,\ \tau \in [0,T_0], \\ \varphi(0) - \varphi(T_0) &= 0, \\ \int_0^{T_0} \langle \varphi(\tau), \varphi(\tau)\rangle d\tau - 1 &= 0, \end{cases}$$

and

$$\begin{cases} \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) &= 0,\ \tau \in [0,T_0], \\ \varphi^*(0) - \varphi^*(T_0) &= 0. \end{cases}$$

If $h \in C^1([0,T_0],\mathbb{R}^n)$ is a solution to

$$\begin{cases} \dot{h}(\tau) - A(\tau)h(\tau) &= g(\tau),\ \tau \in [0,T_0], \\ h(0) - h(T_0) &= 0, \end{cases}$$

with $g \in C^0([0,T_0],\mathbb{R}^n)$, then

$$\int_0^{T_0} \langle \varphi^*(\tau), g(\tau)\rangle \, d\tau = 0.$$

When it holds, there is a unique solution h satisfying

$$\int_0^{T_0} \langle \varphi^*(\tau), h(\tau)\rangle \, d\tau = 0 \qquad (\textit{Fredholm solvability condition})$$

Fold (LPC): $\mu_1 = \mu_n = 1$ (double non-semisimple)

- Critical center manifold W_0^c:

$$x = H(\tau, \xi), \quad \tau \in [0, T_0],\ \xi \in \mathbb{R},$$

  where

$$H(\tau, \xi) = x_0(\tau) + \xi v(\tau) + \frac{1}{2} h_2(\tau)\xi^2 + O(\xi^3)$$

  with $v(T_0) = v(0),\ h_2(T_0) = h_2(0)$.

- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} & = & 1 + \xi + a\xi^2 + O(\xi^3), \\ \dfrac{d\xi}{dt} & = & b\xi^2 + O(\xi^3), \end{cases}$$

  where $a, b \in \mathbb{R}$, while the $O(\xi^3)$-terms are T_0-periodic in τ.

LPC: Generalized and adjoint eigenfunctions

$$\begin{cases} \dot{v}(\tau) - A(\tau)v(\tau) + f(x_0(\tau),0) &= 0,\ \tau \in [0,T_0], \\ v(0) - v(T_0) &= 0, \\ \displaystyle\int_0^{T_0} \langle v(\tau), f(x_0(\tau),0)\rangle d\tau &= 0, \end{cases}$$

implying

$$\int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau),0)\rangle \, d\tau = 0,$$

where φ^* satisfies

$$\begin{cases} \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) &= 0,\ \tau \in [0,T_0], \\ \varphi^*(0) - \varphi^*(T_0) &= 0, \\ \displaystyle\int_0^{T_0} \langle \varphi^*(\tau), v(\tau)\rangle d\tau - 1 &= 0. \end{cases}$$

LPC: Computation of b

- *Homological equation*:

$$\frac{\partial H(\tau,\xi)}{\partial \tau}\frac{d\tau}{dt}+\frac{\partial H(\tau,\xi)}{\partial \xi}\frac{d\xi}{dt}=f(H(\tau,\xi))$$

- Collect in the homological equation

$$\begin{aligned}
\xi^0 &: \quad \dot{x}_0 = f(x_0,0),\\
\xi^1 &: \quad \dot{v}-A(\tau)v+f(x_0,0)=0,\\
\xi^2 &: \quad \dot{h}_2-A(\tau)h_2=B(\tau;v,v)-2af(x_0,0)-2\dot{v}-2bv.
\end{aligned}$$

- Since $\mathcal{N}\left(\frac{d}{d\tau}-A(\tau)\right)=\text{span}\{\varphi=\dot{x}_0\}$ in the subspace of T_0-periodic functions in $C^1([0,T_0],\mathbb{R}^n)$, we must have

$$b=\frac{1}{2}\int_0^{T_0}\langle\varphi^*(\tau),B(\tau;v(\tau),v(\tau))-2A(\tau)v(\tau)\rangle d\tau$$

(Fredholm's solvability condition)

Flip (PD): $\mu_1 = -1, \mu_n = 1$ (both simple)

- Critical center manifold W_0^c:

$$x = H(\tau,\xi), \quad \tau \in [0, 2T_0],\ \xi \in \mathbb{R},$$

where

$$H(\tau,\xi) = x_0(\tau) + \xi v(\tau) + \frac{1}{2} h_2(\tau)\xi^2 + \frac{1}{6} h_3(\tau)\xi^3 + O(\xi^4)$$

with $v(T_0) = -v(0),\ h_2(T_0) = h_2(0)$, and $h_3(T_0) = -h_3(0)$.

- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} & = & 1 + a\xi^2 + O(\xi^4), \\ \dfrac{d\xi}{dt} & = & c\xi^3 + O(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$, while the $O(\xi^4)$-terms are $2T_0$-periodic in τ.

PD: Eigenfunctions

- Eigenfunction:

$$\left\{\begin{array}{rcl} \dot{v}(\tau)-A(\tau)v(\tau) & = & 0,\ \tau\in[0,T_0], \\ v(0)+v(T_0) & = & 0, \\ \displaystyle\int_0^{T_0}\langle v(\tau),v(\tau)\rangle d\tau-1 & = & 0, \end{array}\right.$$

  extended to $[T_0, 2T_0]$ by

$$v(t+T_0)=-v(t),\ \ t\in[0,T_0].$$

- Adjoint eigenfunction:

$$\left\{\begin{array}{rcl} \dot{v}^*(\tau)+A^{\mathrm{T}}(\tau)v^*(\tau) & = & 0,\ \tau\in[0,T_0], \\ v^*(0)+v^*(T_0) & = & 0, \\ \displaystyle\int_0^{T_0}\langle v^*(\tau),v(\tau)\rangle d\tau-1 & = & 0. \end{array}\right.$$

  extended to $[T_0, 2T_0]$ by

$$v^*(t+T_0)=-v^*(t),\ \ t\in[0,T_0].$$

PD: Quadratic terms

Collect in the homological equation

$$\begin{aligned} \xi^0 &: \quad \dot{x}_0 = f(x_0, 0), \\ \xi^1 &: \quad \dot{v} - A(\tau) v = 0, \\ \xi^2 &: \quad \dot{h}_2 - A(\tau) h_2 = B(\tau; v, v) - 2a\dot{x}_0, \quad \tau \in [0, T_0]. \end{aligned}$$

Since $\mathcal{N}\left(\frac{d}{d\tau} - A(\tau)\right) = \text{span}\{\varphi = \dot{x}_0\}$ in the subspace of T_0-periodic functions in $C^1([0, T_0], X)$, the Fredholm solvability condition must hold:

$$\int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) - 2a\dot{x}_0(\tau) \rangle \, d\tau = 0,$$

where φ^* satisfies

$$\begin{cases} \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) = 0, \; \tau \in [0, T_0], \\ \varphi^*(0) - \varphi^*(T_0) = 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), 0) \rangle \, d\tau - 1 = 0, \end{cases}$$

PD: Computation of a and h_2

- The above Fredholm solvability condition implies

$$a = \frac{1}{2}\int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)\rangle d\tau.$$

- Define h_2 on $[0, T_0]$ as the unique solution to

$$\left\{ \begin{array}{rcl} \dot{h}_2(\tau) - A(\tau)h_2(\tau) - B(\tau; v(\tau), v(\tau)) + 2af(x_0(\tau), 0) & = & 0, \\ h_2(0) - h_2(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), h_2(\tau)\rangle \, d\tau & = & 0. \end{array} \right.$$

and extend to $[T_0, 2T_0]$ by periodicity

$$h_2(t + T_0) = h_2(t), \quad t \in [0, T_0].$$

PD: Computation of c

Cubic terms: ξ^3

$$\dot{h}_3 - A(\tau)h_3 = C(\tau; v, v, v) + 3B(\tau; v, h_2) - 6a\dot{v} - 6cv$$

Since $\mathcal{N}\left(\frac{d}{d\tau} - A(\tau)\right) = \text{span}\{v\}$ in the subspace of T_0-*anti*periodic functions in $C^1([0, T_0], \mathbb{R}^n)$,

the Fredholm solvability condition

$$\int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), v(\tau)) + 3B(\tau; v(\tau), h_2(\tau)) - 6aA(\tau)v(\tau) - 6cv(\tau)\rangle d\tau = 0$$

must hold, implying

$$c = \frac{1}{6}\int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), v(\tau)) + 3B(\tau; v(\tau), h_2(\tau)) - 6aA(\tau)v(\tau)\rangle d\tau$$

Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}, \mu_n = 1, \ (e^{ik\theta_0} \neq 1, k = 1,2,3,4)$

- Critical center manifold W_0^c:

$$x = H(\tau, \xi, \bar{\xi}), \ \ \tau \in [0, T_0], \ \xi \in \mathbb{C}$$

where

$$\begin{aligned} H(\tau,\xi,\bar{\xi}) &= x_0(\tau) + \xi v(\tau) + \bar{\xi}\bar{v}(\tau) + \frac{1}{2}h_{20}(\tau)\xi^2 + h_{11}(\tau)\xi\bar{\xi} + \frac{1}{2}h_{02}(\tau)\bar{\xi}^2 \\ &+ \frac{1}{6}h_{30}(\tau)\xi^3 + \frac{1}{2}h_{21}(\tau)\xi^2\bar{\xi} + \frac{1}{2}h_{12}(\tau)\xi\bar{\xi}^2 + \frac{1}{6}h_{03}(\tau)\bar{\xi}^3 + O(|\xi|^4) \end{aligned}$$

with $v(T_0) = v(0)$ and $h_{jk}(T_0) = h_{jk}(0)$.

- Critical periodic normal form on W_0^c:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + a|\xi|^2 + O(|\xi|^4), \\ \dfrac{d\xi}{dt} = \dfrac{i\theta_0}{T_0}\xi + d\xi|\xi|^2 + O(|\xi|^4), \end{cases}$$

where $a \in \mathbb{R}, d \in \mathbb{C}$, and the $O(|\xi|^4)$-terms are T_0-periodic in τ.

NS: Complex eigenfunctions

$$\left\{\begin{array}{rcl} \dot{v}(\tau) - A(\tau)v(\tau) + \dfrac{i\theta_0}{T_0}v(\tau) & = & 0,\ \tau \in [0, T_0], \\ v(0) - v(T_0) & = & 0, \\ \displaystyle\int_0^{T_0} \langle v(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

and

$$\left\{\begin{array}{rcl} \dot{v}^*(\tau) + A^{\mathrm{T}}(\tau)v^*(\tau) + \dfrac{i\theta_0}{T_0}v^*(\tau) & = & 0,\ \tau \in [0, T_0], \\ v^*(0) - v^*(T_0) & = & 0, \\ \displaystyle\int_0^{T_0} \langle v^*(\tau), v(\tau)\rangle d\tau - 1 & = & 0. \end{array}\right.$$

Homological equation:

$$\frac{\partial H(\tau,\xi,\bar{\xi})}{\partial \tau}\frac{d\tau}{dt} + \frac{\partial H(\tau,\xi\bar{\xi})}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial H(\tau,\xi,\bar{\xi})}{\partial \bar{\xi}}\frac{d\bar{\xi}}{dt} = f(H(\tau,\xi,\bar{\xi}))$$

NS: Quadratic terms

- ξ^2:

$$\dot{h}_{20} - A(\tau)h_{20} + \frac{2i\theta_0}{T_0}h_{20} = B(\tau; v, v)$$

Since $e^{2i\theta_0}$ is not a multiplier of the critical cycle, the BVP

$$\begin{cases} \dot{h}_{20} - A(\tau)h_{20} + \frac{2i\theta_0}{T_0}h_{20} - B(\tau; v(\tau), v(\tau)) & = & 0, \\ h_{20}(0) - h_{20}(T_0) & = & 0. \end{cases}$$

has a unique solution on $[0, T_0]$.

- $|\xi|^2$:

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - a\dot{x}_0$$

Here

$$\mathcal{N}\left(\frac{d}{d\tau} - A(\tau)\right) = \text{span}\{\varphi = \dot{x}_0\}$$

in the subspace of T_0-periodic functions in $C^1([0, T_0], \mathbb{R}^n)$.

NS: Computation of a and h_{11}

- Define φ^* as the unique solution of

$$\begin{cases} \dot{\varphi}^*(\tau) + A^{\mathrm{T}}(\tau)\varphi^*(\tau) & = & 0, \\ \varphi^*(0) - \varphi^*(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0)\rangle d\tau - 1 & = & 0. \end{cases}$$

- Fredholm solvability:

$$a = \int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), \bar{v}(\tau)\rangle \, d\tau$$

- Then find h_{11} on $[0, T_0]$ from the BVP

$$\begin{cases} \dot{h}_{11}(\tau) - A(\tau)h_{11}(\tau) - B(\tau; v(\tau), \bar{v}(\tau)) + af(x_0(\tau), \alpha_0) & = & 0, \\ h_{11}(0) - h_{11}(T_0) & = & 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), h_{11}(\tau)\rangle \, d\tau & = & 0. \end{cases}$$

NS: Computation of d

- Cubic terms: $\xi^2\bar{\xi}$

$$\dot{h}_{21} - Ah_{21} + \frac{i\theta_0}{T_0}h_{21} = 2B(\tau;h_{11},v) + B(\tau;h_{20},\bar{v}) + C(\tau;v,v,\bar{v}) - 2a\dot{v} - 2dv$$

- Fredholm solvability condition:

$$\begin{aligned} d &= \frac{1}{2}\int_0^{T_0} \langle v^*(\tau), C(\tau;v(\tau),v(\tau),\bar{v}(\tau))\rangle\, d\tau \\ &+ \frac{1}{2}\int_0^{T_0} \langle v^*(\tau), 2B(\tau;h_{11}(\tau),v(\tau)) + B(\tau;h_{20}(\tau),\bar{v}(\tau))\rangle\, d\tau \\ &- a\int_0^{T_0} \langle v^*(\tau), A(\tau)v(\tau)\rangle\, d\tau + \frac{ia\theta_0}{T_0} \end{aligned}$$

Local codim 1 bifurcations of cycles in DDEs

Diekmann, O., Verduyn Lunel, S.M., van Gils, S.A. Walther, H.-O.
Delay Equations: Functional-, Complex-, and Nonlinear Analysis
Applied Mathematical Sciences 110, Springer (1995)

Lentjes, B., Spek, L., Bosschaert, M. M., and Kuznetsov, Yu. A.
Periodic center manifolds for DDEs in the light of suns and stars
J. Dyn. Diff. Equat. (2023), https://doi.org/10.1007/s10884-023-10289-9

Lentjes, B., Spek, L., Bosschaert, M. M. and Kuznetsov, Yu. A.
Periodic normal forms for bifurcations of limit cycles in DDEs
arXiv:2302.08806v2 (2024)

Bosschaert, M. M., Lentjes, B., Spek, L., and Kuznetsov, Yu. A.
Numerical periodic normalization at codim 1 bifurcations of limit cycles in DDEs
[work in progress]

Delay Differential Equations (DDEs)

- Consider now a *discrete delay differential equation* (DDE)

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_m)), \quad t \geq 0,$$

where $x(t) \in \mathbb{R}^n$, $f : \mathbb{R}^{n\times(m+1)} \to \mathbb{R}^n$ is smooth, and the delays satisfy

$$0 < \tau_1 < \cdots < \tau_m = h < \infty$$

- Let the Banach space $X = C([-h,0], \mathbb{R}^n)$ be the space of initial data.
- Write the initial-value problem for the above DDE as

$$\begin{cases} \dot{x}(t) = F(x_t), & t \geq 0, \\ x_0 = \varphi, & \varphi \in X. \end{cases}$$

where the *history* of x at time $t \geq 0$, denoted by $x_t \in X$, is defined by

$$x_t(\theta) := x(t+\theta), \quad \theta \in [-h, 0].$$

What should be generalized for DDEs ?

- Phase space and solutions
- (Adjoint) linearized equations and multipliers
- Center manifold existence
- Normal forms on the center manifold
- Abstract ODE on the center manifold and the homological equation
- (Generalized) eigenfunctions
- Fredholm solvability
- Formulas for the normal form coefficients
- Numerical implementation

Introduction to $\odot\star$-calculus

- Strongly continuous *shift semigroup* on X (solution to DDE with $F = 0$):

$$(T_0(t)\varphi)(\theta) := \begin{cases} \varphi(t+\theta), & -h \le t+\theta \le 0, \\ \varphi(0), & t+\theta \ge 0, \end{cases} \qquad t \ge 0,\ \varphi \in X,\ \theta \in [-h, 0]$$

  with the infinitesimal generator A_0:

$$\mathscr{D}(A_0) = \left\{\varphi \in C^1([-h,0],\mathbb{R}^n) : \varphi'(0) = 0\right\}, \quad A_0\varphi = \varphi',$$

- The dual semigroup $T_0^\star(t)$ on the dual space $X^\star = \mathrm{NBV}([0,h],\mathbb{R}^{n*})$ has the infinitesimal generator $A_0^\star$:

$$\mathscr{D}(A_0^\star) = \Big\{f \in \mathrm{NBV}([0,h],\mathbb{R}^{n\star}) : f(\theta) = f(0^+) + \int_0^\theta g(\sigma)d\sigma \text{ for all } \theta \in (0,h],$$
$$g \in \mathrm{NBV}([0,h],\mathbb{R}^{n\star}) \text{ and } g(h) = 0\Big\}, \quad A_0^\star f = g,$$

- The maximal domain of strong continuity of $T_0^\star(t)$ on $X^\star$: $X^\odot = \mathbb{R}^{n*} \times L^1([0,h],\mathbb{R}^{n*})$

$$\mathscr{D}(A_0^\star) = \mathbb{R}^{n\star} \times \mathrm{NBV}([0,h],\mathbb{R}^{n\star})$$

- The strongly continuous semigroup

$$T_0^{\odot}(t) = T_0^{\star}(t)\big|_{X^{\odot}}$$

has the infinitesimal generator $A_0^{\odot}$ that is a part of $A_0^{\star}$ in $X^{\odot}$.

- The dual semigroup $T_0^{\odot\star}(t)$ on the dual space $X^{\odot\star} = \mathbb{R}^n \times L^{\infty}([-h,0],\mathbb{R}^n)$ has the infinitesimal generator $A_0^{\odot\star}$:

$$\mathscr{D}(A_0^{\odot\star}) = \left\{(\alpha,\varphi) \in X^{\odot\star} : \varphi \in \mathrm{Lip}([-h,0],\mathbb{R}^n) \text{ and } \varphi(0) = \alpha\right\}, \quad A_0^{\odot\star}(\alpha,\varphi) = (0,\varphi').$$

- The maximal domain of strong continuity of $T_0^{\odot\star}(t)$ on $X^{\odot\star}$: $X^{\odot\odot} = \mathbb{R}^n \times C([0,h],\mathbb{R}^n)$
- The strongly continuous semigroup

$$T_0^{\odot\odot}(t) = T_0^{\odot\star}(t)\big|_{X^{\odot}}$$

has the infinitesimal generator $A_0^{\odot\odot}$ that is a part of $A_0^{\odot\star}$ in $X^{\odot}$.

- The *canonical embedding* $j : X \to X^{\odot\star}$ has action

$$j\varphi = (\varphi(0),\varphi), \quad \varphi \in X$$

maps X onto $X^{\odot\odot}$, i.e. X is $\odot$-*reflexive* with respect to the shift semigroup T_0.

- Table [Janssens, S.G., 2010]

space	representation	duality pairing
X $X^\star$	$\varphi \in C([-h,0],\mathbb{R}^n)$ $\eta \in \text{NBV}([0,h],\mathbb{R}^{n*})$	$\langle \eta, \varphi \rangle = \int_0^h d\eta(\theta)\phi(-\theta)$
X $X^\odot$	$\varphi \in C([-h,0],\mathbb{R}^n)$ $(c,g) \in \mathbb{R}^{n*} \times L^1([0,h],\mathbb{R}^{n*})$	$\langle (c,g), \varphi \rangle = c\phi(0) + \int_0^h g(\theta)\varphi(-\theta)\, d\theta$
$X^\odot$ $X^{\odot\star}$	$(c,g) \in \mathbb{R}^{n*} \times L^1([0,h],\mathbb{R}^{n*})$ $(\alpha,\psi) \in \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$	$\langle (\alpha,\psi),(c,g) \rangle = c\alpha + \int_0^h g(\theta)\psi(-\theta)\, d\theta$

- For $w \in \mathbb{R}^n$ define

$$wr^{\odot\star} = (w,0) \in X^{\odot\star}$$

Linear periodic DDEs

- Consider the linearization of the DDE around a *periodic solution* $\gamma(t) = \gamma(t + T_0)$:

$$\text{LDDE} \quad \begin{cases} \dot{y}(t) = L(t)y_t, & t \geq s, \\ y_s = \varphi, & \varphi \in X, \end{cases}$$

where $L(t) = DF(\gamma_t)$ for $t \in \mathbb{R}$.

- There is a one-to-one correspondence between solutions of LDDE and the time-dependent abstract integral equation (AIE)

$$u(t) = T_0(t-s)\varphi + j^{-1}\int_s^t T_0^{\odot\star}(s-\tau)B(\tau)u(\tau)d\tau, \quad \varphi \in X$$

with $t \geq s$. Here $B(t)\varphi = [L(t)\varphi]r^{\odot\star}$ for $\varphi \in X$, and the integral is the weak* Riemann integral.

- The AIE defines an *evolutionary system* $\{U(t,s)\}_{t\geq s}$ on X, so that

$$U(t+T, s+T) = U(t,s), \quad U(s+kT, s) = U(s+T, s)^k$$

and

$$u(t) = U(t,s)\varphi, \;\; t \geq s.$$

- Riesz's representation implies

$$L(t)\varphi = \int_0^h d_2\zeta(t,\theta)\varphi(-\theta) = \langle \zeta(t,\cdot),\varphi\rangle, \quad t\in\mathbb{R},\ \varphi\in X,$$

where $\zeta:\mathbb{R}\times[0,h]\to\mathbb{R}^{n\times n}$ is such that $\zeta(t,\cdot)\in \mathrm{NBV}([0,h],\mathbb{R}^{n\times n})$ and T_0-periodic in the first component.

- We define $U(t,s)$, $U^\star(t,s)$, $U^\odot(t,s)$, and $U^{\odot\star}(t,s)$ as usual.
- For their respective infinitesimal generators $A(\tau), A^\star(\tau), A^\odot(\tau)$, and $A^{\odot\star}(\tau)$, we get

$$\begin{aligned}
&\mathscr{D}(A(\tau)) = \left\{\varphi\in C^1([-h,0],\mathbb{R}^n) : \varphi'(0) = L(\tau)\varphi\right\}, \quad A(\tau)\varphi = \varphi'\\
&\mathscr{D}(A^\star(\tau)) = \mathscr{D}(A_0^\star), \quad A^\star(\tau)f = f' + f(0^+)\zeta(\tau,\cdot)\\
&\mathscr{D}(A^\odot(\tau)) = \{(c,g)\in\mathscr{D}(A_0^\star) : g + c\zeta(\tau,\cdot)\in X^\odot\}, \quad A^\odot(\tau)(c,g) = g + c\zeta(\tau,\cdot)\\
&\mathscr{D}(A^{\odot\star}(\tau)) = \mathscr{D}(A_0^{\odot\star}), \quad A^{\odot\star}(\tau)(\alpha,\varphi) = (L(\tau)\varphi,\varphi').
\end{aligned}$$

- Note that

$$A^{\odot\star}(\tau)j\varphi = A_0^{\odot\star}j\varphi + B(\tau)\varphi = A_0^{\odot\star}j\varphi + [DF(\gamma_\tau)\varphi]r^{\odot\star}$$

for all $\varphi\in X$

Center Manifold Theorem for DDEs

Let $X_0(s) \subset X$ denote the (n_0+1)-dimensional *center subspace* for $U(s+T_0, s)$ defined by the direct sum of all its generalized eigenspaces with a Floquet multiplier on the unit circle and let $X_0 := \{(s, y_0) \in \mathbb{R} \times X : y_0 \in X_0(s)\}$ denote the *center fiber bundle.*

Theorem (Lentjes et al., 2023)

Consider a DDE

$$\begin{cases} \dot{x}(t) = F(x_t), & t \geq 0, \\ x_0 = \varphi, & \varphi \in X. \end{cases}$$

with a C^{k+1}-smooth $F : X \to \mathbb{R}^n$ for some finite $k \geq 1$. Let γ_t be a T_0-periodic solution of this system having $n_0 + 1 \geq 2$ Floquet multipliers satisfying $|\mu| = 1$. Then there exists a locally defined T_0-periodic C^k-smooth (n_0+1)-dimensional invariant manifold $\mathcal{W}_0^c \subset X$ defined around $\Gamma = \{u \in X : u = \gamma_t,\ t \in [0, T_0]\}$ and tangent to the center fiber bundle X_0. □

Moreover, a sufficiently smooth system

$$\begin{cases} \dot{x}(t) = F(x_t, \alpha), & t \geq 0,\ \alpha \in \mathbb{R}^m, \\ x_0 = \varphi, & \varphi \in X, \end{cases}$$

such that $F(\varphi, 0)$ coincides with $F(\varphi)$, has a locally defined periodic smooth (n_0+1)-dimensional invariant manifold $\mathcal{W}_\alpha^c$ that is a smooth continuation of $\mathcal{W}_0^c$.

Critical normal form coefficients for DDEs

- The solution $u(t) = x_t \in \mathscr{W}_0^c$ satisfies the abstract ODE

$$\frac{d}{dt} ju(t) = A_0^{\odot\star} ju(t) + G(u(t)),$$

where $G \in C^{k+1}(X, X^{\odot\star})$ is defined by

$$G(\varphi) := [F(\varphi)]\, r^{\odot\star}, \quad \varphi \in X.$$

- Taylor expansion and multilinear forms

$$\begin{aligned} G(\gamma_\tau + \psi(\tau)) &= G(\gamma_\tau) + [DF(\gamma_\tau)\psi(\tau)]\, r^{\odot\star} \\ &+ \frac{1}{2} B(\tau; \psi(\tau), \psi(\tau))\, r^{\odot\star} + \frac{1}{6} C(\tau; \psi(\tau), \psi(\tau), \psi(\tau))\, r^{\odot\star} + O(\|\psi\|^4) \end{aligned}$$

where

$$\begin{aligned} B(\tau; \phi(\tau), \psi(\tau)) &= D^2 F(\gamma_\tau)(\phi(\tau), \psi(\tau)) \\ C(\tau; \phi(\tau), \psi(\tau), \zeta(\tau)) &= D^3 F(\gamma_\tau)(\phi(\tau), \psi(\tau), \zeta(\tau)) \end{aligned}$$

Fredholm-like technique

- Let $C_{T_0}(\mathbb{R}, Y)$ be the space of T_0-periodic continuous Y-valued functions. Define

(1) $\mathscr{A} : \mathscr{D}(\mathscr{A}) \to C_{T_0}(\mathbb{R}, X)$ by

$$\mathscr{D}(\mathscr{A}) = \left\{\varphi \in C^1_{T_0}(\mathbb{R}, X) : \varphi(\tau) \in \mathscr{D}(A(\tau)), \forall \tau \in \mathbb{R}\right\}, \mathscr{A}\varphi = (\tau \mapsto A(\tau)\varphi(\tau) - \dot{\varphi}(\tau))$$

(2) $\mathscr{A}^\star : \mathscr{D}(\mathscr{A}^\star) \to C_{T_0}(\mathbb{R}, X^\star)$ by

$$\mathscr{D}(\mathscr{A}^\star) = \left\{\varphi^\star \in C^1_{T_0}(\mathbb{R}, X^\star) : \varphi^\star(\tau) \in \mathscr{D}(A^\star(\tau)), \forall \tau \in \mathbb{R}\right\},$$
$$\mathscr{A}^\star\varphi^\star = (\tau \mapsto A^\star(\tau)\varphi^\star(\tau) + \dot{\varphi}^\star(\tau))$$

(3) $\mathscr{A}^\odot : \mathscr{D}(\mathscr{A}^\odot) \to C_{T_0}(\mathbb{R}, X^\odot)$ by

$$\mathscr{D}(\mathscr{A}^\odot) = \left\{\varphi^\odot \in C^1_{T_0}(\mathbb{R}, X^\odot) : \varphi^\odot(\tau) \in \mathscr{D}(A^\odot(\tau)), \forall \tau \in \mathbb{R}\right\}, \ \mathscr{A}^\odot\varphi^\odot = \mathscr{A}^\star\varphi^\odot,$$

(4) $\mathscr{A}^{\odot\star} : \mathscr{D}(\mathscr{A}^{\odot\star}) \to C_{T_0}(\mathbb{R}, X^{\odot\star})$ by

$$\mathscr{D}(\mathscr{A}^{\odot\star}) = \left\{\varphi^{\odot\star} \in C^1_{T_0}(\mathbb{R}, X^{\odot\star}) : \varphi^{\odot\star}(\tau) \in \mathscr{D}(A^{\odot\star}(\tau)) \forall \tau \in \mathbb{R}\right\},$$
$$\mathscr{A}^{\odot\star}\varphi^{\odot\star} = (\tau \mapsto A^{\odot\star}(\tau)\varphi^{\odot\star}(\tau) - \dot{\varphi}^{\odot\star}(\tau))$$

- The linear operators $\mathscr{A}, \mathscr{A}^\star, \mathscr{A}^\odot, \mathscr{A}^{\odot\star}$ are unbounded non-closed but closable. The spectra of all these operators coincide with the set of all Floquet exponents
- Introduce the *pairings*

$$\langle \varphi^\star, \varphi \rangle_{T_0} := \int_0^{T_0} \langle \varphi^\star(\tau), \varphi(\tau) \rangle d\tau, \quad \langle \varphi^{\odot\star}, \varphi^\odot \rangle_{T_0} := \int_0^{T_0} \langle \varphi^{\odot\star}(\tau), \varphi^\odot(\tau) \rangle d\tau$$

for $(\varphi^\star, \varphi) \in C_{T_0}(\mathbb{R}, X^\star) \times C_{T_0}(\mathbb{R}, X)$ and $(\varphi^{\odot\star}, \varphi^\odot) \in C_{T_0}(\mathbb{R}, X^{\odot\star}) \times C_{T_0}(\mathbb{R}, X^\odot)$

- The pairings are nondegenerate and it holds that

$$\langle \mathscr{A}^\star \varphi^\star, \varphi \rangle_{T_0} = \langle \varphi^\star, \mathscr{A} \varphi \rangle_{T_0}, \quad \langle \mathscr{A}^{\odot\star} \varphi^{\odot\star}, \varphi^\odot \rangle_{T_0} = \langle \varphi^{\odot\star}, \mathscr{A}^\odot \varphi^\odot \rangle_{T_0}$$

for all $\varphi \in \mathscr{D}(\mathscr{A}), \varphi^\star \in \mathscr{D}(\mathscr{A}^\star), \varphi^\odot \in \mathscr{D}(\mathscr{A}^\odot)$ and $\varphi^{\odot\star} \in \mathscr{D}(\mathscr{A}^{\odot\star})$.

Theorem (Fredholm Solvability)

If the equation

$$\mathscr{A}^{\odot\star}(v_0, v) = (w_0, w) \in C_{T_0}(\mathbb{R}, X^{\odot\star})$$

has a solution $(v_0, v) \in \mathscr{D}(\mathscr{A}^{\odot\star})$, *then* $\langle (w_0, w), \varphi^\odot \rangle_{T_0} = 0$, *where* $\varphi^\odot$ *is an eigenfunction of* $\mathscr{A}^\star$ *corresponding to zero eigenvalue. If zero eigenvalue is simple, the bordered linear system*

$$\mathscr{A}^{\odot\star}(v_0, v) = (w_0, w), \quad \langle (v_0, v), \varphi^\odot \rangle_{T_0} = 0, \tag{1}$$

has a unique solution $(v_0, v) =: (\mathscr{A}^{\odot\star})^{\mathrm{INV}}(w_0, w)$.

Fold (LPC): $\mu_1 = \mu_n = 1$ (double non-semisimple)

- Critical center manifold $\mathcal{W}_0^c$:

$$u = \mathcal{H}(\tau,\xi) \quad \tau \in [0, T_0],\ \xi \in \mathbb{R},$$

where

$$\mathcal{H}(\tau,\xi) = \gamma_\tau + \xi\varphi_1(\tau) + \frac{1}{2}h_2(\tau)\xi^2 + O(\xi^3)$$

with $\varphi_1(T_0) = \varphi_1(0),\ h_2(T_0) = h_2(0)$.

- Critical periodic normal form on $\mathcal{W}_0^c$:

$$\begin{cases} \dfrac{d\tau}{dt} & = & 1 + \xi + a\xi^2 + O(\xi^3), \\ \dfrac{d\xi}{dt} & = & b\xi^2 + O(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$, while the $O(\xi^3)$-terms are T_0-periodic in τ.

LPC: Generalized and adjoint eigenfunctions

- The eigenfunctions $\varphi_0(\tau) = \dot{\gamma}_\tau$ and $\varphi_1(\tau)$ solve

$$\begin{cases} \mathscr{A}\varphi_0 & = & 0 \\ \varphi_0(0) - \varphi_0(T_0) & = & 0 \end{cases} \text{ or } \begin{cases} \left(\frac{d}{d\tau} - A(\tau)\right)\varphi_0(\tau) & = & 0, \; \tau \in [0, T_0], \\ \varphi_0(0) - \varphi_0(T_0) & = & 0 \end{cases}$$

and

$$\begin{cases} \mathscr{A}\varphi_1 & = & \varphi_0 \\ \varphi_1(0) - \varphi_1(T_0) & = & 0 \end{cases} \text{ or } \begin{cases} \left(\frac{d}{d\tau} - A(\tau)\right)\varphi_1(\tau) + \varphi_0(\tau) & = & 0, \; \tau \in [0, T_0], \\ \varphi_1(0) - \varphi_1(T_0) & = & 0, \end{cases}$$

implying $\langle \varphi^\odot, \varphi_0 \rangle_{T_0} = 0$, where $\varphi^\odot$ satisfies

$$\begin{cases} \mathscr{A}^\star \varphi^\odot & = & 0, \\ \varphi^\odot(0) - \varphi^\odot(T_0) & = & 0, \\ \langle \varphi^\odot, \varphi_1 \rangle_{T_0} - 1 & = & 0. \end{cases} \text{ or } \begin{cases} \left(\frac{d}{d\tau} + A^\star(\tau)\right)\varphi^\odot(\tau) & = & 0, \; \tau \in [0, T_0], \\ \varphi^\odot(0) - \varphi^\odot(T_0) & = & 0, \\ \langle \varphi^\odot, \varphi_1 \rangle_{T_0} - 1 & = & 0. \end{cases}$$

LPC: Computation of b

- Homological equation

$$j\Big(\frac{\partial \mathcal{H}(\tau,\xi)}{\partial \tau}\dot{\tau} + \frac{\partial \mathcal{H}(\tau,\xi)}{\partial \xi}\dot{\xi}\Big) = A_0^{\odot\star} j(\mathcal{H}(\tau,\xi)) + G(\mathcal{H}(\tau,\xi))$$

- Collect

$$\begin{aligned}
\xi^0 &: \quad \frac{d}{d\tau} j\gamma_\tau = A_0^{\odot\star} j\gamma_\tau + G(\gamma_\tau),\\
\xi^1 &: \quad \Big(\frac{d}{d\tau} - A^{\odot\star}(\tau)\Big) j\varphi_1(\tau) = -j\dot{\gamma}_\tau,\\
\xi^2 &: \quad \Big(\frac{d}{d\tau} - A^{\odot\star}(\tau)\Big) jh_2(\tau) = B(\tau;\varphi_1(\tau),\varphi_1(\tau))r^{\odot\star} - 2aj\dot{\gamma}_\tau - 2j\dot{\varphi}_1(\tau) - 2bj\varphi_1(\tau).
\end{aligned}$$

- Since $\mathcal{N}\left(\frac{d}{d\tau} - A^{\odot\star}(\tau)\right) = \mathrm{span}\{\tau \mapsto j\dot{\gamma}_\tau\}$ in the subspace of T_0-periodic functions in $C^1([0,T_0],\mathbb{R}^n)$, the Fredholm solvability condition implies

$$b = \frac{1}{2}\int_0^{T_0} \langle B(\tau;\varphi_1(\tau),\varphi_1(\tau))r^{\odot\star} - 2A^{\odot\star}(\tau)j\varphi_1(\tau), \varphi^{\odot}(\tau)\rangle d\tau$$

Flip (PD): $\mu_1 = -1, \mu_n = 1$ (both simple)

- Critical center manifold $\mathscr{W}_0^c$:

$$u = \mathscr{H}(\tau, \xi), \quad \tau \in [0, 2T_0], \ \xi \in \mathbb{R},$$

where

$$\mathscr{H}(\tau, \xi) = \gamma_\tau + \xi\varphi(\tau) + \frac{1}{2}h_2(\tau)\xi^2 + \frac{1}{6}h_3(\tau)\xi^3 + O(\xi^4)$$

with $\varphi(T_0) = -\varphi(0),\ h_2(T_0) = h_2(0)$, and $h_3(T_0) = -h_3(0)$.

- Critical periodic normal form on $\mathscr{W}_0^c$:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + a\xi^2 + O(\xi^4), \\ \dfrac{d\xi}{dt} = c\xi^3 + O(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$, while the $O(\xi^4)$-terms are $2T_0$-periodic in τ.

PD: Eigenfunctions

- Eigenfunction corresponding to $\mu_1 = -1$:

$$\begin{cases} \mathscr{A}\varphi = 0 \\ \varphi(0)+\varphi(T_0) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \left(\frac{d}{d\tau} - A(\tau)\right)\varphi(\tau) = 0, \ \tau \in [0, T_0], \\ \varphi(0)+\varphi(T_0) = 0 \end{cases}$$

extended to $[T_0, 2T_0]$ by

$$\varphi(t+T_0) = -\varphi(t), \quad t \in [0, T_0].$$

- Adjoint eigenfunction corresponding to $\mu_1 = -1$:

$$\begin{cases} \mathscr{A}^\star\varphi^\odot = 0, \\ \varphi^\odot(0)+\varphi^\odot(T_0) = 0, \\ \langle\varphi^\odot, \varphi\rangle_{T_0} - 1 = 0. \end{cases} \quad \text{or} \quad \begin{cases} \left(\frac{d}{d\tau} + A^\star(\tau)\right)\varphi^\odot(\tau) = 0, \ \tau \in [0, T_0], \\ \varphi^\odot(0)+\varphi^\odot(T_0) = 0, \\ \langle\varphi^\odot, \varphi\rangle_{T_0} - 1 = 0. \end{cases}$$

extended to $[T_0, 2T_0]$ by

$$\varphi^\odot(t+T_0) = -\varphi^\odot(t), \quad t \in [0, T_0].$$

- Eigenfunction corresponding to $\mu_n = 1$ is $\varphi_0 = \dot{\gamma}_\tau$.

PD: Quadratic terms

Collect in the homological equation

$$
\begin{aligned}
\xi^0 &: \quad \frac{d}{d\tau} j\gamma_\tau = A_0^{\odot\star} j\gamma_\tau + G(\gamma_\tau),\\
\xi^1 &: \quad \left(\frac{d}{d\tau} - A^{\odot\star}(\tau)\right) j\varphi(\tau) = 0,\\
\xi^2 &: \quad \left(\frac{d}{d\tau} - A^{\odot\star}(\tau)\right) jh_2(\tau) = B(\tau;\varphi(\tau),\varphi(\tau))r^{\odot\star} - 2aj\dot{\gamma}_\tau.
\end{aligned}
$$

Since $\mathcal{N}\left(\frac{d}{d\tau} - A^{\odot\star}(\tau)\right) = \operatorname{span}\{\tau \mapsto j\dot{\gamma}_\tau\}$ in the subspace of T_0-periodic functions in $C^1_{T_0}(\mathbb{R}, X)$, the Fredholm solvability condition must hold:

$$
\int_0^{T_0} \langle B(\tau;\varphi(\tau),\varphi(\tau))r^{\odot\star} - 2aj\dot{\gamma}_\tau, \psi^\odot(\tau)\rangle d\tau = 0,
$$

where $\psi^\odot$ satisfies

$$
\left\{
\begin{aligned}
\mathcal{A}^\star \psi^\odot &= 0,\\
\psi^\odot(0) - \psi^\odot(T_0) &= 0,\\
\langle \psi^\odot, \varphi_0\rangle_{T_0} - 1 &= 0.
\end{aligned}
\right.
\quad \text{or} \quad
\left\{
\begin{aligned}
\left(\frac{d}{d\tau} + A^\star(\tau)\right)\psi^\odot(\tau) = 0 &= 0,\ \tau \in [0, T_0],\\
\psi^\odot(0) - \psi^\odot(T_0) &= 0,\\
\langle \psi^\odot, \varphi_0\rangle_{T_0} - 1 &= 0.
\end{aligned}
\right.
$$

PD: Computation of a and h_2

- The above Fredholm solvability condition implies

$$a = \frac{1}{2}\int_0^{T_0} \langle B(\tau;\varphi_1(\tau),\varphi_1(\tau))r^{\odot\star}, \psi^{\odot}(\tau)\rangle d\tau$$

- Define h_2 on $[0, T_0]$ as the unique solution to

$$\begin{cases} \dot{h}_2(\tau) - A(\tau)h_2(\tau) - B(\tau;\varphi(\tau),\varphi(\tau))r^{\odot\star} + 2aj\varphi_0 & = & 0, \\ h_2(0) - h_2(T_0) & = & 0, \\ \int_0^{T_0} \langle \psi^{\odot}(\tau), h_2(\tau)\rangle \, d\tau & = & 0. \end{cases}$$

and extend to $[T_0, 2T_0]$ by periodicity

$$h_2(t+T_0) = h_2(t), \quad t \in [0, T_0].$$

PD: Computation of c

Cubic terms: ξ^3

$$\left(\frac{d}{d\tau}-A^{\odot\star}(\tau)\right)jh_3(\tau)=\left[C(\tau;\varphi(\tau),\varphi(\tau),\varphi(\tau))+3B(\tau;\varphi(\tau),h_2(\tau))\right]r^{\odot\star}-6aj\dot{\varphi}(\tau)-6cj\varphi(\tau).$$

Since $\mathcal{N}\left(\frac{d}{d\tau}-A^{\odot\star}(\tau)\right)=\operatorname{span}\{\tau\mapsto j\varphi(\tau)\}$ in the subspace of T_0-*anti*periodic functions in $C^1_{T_0}(\mathbb{R},X)$, the Fredholm solvability condition

$$\int_0^{T_0}\langle[C(\tau;\varphi(\tau),\varphi(\tau),\varphi(\tau))+3B(\tau;\varphi(\tau),h_2(\tau))]r^{\odot\star}-6aj\dot{\varphi}(\tau)-6cj\varphi(\tau),\varphi^{\odot}(\tau)\rangle d\tau=0$$

must hold, which can be solved for c as

$$c=\frac{1}{3}\int_0^{T_0}\langle[C(\tau;\varphi(\tau),\varphi(\tau),\varphi(\tau))+3B(\tau;\varphi(\tau),h_2(\tau))]r^{\odot\star}-6aA^{\odot\star}(\tau)j\varphi(\tau),\varphi^{\odot}(\tau)\rangle d\tau.$$

Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}, \mu_n = 1, \ (e^{ik\theta_0} \neq 1, k = 1,2,3,4)$

- Critical center manifold $\mathscr{W}_0^c$:

$$x = \mathscr{H}(\tau,\xi,\bar{\xi}), \ \tau \in [0,T_0], \ \xi \in \mathbb{C}$$

where

$$\begin{aligned}\mathscr{H}(\tau,\xi,\bar{\xi}) &= \gamma_\tau + \xi\varphi(\tau) + \bar{\xi}\bar{\varphi}(\tau) + \frac{1}{2}h_{20}(\tau)\xi^2 + h_{11}(\tau)\xi\bar{\xi} + \frac{1}{2}h_{02}(\tau)\bar{\xi}^2 \\ &+ \frac{1}{6}h_{30}(\tau)\xi^3 + \frac{1}{2}h_{21}(\tau)\xi^2\bar{\xi} + \frac{1}{2}h_{12}(\tau)\xi\bar{\xi}^2 + \frac{1}{6}h_{03}(\tau)\bar{\xi}^3 + O(|\xi|^4)\end{aligned}$$

with $v(T_0) = v(0)$ and $h_{jk}(T_0) = h_{jk}(0)$.

- Critical periodic normal form on $\mathscr{W}_0^c$:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + a|\xi|^2 + O(|\xi|^4), \\ \dfrac{d\xi}{dt} = \dfrac{i\theta_0}{T_0}\xi + d\xi|\xi|^2 + O(|\xi|^4), \end{cases}$$

where $a \in \mathbb{R}, d \in \mathbb{C}$, and the $O(|\xi|^4)$-terms are T_0-periodic in τ.

NS: Complex eigenfunctions

- Eigenfunction corresponding to $\mu_1 = e^{i\theta_0}$:

$$\begin{cases} \mathscr{A}\varphi + \frac{i\theta_0}{T_0}\varphi &= 0 \\ \varphi(0) - \varphi(T_0) &= 0 \end{cases} \quad \text{or} \quad \begin{cases} \left(\frac{d}{d\tau} - A(\tau)\right)\varphi(\tau) + \frac{i\theta_0}{T_0}\varphi(\tau) &= 0, \ \tau \in [0, T_0], \\ \varphi(0) - \varphi(T_0) &= 0 \end{cases}$$

- Adjoint eigenfunction corresponding to $\mu_1 = e^{i\theta_0}$:

$$\begin{cases} \mathscr{A}^\star\varphi^\odot - \frac{i\theta_0}{T_0}\varphi^\odot &= 0, \\ \varphi^\odot(0) - \varphi^\odot(T_0) &= 0, \\ \langle\varphi^\odot, \varphi\rangle_{T_0} - 1 &= 0. \end{cases} \quad \text{or} \quad \begin{cases} \left(\frac{d}{d\tau} + A^\star(\tau)\right)\varphi^\odot(\tau) - \frac{i\theta_0}{T_0}\varphi^\odot(\tau) &= 0, \ \tau \in [0, T_0], \\ \varphi^\odot(0) - \varphi^\odot(T_0) &= 0, \\ \langle\varphi^\odot, \varphi\rangle_{T_0} - 1 &= 0. \end{cases}$$

- Eigenfunction corresponding to $\mu = 1$ is $\varphi_0 = \dot{\gamma}_\tau$.

Homological equation:

$$j\left(\frac{\partial\mathscr{H}(\tau,\xi,\bar{\xi})}{\partial\tau}\dot{\tau} + \frac{\partial\mathscr{H}(\tau,\xi,\bar{\xi})}{\partial\xi}\dot{\xi} + \frac{\partial\mathscr{H}(\tau,\xi,\bar{\xi})}{\partial\bar{\xi}}\dot{\bar{\xi}}\right) = A_0^{\odot\star} j(\mathscr{H}(\tau,\xi,\bar{\xi})) + G(\mathscr{H}(\tau,\xi,\bar{\xi})).$$

NS: Quadratic terms

- ξ^2:

$$\left(\frac{d}{d\tau} - A^{\odot\star}(\tau) + \frac{2i\theta_0}{T_0}\right) jh_{20}(\tau) = B(\tau;\varphi(\tau),\varphi(\tau))\, r^{\odot\star}$$

Since $e^{2i\theta_0}$ is not a multiplier of the critical cycle, the BVP

$$\begin{cases} \left(\frac{d}{d\tau} - A^{\odot\star}(\tau) + \frac{2i\theta_0}{T_0}\right) jh_{20}(\tau) - B(\tau;\varphi(\tau),\varphi(\tau))\, r^{\odot\star} &= 0, \\ h_{20}(0) - h_{20}(T_0) &= 0. \end{cases}$$

has a unique solution h_{20} on $[0, T_0]$.

- $|\xi|^2$:

$$\left(\frac{d}{d\tau} - A^{\odot\star}(\tau)\right) jh_{11}(\tau) = B(\tau;\varphi(\tau),\bar{\varphi}(\tau))\, r^{\odot\star} - aj\dot{\gamma}_\tau.$$

Here

$$\mathcal{N}\left(\frac{d}{d\tau} - A^{\odot\star}(\tau)\right) = \mathrm{span}\{\tau \mapsto j\dot{\gamma}_\tau\}$$

in the subspace of T_0-periodic functions in $C^1_{T_0}(\mathbb{R}, X)$.

NS: Computation of a and h_{11}

- Define $\psi^{\odot}$ as the unique solution of

$$\begin{cases} \mathscr{A}^{\star}\psi^{\odot} &= 0, \\ \psi^{\odot}(0)-\psi^{\odot}(T_0) &= 0, \\ \langle \psi^{\odot},\varphi_0\rangle_{T_0}-1 &= 0. \end{cases} \quad \text{or} \quad \begin{cases} \left(\frac{d}{d\tau}+A^{\star}(\tau)\right)\psi^{\odot}(\tau)=0 &= 0,\ \tau\in[0,T_0], \\ \psi^{\odot}(0)-\psi^{\odot}(T_0) &= 0, \\ \langle \psi^{\odot},\varphi_0\rangle_{T_0}-1 &= 0. \end{cases}$$

- Fredholm solvability:

$$a=\int_0^{T_0}\langle B(\tau;\varphi(\tau),\bar{\varphi}(\tau))r^{\odot\star},\psi^{\odot}\rangle\, d\tau$$

- Then find h_{11} on $[0,T_0]$ from the BVP

$$\begin{cases} \left(\frac{d}{d\tau}-A^{\odot\star}(\tau)\right)jh_{11}(\tau)-B(\tau;\varphi(\tau),\bar{\varphi}(\tau))r^{\odot\star}+aj\varphi_0(\tau) &= 0, \\ h_{11}(0)-h_{11}(T_0) &= 0, \\ \langle \psi^{\odot},h_{11}\rangle_{T_0} &= 0. \end{cases}$$

NS: Computation of d

- Cubic terms: $\xi^2\bar{\xi}$

$$\begin{aligned}\left(\frac{d}{d\tau}-A^{\odot\star}(\tau)+\frac{i\theta_0}{T_0}\right)jh_{21}(\tau) &= [2B(\tau;h_{11}(\tau),\varphi(\tau))+B(\tau;h_{20}(\tau),\overline{\varphi}(\tau))\\ &+ C(\tau;\varphi(\tau),\varphi(\tau),\overline{\varphi}(\tau))]\,r^{\odot\star}-2aj\dot{\varphi}(\tau)\;-\;2dj\varphi(\tau),\end{aligned}$$

- Fredholm solvability condition implies

$$\begin{aligned} d &= \frac{1}{2}\int_0^{T_0}\langle C(\tau;\varphi(\tau),\varphi(\tau),\overline{\varphi}(\tau))\,r^{\odot\star},\varphi^{\odot}(\tau)\rangle\,d\tau\\ &+\frac{1}{2}\int_0^{T_0}\langle[2B(\tau;h_{11}(\tau),\varphi(\tau))+B(\tau;h_{20}(\tau),\overline{\varphi}(\tau))]\,r^{\odot\star},\varphi^{\odot}(\tau)\rangle\,d\tau\\ &-a\int_0^{T_0}\langle A^{\odot\star}(\tau)j\varphi(\tau),\varphi^{\odot}(\tau)\rangle\,d\tau+\frac{ia\theta_0}{T_0}\end{aligned}$$

Numerical implementation

- Multilinear functions

  Introduce the linear evaluation operator $\Xi : X \to \mathbb{R}^{n\times(m+1)}$ as

$$\Xi\varphi := (\varphi(-\tau_0), \dots, \varphi(-\tau_m))$$

  Then we have $D^r F(\gamma_\tau) : X^r \to \mathbb{R}^n$ given by

$$D^r F(\gamma_\tau)(\varphi_1, \dots, \varphi_r) = \sum_{j_1,\dots,j_r=1}^{n} \sum_{k_1,\dots,k_r=0}^{m} D^r_{j_1 k_1,\dots,j_r k_r} f_\gamma(\tau) \Phi_{1,j_1 k_1} \cdots \Phi_{r,j_r k_r}$$

  where $\Phi := \Xi\varphi$ for all $\varphi \in X$ and $f_\gamma(\tau) := f(\gamma(\tau), \gamma(\tau-\tau_1), \gamma(\tau-\tau_2), \dots, \gamma(\tau-\tau_m))$

- Linear inhomogeneous DDE:

$$\begin{cases} \dot{y}(t) - g(y(t), y(t-\tau_1), \cdots, y(t-\tau_m)) &= h(t, t-\tau_1, \cdots, t-\tau_m), \quad t \in [0, T_0] \\ y_{T_0} - y_0 &= 0 \end{cases}$$

$$g(y(t), y(t-\tau_1), \cdots, y(t-\tau_m)) = \Lambda_0(t) y(t) + \sum_{j=1}^{m} \Lambda_j(t) y(t-\tau_j)$$

  with T_0-periodic $t \mapsto \Lambda_j(t) \in \mathbb{R}^{n\times n}$ and $t \mapsto h(t, t-\tau_1, \cdots, t-\tau_m)$.

- Mesh points: $0 < t_1 < t_2 < \cdots < t_L = T_0$
  Basis points: $t_{i,j} = t_i + \frac{j}{M}(t_{i+1} - t_i), i = 0, 1, \ldots, L-1, j = 1, \ldots, M-1$
  Approximation:
  $$y(t) = \sum_{j=0}^{M} y^{i,j} P_{i,j}(t), t \in [t_i, t_{i+1}],$$
  where $P_{i,j}(t)$ are the Lagrange basis polynomials.
- Defining system (with $n(LM+1)+1$ scalar equations)
  $$\begin{cases} \dot{y}\left(c_{i,j}\right) - g\left(y\left(c_{i,j}\right), y\left(\left(c_{i,j} - \tau_1\right) \bmod T_0\right), \ldots, y\left(\left(c_{i,j} - \tau_m\right) \bmod T_0\right)\right) = \\ \qquad h\left(c_{i,j}, \left(c_{i,j} - \tau_1\right) \bmod T_0, \ldots, \left(c_{i,j} - \tau_m\right) \bmod T_0\right) \\ \qquad y^{0,0} - y^{L,0} = 0 \end{cases}$$
  where $c_{i,j}$ are the roots of the M-th degree Gauss-Legendre polynomial on $[-1, 1]$ translated to $[t_i, t_{i+1}]$.
- Unknowns $\left(\left\{y^{i,j}\right\}_{j=1,\ldots,M}^{i=0,1,\ldots,L-1}, y^{L,0}\right) \in \mathbb{R}^{n(LM+1)}$. When the solution is not unique, an extra bordering condition is appended.

LPC computations

- The generalized eigenfunction $\psi = \varphi_1$ satisfies the equation

$$\left(\frac{d}{d\tau} - A^{\odot\star}(\tau)\right) j\psi(\tau) = -j\dot{\gamma}_\tau$$

that is equivalent to

$$\begin{cases} \frac{d}{d\tau}\psi(\tau)(0) - \langle \zeta(\tau,\cdot), \psi(\tau)\rangle & = & -\dot{\gamma}(\tau) \\ \frac{\partial}{\partial\tau}\psi(\tau)(\theta) - \frac{\partial}{\partial\theta}\psi(\tau)(\theta) & = & -\dot{\gamma}(\tau+\theta) \\ \psi(T_0)(\theta) - \psi(0)(\theta) & = & 0, \quad \theta \in [-h, 0] \end{cases}$$

- The general solution of the second component $\psi(\tau)(\theta) = \tilde{\psi}(\tau+\theta) + \theta\dot{\gamma}(\tau+\theta)$
- The first component, together with the periodicity condition, yields

$$\begin{cases} \frac{d}{d\tau}\tilde{\psi}(\tau) - \langle \zeta(\tau,\cdot), \tilde{\psi}\rangle = -\dot{\gamma}(\tau) + \langle \zeta(\tau,\cdot), \theta \mapsto \theta\dot{\gamma}(\theta)\rangle \\ \tilde{\psi}(T_0) - \tilde{\psi}(0) = 0 \end{cases}$$

which is a linear inhomogeneous DDE considered above.

- Quadratic coefficient of the center manifold

$$\left(\frac{d}{d\tau}-A^{\odot\star}(\tau)\right)jh_2(\tau)=B(\tau;\psi(\tau),\psi(\tau))r^{\odot\star}-2aj\dot{\gamma}_\tau-2j\dot{\psi}(\tau)-2bj\psi(\tau).$$

that is equivalent to

$$\begin{cases} \frac{d}{d\tau}h_2(\tau)(0)-\langle\zeta(\tau,\cdot),h_2(\tau)\rangle &= B(\tau;\psi(\tau),\psi(\tau))-2a\dot{\gamma}(\tau)-2\dot{\tilde{\psi}}(\tau)-2b\tilde{\psi}(\tau) \\ \frac{\partial}{\partial\tau}h_2(\tau)(\theta)-\frac{\partial}{\partial\theta}h_2(\tau)(\theta) &= -2a\dot{\gamma}(\tau+\theta)-2\dot{\psi}(\tau)(\theta)-2b\psi(\tau)(\theta) \\ h_2(T_0)(\theta)-h_2(0)(\theta) &= 0,\quad \theta\in[-h,0] \end{cases}$$

- The general solution of the second component

$$h_2(\tau)(\theta)=\tilde{h}_2(\tau+\theta)+2a\theta\dot{\gamma}(\tau+\theta)+2\theta\dot{\tilde{\psi}}(\tau+\theta)+\theta^2\ddot{\gamma}(\tau+\theta)+b\left(2\theta\tilde{\psi}(\tau+\theta)+\theta^2\dot{\gamma}(\tau+\theta)\right)$$

- The first component, together with the periodicity condition, yields

$$\begin{cases} \frac{d}{d\tau}\tilde{h}_2(\tau)-\left\langle\zeta(\tau,\cdot),\tilde{h}_2\right\rangle &= B(\tau;\psi(\tau),\psi(\tau))-2a\dot{\gamma}(\tau)+2a\left\langle\zeta(\tau,\cdot),\theta\mapsto\theta\dot{\gamma}(\theta)\right\rangle \\ & \quad -2\dot{\tilde{\psi}}(\tau)+\left\langle\zeta(\tau,\cdot),\theta\mapsto 2\theta\dot{\tilde{\psi}}(\tau+\theta)+\theta^2\ddot{\gamma}(\tau+\theta)\right\rangle \\ & \quad -2b\left(\tilde{\psi}(\tau)+\left\langle\zeta(\tau,\cdot),\theta\mapsto 2\theta\tilde{\psi}(\tau+\theta)+\theta^2\dot{\gamma}(\tau+\theta)\right\rangle\right) \\ \tilde{h}_2(T_0)-\tilde{h}_2(0) &= 0 \end{cases}$$

- Let p be the null vector of the *discretization* of the right-hand side, then

$$b=\frac{\left\langle p,B(\tau;\psi(\tau),\psi(\tau))-2\dot{\tilde{\psi}}(\tau)+\left\langle\zeta(\tau,\cdot),\theta\mapsto 2\theta\dot{\tilde{\psi}}(\tau+\theta)+\theta^2\ddot{\gamma}(\tau+\theta)\right\rangle\right\rangle}{2\left\langle p,\tilde{\psi}(\tau)+\left\langle\zeta(\tau,\cdot),\theta\mapsto 2\theta\tilde{\psi}(\tau+\theta)+\theta^2\dot{\gamma}(\tau+\theta)\right\rangle\right\rangle}$$

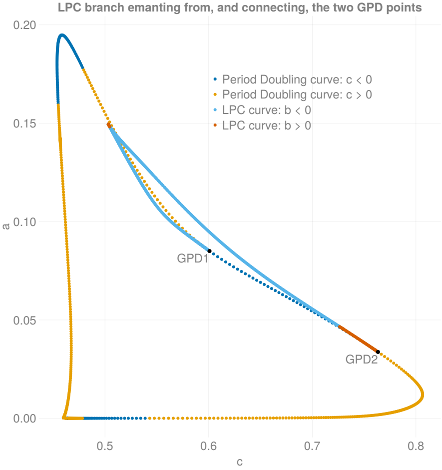
A lumped model of neocortex with two delays

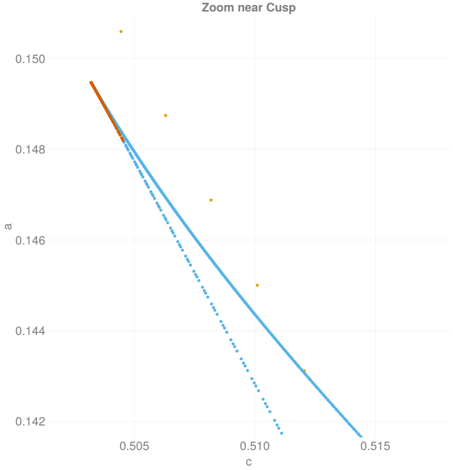
- In [Visser2012], the following model of two interacting layers of neurons is considered

$$\begin{cases} \dot{x}_1(t) & = & -x_1(t) - ag(bx_1(t-\tau_1)) + cg(dx_2(t-\tau_2)) \\ \dot{x}_2(t) & = & -x_2(t) - ag(bx_2(t-\tau_1)) + cg(dx_1(t-\tau_2)) \end{cases}$$

- $g(x) = (\tanh(x-1) + \tanh(1))\cosh(1)^2$
- We set $b = 2.0$, $d = 1.2$, $\tau_1 = 11.6$, $\tau_2 = 20.3$
- The unfolding parameters are a and c

Numerical bifurcation diagram





Active Control System

- In [Peng2013], the following active control system

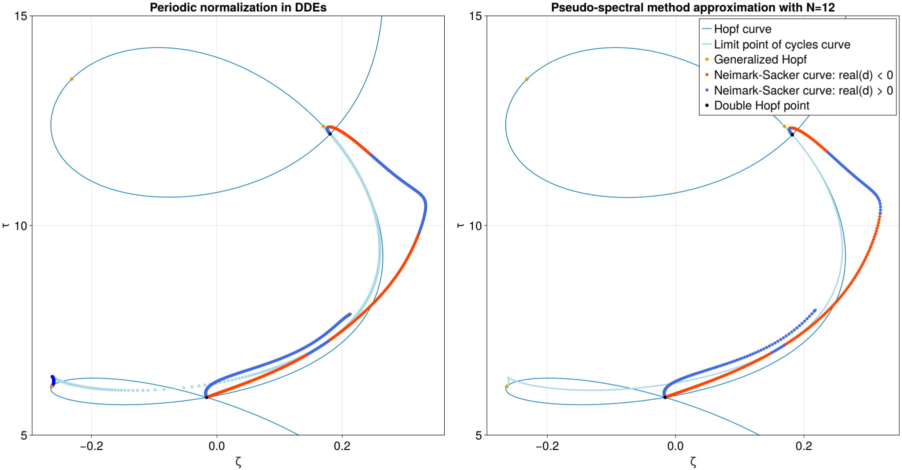
$$\begin{cases} \dot{x}(t) = \tau y(t) \\ \dot{y}(t) = \tau \left(-x(t) - g_u x(t-1) - 2\zeta y(t) - g_v y(t-1) + \beta x^3(t-1)\right) \end{cases}$$

  which is used to control the response of structures to internal or external excitation
- The parameters

$$g_u = 0.1, \quad g_v = 0.52, \quad \beta = 0.1$$

  are fixed
- The unfolding parameters are ζ and τ

Numerical bifurcation diagram



Open questions

- Switching to codim 1 bifurcations of cycles at codim 2 bifurcations (ODEs & DDEs). The critical normal forms for ODEs [MATCONT]

De Witte, V., Della Rossa, F., Govaerts, W., and Kuznetsov, Yu.A.
Numerical periodic normalization for codim 2 bifurcations of limit cycles: Computational formulas, numerical implementation, and examples.
SIAM J. Appl. Dyn. Syst. **12** (2013), 722-788

De Witte, V., Govaerts, W., Kuznetsov, Yu.A., and Meijer, H. G. E.
Analysis of bifurcations of limit cycles with Lyapunov exponents and numerical normal forms.
Physica D **269** (2014), 126-141

- General context of sun-star calculus for other classes of delay equations: difficulties with smoothness.
- Generalizing to abstract DDEs (neural fields): No periodic center manifold available yet, but promising work by [Janssens, 2020].