Home assignment for 25-10-2011
("Dynamical Systems", MasterMath Fall 2011)

The aim of this assignment is to determine the phase portraits of a one-parameter family of two-dimensional continuous time dynamical systems. The differential equations that generate the systems are

$$\left\{ \begin{array}{rcl} \dot{x} & = & x\left[-1+(a+1)x+2y-a(x^2+xy+y^2)\right], \\ \dot{y} & = & y\left[1-a+2(a-1)x+(2a-1)y-a(x^2+xy+y^2)\right]. \end{array} \right. \tag{1}$$

The real parameter a can take both negative and positive values. The state variables x and y are restricted to the triangle

$$\Delta := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 1\}. \tag{2}$$

1. Show that Δ is invariant (both forward and backward) under the flow generated by (1).

2. We define a transformation T by

   $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 1-x-y \end{pmatrix}. \tag{3}$$

   It follows that

   $$T^2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-x-y \\ x \end{pmatrix}$$

   and that T^3 is the identity. The system (1) is *equivariant* with respect to T (you do not have to check this). It follows that the dynamics on each of the three line segments that constitute the boundary $\partial\Delta$ is the same. Determine the dynamics on $\{(x,0) : 0 \leq x \leq 1\}$ and present your result in the form of a *bifurcation diagram*, i.e. draw a one-dimensional phase portrait for each representative value of a.

3. For a certain range of a-values, the equilibrium $(0,0)$ of (1) is a saddle point. Determine the stable- and the unstable-manifold of this saddle point.

4. Check that $(\bar{x},\bar{y}) = \left(\frac{1}{3},\frac{1}{3}\right)$ is an equilibrium of (1). (It is not straightforward to check that there are no other internal equilibria, but this is true and you may take this for granted.) Derive the linearization of (1) around $(\bar{x},\bar{y})$ and determine the corresponding eigenvalues. Characterize $(\bar{x},\bar{y})$ in terms of its type (i.e. saddle, node, focus) and stability character, as a function of a for $a \neq 0$.

5. Show that the system (1) is *Hamiltonian* for $a = 0$ and draw its phase portrait for this particular value of a. What is the type of $(\bar{x}, \bar{y})$ at this parameter value ?

6. Define $V : \mathbb{R}^2 \to \mathbb{R}$ by
$$V(x, y) := xy(1 - x - y). \tag{4}$$
Let $\dot{V}$ denote the derivative of V along orbits of (1). Show that
$$\dot{V}(x, y) = -3axy(1 - x - y)\left[\left(x - \frac{1}{3}\right)^2 + \left(x - \frac{1}{3}\right)\left(y - \frac{1}{3}\right) + \left(y - \frac{1}{3}\right)^2\right]. \tag{5}$$
Consider an orbit of (1) through an interior point of Δ different from $(\bar{x}, \bar{y})$. What can you say about the ω-limit set and the α-limit set of such an orbit ? Explain in detail the arguments that you use in order to arrive at your conclusion.

7. Summarize your results in the form of a bifurcation diagram for (1), i.e. draw a two-dimensional phase portrait for each representative value of a.