Take-home examination problem for the course "Dynamical Systems Generated by ODEs and Maps" (MasterMath Fall 2011)

Write an essay about the Lorenz system

$$\begin{cases} \dot{x} &=& -\sigma x + \sigma y, \\ \dot{y} &=& rx - y - xz, \\ \dot{z} &=& -bz + xy. \end{cases}$$

Explain the concepts and notions that you need and formulate results about the dynamics, in particular about how dynamical properties change when one parameter is varied.

Pay attention to at least some of the following subjects:

- pitchfork and Hopf bifurcations of equilibria (and how to compute their direction);

- existence of a bounded positively-invariant domain that attracts all orbits;

- sensitive dependence on initial conditions;

- discontinuous scalar maps, symbolic dynamics, and kneading sequences;

- geometric Lorenz attractor and chaotic behaviour;

Provide some proofs, technical details, and literature references.