Homoclinic saddle to saddle-focus transitions in 3D and 4D ODEs

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Joint work with Manu Kalia & Hil Meijer
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General Methodology

Combine analytic and numerical bifurcation methods!
Homoclinic orbits

- Consider a smooth family of smooth ODE systems
  \[ \dot{u} = F(u, p), \quad u \in \mathbb{R}^n, \; p \in \mathbb{R}^m \]
- Suppose that at \( p_0 \in \mathbb{R}^m \) there is a hyperbolic equilibrium \( u_0 \in \mathbb{R}^n \).
- Assume that \( u_0 \) has a primary homoclinic orbit \( \Gamma_0 \), i.e. for a non-equilibrium solution \( t \mapsto u(t) \) holds
  \[ \lim_{t \to \pm \infty} u(t) = u_0 \]
- Generically, the corresponding \( p_0 \)-values occupy a codimension 1 homoclinic bifurcation manifold in \( \mathbb{R}^m \).
• Leading eigenvalues of $u_0$ (possibly, after reversing time):

\[ \begin{array}{c}
\text{(a) Saddle} \quad \text{(b) Saddle-focus} \quad \text{(c) Focus-focus}
\end{array} \]

- Saddle quantity (index):

\[
\sigma := \Re(\lambda^s) + \Re(\lambda^u) \quad \left(\nu := -\frac{\Re(\lambda^s)}{\Re(\lambda^u)}\right)
\]

where $\lambda^s$ and $\lambda^u$ are leading stable and unstable eigenvalues of $u_0$. 
Limit cycles due to codim 1 homoclinic bifurcations

<table>
<thead>
<tr>
<th></th>
<th>$\sigma &lt; 0$</th>
<th>$\sigma &gt; 0$</th>
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<tbody>
<tr>
<td><strong>Saddle</strong></td>
<td>one cycle</td>
<td>one cycle</td>
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<tr>
<td><strong>Saddle-focus</strong></td>
<td>one cycle</td>
<td>$\infty$ cycles</td>
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<tr>
<td><strong>Focus-focus</strong></td>
<td>$\infty$ cycles</td>
<td>$\infty$ cycles</td>
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L.P. Shilnikov (1934-2011)
Codim 2 saddle-focus to saddle transitions

2DL (double eigenvalue) Belyakov bifurcation:

3DL bifurcation:

\[ \mu_1 < 0 \]

\[ \mu_1 = 0 \]

\[ \mu_1 > 0 \]
2DL Belyakov bifurcation

L.A. Belyakov
The bifurcation set in a system with a homoclinic saddle curve

Yu.A. Kuznetsov, O. De Feo, and S. Rinaldi
Belyakov homoclinic bifurcations in a tritrophic food chain model

3D ODE system near the equilibrium \( x = 0 \):

\[
\begin{align*}
\dot{x}_1 &= \gamma(\mu)x_1 + x_2 + f_1(x, \mu) \\
\dot{x}_2 &= -\mu_1 x_1 + \gamma(\mu)x_2 + f_2(x, \mu) \\
\dot{x}_3 &= \beta(\mu)x_3
\end{align*}
\]

where \( \mu = (\mu_1, \mu_2) \in \mathbb{R}^2 \) and \( \lambda_{1,2}^s = \gamma(0) < 0, \lambda^u = \beta(0) > -\gamma(0) \).
Geometric setting

Homoclinic orbit at $\mu = 0$

Double homoclinic orbit
Model Poincaré maps

- 2D Model Poincaré Map on $\Sigma_s$:

\[
\begin{pmatrix}
  x_2 \\
  x_3
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 + \frac{A}{\sqrt{\mu_1}} x_2 x_3^\nu \sin\left(-\frac{\sqrt{\mu_1}}{\beta} \ln x_3 + \Theta\right) \\
  \mu_2 + \frac{C}{\sqrt{\mu_1}} x_2 x_3^\nu \sin\left(-\frac{\sqrt{\mu_1}}{\beta} \ln x_3\right)
\end{pmatrix}
\]

where $\nu = -\frac{\gamma}{\beta} < 1$ (chaotic case)

- 1D Model Poincaré Map:

\[
x_3 \rightarrow \mu_2 + \frac{C}{\sqrt{\mu_1}} x_3^\nu \sin\left(-\frac{\sqrt{\mu_1}}{\beta} \ln x_3\right)
\]
(Partial) bifurcation diagram ($\nu < 1$)

\[ LP_{n}^{(1)} \text{ and } PD_{n}^{(1)} : \mu_{2}^{(n)} = \frac{(-1)^{n}C}{e\gamma} \exp\left(\frac{\gamma\pi n}{\sqrt{\mu_{1}}}\right)(1 + o(\mu_{1})) \]

\[ Hom_{m}^{(2)} : \mu_{2}^{(m)} = \exp\left(-\frac{\beta\pi m}{\sqrt{\mu_{1}}}\right)(1 + o(\mu_{1})) \]
3DL bifurcation

M. Kalia, Yu.A. Kuznetsov, and H.G.E. Meijer

Homoclinic saddle to saddle-focus transitions in 4D systems

arXiv:1712.03212[submitted to Nonlinearity]

4D ODE $C^1$ orbitally equivalent system near the equilibrium $x = 0$:

\[
\begin{align*}
\dot{x}_1 &= \gamma(\mu)x_1 - x_2, \\
\dot{x}_2 &= x_1 + \gamma(\mu)x_2, \\
\dot{x}_3 &= (\gamma(\mu) - \mu_1)x_3, \\
\dot{x}_4 &= \beta(\mu)x_4,
\end{align*}
\]

where $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ and

\[
\Re(\lambda^s_{1,2,3}) = \gamma(0) < 0, \quad \lambda^u = \beta(0) > -\gamma(0)
\]

so that

\[
\nu = -\frac{\gamma}{\beta} < 1 \quad \text{(chaotic case)}
\]
Geometric setting

Homoclinic orbit

Double homoclinic orbit
Model Poincaré maps

• 3D Model Poincaré Map on $\Sigma_s$:

$$G: \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 1 + \alpha_1 x_1 x_4^\nu \cos \left( -\frac{1}{\beta} \ln x_4 + \phi_1 \right) + \alpha_2 x_2 x_4^{\nu+\mu_1/\beta} \\ 1 + \alpha_3 x_1 x_4^\nu \sin \left( -\frac{1}{\beta} \ln x_4 + \phi_2 \right) + \alpha_4 x_2 x_4^{\nu+\mu_1/\beta} \\ \mu_2 + C_1 x_1 x_4^\nu \sin \left( -\frac{1}{\beta} \ln x_4 \right) + C_2 x_2 x_4^{\nu+\mu_1/\beta} \end{pmatrix}$$

• 1D Model Poincaré Map:

$$F: x_4 \mapsto \mu_2 + C_1 x_4^\nu \sin \left( -\frac{1}{\beta} \ln x_4 \right) + C_2 x_4^{\nu+\mu_1/\beta}$$
Lemma (Bifurcations of 1D model map)

Generically, the model map $F$ with $\nu < 1$ has an infinite number of fold curves $LP_{n}^{(1)}$ accumulating to the half axis $\mu_2 = 0$ with $\mu_1 \geq 0$. Each curve resembles a ‘horn’ with a cusp point

$$CP_{n}^{(1)} : \left( \begin{array}{c} \mu_1^{(n)} \\ \mu_2^{(n)} \end{array} \right) = \left( \begin{array}{c} \frac{1}{4\pi n} \left[ \ln a + O\left(\frac{1}{n}\right) \right] \\ -e^{-\beta \nu (2\pi n + \theta + \phi)} \frac{\text{sign}(C_2) C_1}{\beta \nu \sqrt{1 + \beta^2 \nu^2}} a^{-(\theta + \phi)/4\pi n} + O\left(\frac{1}{\sqrt{n}}\right) \end{array} \right)$$

where

$$a = \frac{\beta^2 \nu^2}{1 + \beta^2 \nu^2} \frac{C_2^2}{C_1^2}, \quad \sin \phi = (1 + \beta^2 \nu^2)^{-1/2}, \quad \theta = \begin{cases} \pi/2, & \text{if } C_2 < 0, \\ 3\pi/2, & \text{if } C_2 > 0. \end{cases}$$

Moreover, there exists an infinite number of period-doubling curves $PD_{n}^{(1)}$ having – away from the cusp points $CP_{n}^{(1)}$ – the same asymptotic properties as the fold bifurcation curves $LP_{n}^{(1)}$. 
Primary LP and PD curves of 1D map $F$

$C_1 = 1.2, C_2 = 0.7$

$C_1 = 1.2, C_2 = \pm 0.7$

$\beta = 0.2, \quad \nu = 0.5$
Spring and saddle areas

Depending on \((C_1, C_2)\), curves \(PD^{(1)}_n\) could either be smooth or develop small loops around the corresponding cusp points.

\[ C_1 = 1.2, C_2 = 0.7 \text{ (Saddle area)} \]

\[ C_1 = 0.8, C_2 = -1.1 \text{ (Spring area)} \]
Theorem (Bifurcations of 3D model map)

Generically, the model map $G$ with $\nu < 1$ has an infinite number of fold and period-doubling curves accumulating to the half axis $\mu_2 = 0$ with $\mu_1 \geq 0$ and having the same asymptotic properties as the curves $LP_n^{(1)}$ and $PD_n^{(1)}$ above.

Moreover, there exists an infinite sequence of ‘parabolas’ $\text{Hom}_m^{(2)}$ that accumulate onto the half axis $\mu_2 = 0$ with $\mu_1 \geq 0$ and correspond to the double homoclinic orbits. The turning points of the parabolas are

$$
\begin{pmatrix}
\mu_1^{(m)} \\
\mu_2^{(m)}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{4\pi m} \left( \ln \left( \frac{C_2}{C_1} \right) + O \left( \frac{1}{m} \right) \right) \\
e^{-\beta(2\pi m+\theta)} \left( 1 + O \left( \frac{1}{m} \right) \right)
\end{pmatrix}
$$

where

$$
\theta = \begin{cases} 
\pi/2, & \text{if } C_2 < 0, \\
3\pi/2, & \text{if } C_2 > 0.
\end{cases}
$$
Secondary homoclinic curves defined by 3D map $G$

Continuations: $m = 3, 4...6$

$C_1 = 1.2, C_2 = 0.7, m = 3, 4..., 90$

$\beta = 0.2, \ \nu = 0.5$
Codim 2 bifurcations in 3D map $G$

\[ \nu = 0.5, \beta = 0.2 \]

\[ C_1 = 0.8, \ C_2 = 1.2, \ \alpha_1 = 0.8, \ \alpha_2 = 1.3, \ \alpha_3 = 0.6, \ \alpha_4 = 1.1 \]

\[ \phi_1 = \phi_2 = \pi/6 \]
An example: Modified Lorenz-Stenflo equations

L. Stenflo
Generalized Lorenz equations for acoustic-gravity waves in the atmosphere


\[
\begin{align*}
\dot{x} &= \sigma (y-x) + su \\
\dot{y} &= rx - xz - y + \epsilon_1 z \\
\dot{z} &= xy - bz \\
\dot{u} &= -x - \sigma u + \epsilon_2 y
\end{align*}
\]

\[\sigma = 0.1, \ s = 33, \ \epsilon_1 = 0.1, \ \epsilon_2 = 0.3\]
Chaotic 3DL bifurcation in modified LS model

$3DL: (r, b) \approx (15.302531, 1.9884) \quad (\nu \approx 0.71605 < 1)$
Open questions

• Exact condition for the Saddle/Spring area transition in 2D model map.

• Analytic proof of strong resonances in 3D map.

• Can $C^k$-linearization with $k > 1$ and related non-resonance conditions be avoided?

• Can the use of Homoclinic Center Manifold Theorem and related gap conditions be avoided as well?

• More realistic examples.

• What happens in the volume-preserving case, where 3DL has codim 1 and is always chaotic?