Home assignment for 06-12-2011 ("Dynamical Systems", MasterMath Fall 2011)

The aim of this assignment is to study the Andronov-Hopf bifurcation in the famous *Lorenz system*

$$\begin{cases} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= -xz + rx - y, \\ \dot{z} &= xy - bz, \end{cases} \tag{1}$$

using symbolic manipulation software, e.g. MAPLE or Mathematica, if necessary.

1. Show that for fixed $b > 0$, $\sigma > b + 1$, and

$$r = r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}, \tag{2}$$

the positive equilibrium of (1) exhibits an Andronov-Hopf bifurcation.

2. Prove that this bifurcation is *subcritical* and, therefore, gives rise to a unique saddle limit cycle for $r < r_H$.

   (i) Write (1) as a single third-order equation

$$\dddot{x} + (\sigma + b + 1)\ddot{x} + b(1 + \sigma)\dot{x} + b\sigma(1 - r)x = \frac{(1 + \sigma)\dot{x}^2}{x} + \frac{\dot{x}\ddot{x}}{x} - x^2\dot{x} - \sigma x^3.$$

   (ii) Translate the origin to the equilibrium by introducing the new coordinate $\xi = x - x_0$, where $x_0 = \sqrt{b(r-1)}$, thus obtaining the equation

$$\dddot{\xi} + (\sigma + b + 1)\ddot{\xi} + [b(1 + \sigma) + x_0^2]\dot{\xi} + [b\sigma(1 - r) + 3\sigma x_0^2]\xi = f(\xi, \dot{\xi}, \ddot{\xi}), \tag{3}$$

   where

$$f(\xi, \dot{\xi}, \ddot{\xi}) = -3\sigma x_0 \xi^2 - 2x_0 \xi\dot{\xi} + \frac{1 + \sigma}{x_0}\dot{\xi}^2 + \frac{1}{x_0}\dot{\xi}\ddot{\xi} - \sigma\xi^3 - \xi^2\dot{\xi} - \frac{1 + \sigma}{x_0^2}\xi\dot{\xi}^2 - \frac{1}{x_0^2}\xi\dot{\xi}\ddot{\xi} + \ldots$$

   and the dots stand for all higher-order terms in $(\xi, \dot{\xi}, \ddot{\xi})$.

(iii) Rewrite (3) as a system

$$\dot{U} = AU + \frac{1}{2}B(U,U) + \frac{1}{6}C(U,U,U) + O(\|U\|^4), \quad U = \begin{pmatrix} \xi \\ \dot{\xi} \\ \ddot{\xi} \end{pmatrix} \in \mathbb{R}^3, \qquad (4)$$

where A is a 3×3 matrix and B and C are the bilinear- and trilinear- forms, respectively. Note that $A, B,$ and C depend on (r, σ, b). Find an eigenvector $q \in \mathbb{C}^3$ and an adjoint eigenvector $p \in \mathbb{C}^3$ of A corresponding to its purely imaginary eigenvalues $i\omega_0$ and $-i\omega_0$ (when (2) is satisfied), and such that

$$\langle p, q \rangle = 1.$$

(iv) Verify that the critical pair of the complex-conjugate eigenvalues $\lambda_{1,2}$ of A crosses the imaginary axis at $r = r_H$ with a positive velocity w.r.t. parameter r, i.e.

$$\left. \frac{\partial}{\partial r} \text{Re } \lambda_{1,2} \right|_{r=r_H} = \text{Re } \langle p, A'q \rangle > 0,$$

where

$$A' = \left. \frac{\partial A}{\partial r} \right|_{r=r_H}.$$

(v) Compute the first Lyapunov coefficient l_1 for (4) at $r = r_H$ using the formula

$$l_1 = \frac{1}{2\omega_0} \text{Re} \langle p, C(q,q,\bar{q}) - 2B(q, A^{-1}B(q,\bar{q})) + B(\bar{q}, (2i\omega_0 I - A)^{-1}B(q,q)) \rangle.$$

Substitute $\sigma = s + b + 1$ in the resulting expression and show that $l_1 > 0$ for all positive s and b.