Chapter 7

Symbolic dynamics and global bifurcations

So far in this book we mainly concentrated on the (changes in) *local behaviour* near steady states (including periodic orbits of maps and, in principle but not so easy in practice, periodic orbits of ODEs via the Poincaré map). The idea may arise that whenever orbits stay bounded the persistent behaviour is either constant in time or shows regular oscillations. The main aim of this chapter is to show that this idea is wrong: More complicated yet persistent behaviour of dynamical systems is possible and does occur. The main topics are

(1) scalar maps and symbolic dynamics;

(2) Smale's horseshoe and how it is induced by homoclinic orbits of planar maps and three-dimensional ODEs.

In two Appendices we will discuss bifurcations of periodic orbits from homoclinic orbits in n-dimensional ODEs, and bifurcations leading to the appearance of the famous Lorenz attarctor.

7.1 One-dimensional maps

7.1.1 Periodic orbits of continuous maps

Consider a (possibly noninvertible) continuous map

$$f : I \to I, \quad I = [0, 1], \tag{7.1}$$

and the associated discrete-time dynamical system $\{\mathbb{N}, I, f^k\}$, where $\mathbb{N}$ is the set of all nonnegative integers. Let $J, K \subset I$ be two closed intervals.

Definition 7.1 *We say that J **covers** K **under** f and write $J \rightharpoonup K$, if there exists a closed interval $L \subset J$ such that $f(L) = K$.*

Lemma 7.2 *If $J \rightharpoonup J$ under f, then f has a fixed point $x \in J$.*

Proof: Let $J = [a, b]$. By definition, there is an interval $L \subset J$ such that $f(L) = J$. This implies that there exist $c, d \in L$ satisfying $f(c) = a \leq c$ and $f(d) = b \geq d$ (see Figure 7.1 for an illustration). Thus, by the Intermediate Value Theorem, the continuous function $g(x) = f(x) - x$ has a zero $x \in L \subset J$. □

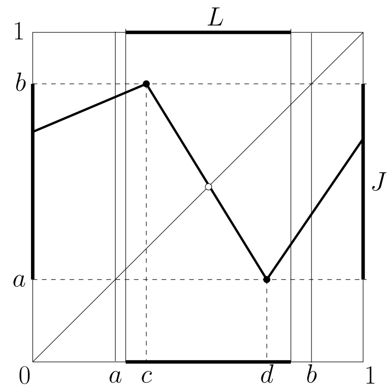

Figure 7.1: Lemma 7.2.

By induction one can prove the following result.

Lemma 7.3 *If $I_0 \rightharpoonup I_1 \rightharpoonup I_2 \rightharpoonup \cdots \rightharpoonup I_n$, then there exists a closed interval $J \subset I_0$ such that $f^k(J) \subset I_k$ for $k = 1, 2, \ldots, n-1$, and $f^n(J) = I_n$.* □

Lemma 7.4 *If $I_0 \rightharpoonup I_1 \rightharpoonup I_2 \rightharpoonup \cdots \rightharpoonup I_{n-1} \rightharpoonup I_0$, then there exists $x \in I_0$ such that $x = f^n(x)$ and $f^k(x) \in I_k$ for $k = 0, 1, \ldots, n-1$.*

Proof: From Lemma 7.3 it follows that there exists a closed interval $J \subset I_0$ such that $f^n(J) = I_0$ and $f^k(J) \subset I_k$ for $k = 0, 1, \ldots, n-1$. Applying Lemma 7.2 to the map f^n we get a fixed point $x \in J \subset I_0$ of f^n. Clearly, $f^k(x) \in f^k(J) \subset I_k$ for $k = 0, 1, 2, \ldots, n-1$. □
Note that n is not necessarily the minimal period of the cycle $\{x, \ldots, f^{n-1}(x)\}$ since we did not assume that $I_0 \cap I_k = \emptyset$ for $k = 1, 2, \ldots, n-1$.

Let us consider a collection $\{I_0, I_1, \ldots\}$ of closed intervals $I_i \subset I$ with pairwise disjoint interiors. The relation "$\rightharpoonup$" yields the edges of a directed graph, whose vertices are the intervals I_i (see Figure 7.2). This graph is called a *Markov graph* for f associated to $\{I_0, I_1, \ldots\}$. Lemma 7.4 implies that any loop in the Markov graph brings in a periodic orbit of f.

Theorem 7.5 (Li-Yorke part of Sharkovsky Theorem) *Suppose a continuous map $f : I \to I$ has a cycle of minimal period 3. Then f has a cycle of minimal period n for all $n \geq 1$.*

Proof: Consider the period-3 orbit $\{p_1, p_2, p_3\}$,

$$p_2 = f(p_1), \; p_3 = f(p_2), \; p_1 = f(p_3),$$

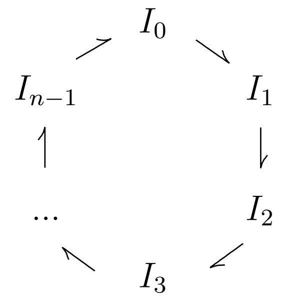

Figure 7.2: A loop in the Markov graph corresponding to Lemma 7.4.

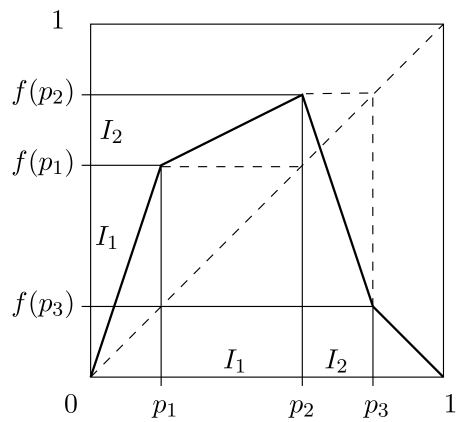

Figure 7.3: Period three implies all periods.

and assume first that $p_1 < p_2 < p_3$ (see Figure 7.3). Introduce two intervals with disjoint interiors:

$$I_1 = [p_1, p_2], \quad I_2 = [p_2, p_3].$$

Then, I_1 covers I_2 and I_2 covers both I_1 and I_2. Thus, the Markov graph associated to $\{I_1, I_2\}$ contains the graph

$$I_1 \rightleftharpoons I_2 \circlearrowleft$$

that has for any integer $n > 0$ a loop

$$I_1 \rightharpoonup I_2 \rightharpoonup I_2 \rightharpoonup \cdots \rightharpoonup I_2 \rightharpoonup I_1$$

with $(n-1)$ occurrences of I_2. By Lemma 7.4, the map f^n has a fixed point x. This fixed point cannot correspond to a cycle with a minimal period $k < n$ except $k = 3$, which existence is assumed.

Indeed, assume that the orbit of x is a cycle of minimal period $k < n$. Then necessarily $x \in I_1 \cap I_2$, since $x \in I_1$ implies $x = f^k(x) \in I_2$. Thus $x \in I_1 \cap I_2 = \{p_2\}$ and $k = 3$. This immediately excludes the possibility that the assumption holds when $n = 2$. Next note that $f^2(x) = f^2(p_2) = p_1 \notin I_2$ while for $n > 2$ our construction guarantees that $f^2(x) \in I_2$. So also for $n > 2$ the assumption leads to a contradiction.

The only other possible case, $p_1 < p_3 < p_2$, can be treated similarly by defining $I_1 = [p_3, p_2]$ and $I_2 = [p_1, p_3]$. $\square$

Remark: The proof reveals that one does not actually need a period-3 orbit. If for some $p \in I$ we have either $f^3(p) \geq p > f(p) > f^2(p)$ or $f^3(p) \leq p < f(p) < f^2(p)$ then the map f has cycles of minimal period n for all $n \geq 1$ (so including $n = 3$).

Example 7.6 (Cycle of period 3 of the logistic map)

Consider the scalar map

$$x \mapsto f_{(\alpha)}(x) = \alpha x(1-x), \quad x \in [0,1]. \tag{7.2}$$

This map is called the *logistic map.*

Proposition 7.7 *At $\alpha_1 = 1 + 2\sqrt{2} < 4$ the map $f^3_{(\alpha)}$ exhibits a generic fold bifurcation, generating a stable period-3 cycle and an unstable period-3 cycle of (7.2) as α increases.*

Proof: Introduce the function

$$G(x,\alpha) = f^3_{(\alpha)}(x).$$

One gets

$$G(x,\alpha) = \alpha^3 x(1-x)(1-\alpha x+\alpha x^2)(1-\alpha^2 x+\alpha^3 x^2-2\alpha^3 x^3+\alpha^2 x^2+\alpha^3 x^4).$$

The fold bifurcation condition for a period-3 cycle $\{x, f_\alpha(x), f^2_\alpha(x)\}$ translates into the following system of polynomial equations:

$$\begin{cases} G(x,\alpha) - x &= 0, \\ G_x(x,\alpha) - 1 &= 0. \end{cases}$$

Eliminating x from this system, we obtain a polynomial equation

$$(\alpha^2 - 2\alpha - 7)(\alpha - 1)^2(\alpha^2 + \alpha + 1)^2(\alpha^2 - 5\alpha + 7)^2 = 0,$$

whose real solutions coincide with those of the equation

$$(\alpha^2 - 2\alpha - 7)(\alpha - 1) = 0.$$

These solutions are

$$\alpha_1 = 1 + 2\sqrt{2},\ \alpha_2 = 1 - 2\sqrt{2},\ \alpha_3 = 1.$$

The second root is negative, while the third is related to the transcritical bifurcation of the fixed point $x = 0$. Thus, α_1 is the only possible critical value.

The fixed points of $f^3_{(\alpha_1)}$ satisfy the polynomial equation $G(x, \alpha_1) = x$, which has two trivial solutions, $x = 0$ and

$$x = 1 - \frac{1}{\alpha_1} = \frac{8}{7} - \frac{2\sqrt{2}}{7},$$

corresponding to the fixed points of $f_{(\alpha_1)}$. Thus, the equation $G(x, \alpha_1) - x = 0$ is equivalent to

$$x(7x - 8 + 2\sqrt{2})(343x^3 - (490 + 49\sqrt{2})x^2 + (91 + 112\sqrt{2})x + 31 - 41\sqrt{2}) = 0.$$

Three real roots $x_1 < x_2 < x_3$ of the third factor

$$343x^3 - (490 + 49\sqrt{2})x^2 + (91 + 112\sqrt{2})x + 31 - 41\sqrt{2}$$

provide a period-3 cycle at $\alpha_1 = 1 + 2\sqrt{2}$. One can express x_1 in radicals and then verify that

$$G_{xx}(x_1, \alpha_1) > 0, \; G_{\alpha}(x_1, \alpha_1) < 0.$$

Thus, two period-3 cycles of $f_{(\alpha)}$, one stable and one unstable, bifurcate for small $|\alpha - \alpha_1|$ with $\alpha > \alpha_1$. □

Thus, at least for small positive $(\alpha - \alpha_1)$, the logistic map (7.2) has cycles of all periods. ◇

Definition 7.8 *The **Sharkovsky ordering** of the natural numbers is defined by*

$$\begin{array}{l} 3 \prec 5 \prec 7 \prec \cdots \prec 2k+1 \prec \cdots \\ 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \cdots \prec 2 \cdot (2k+1) \prec \cdots \\ 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec \cdots \prec 2^2 \cdot (2k+1) \prec \cdots \\ \cdots \\ 2^n \cdot 3 \prec 2^n \cdot 5 \prec 2^n \cdot 7 \prec \cdots \prec 2^n \cdot (2k+1) \prec \cdots \\ \cdots \\ \cdots \prec 2^n \prec 2^{n-1} \prec \cdots \prec 2 \prec 1 \end{array}$$

Since any natural number can be written in the form $2^n \cdot (2k+1)$ for some integers $k, n \geq 0$, all natural numbers occur in this list.

Theorem 7.9 (Sharkovsky, 1964) *If a continuous map $f : I \to I$ has a cycle of minimal period p, then it has a cycle of minimal period q, for any $q \succ p$.* □

This theorem implies Theorem 7.5 and can be proven using similar techniques. The Sharkovsky Theorem is sharp, e.g. there are continuous maps from I into I which have period-5 cycles but no period-3 cycles.

7.1.2 One-sided symbolic dynamics and chaotic maps

Let us introduce two scalar continuous maps in $I = [0, 1]$ – one of which is quadratic while the other is piecewise-linear – that will play an important role in what follows.

Example 7.10 (The logistic map versus the tent map)

Consider the logistic map (7.2) with $\alpha = 4$:

$$x \mapsto F(x) = 4x(1-x), \quad x \in I, \tag{7.3}$$

and the *tent map*:

$$y \mapsto T(y) = 1 - |1 - 2y|, \quad y \in I. \tag{7.4}$$

(see Figure 7.4). Clearly,

$$T(y) = \begin{cases} 2y & \text{for } 0 \le y \le \frac{1}{2}, \\ 2 - 2y & \text{for } \frac{1}{2} < y \le 1, \end{cases},$$

so that $F(I) = T(I) = I$ and both maps expand and fold I.

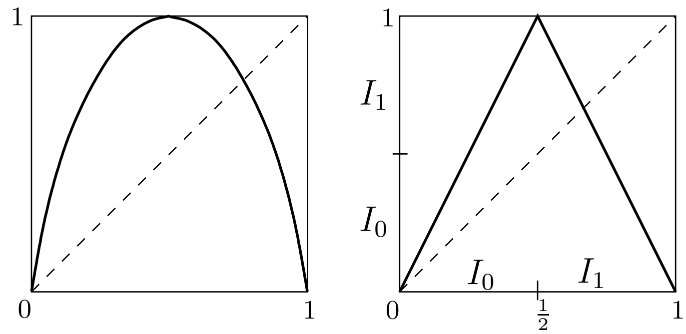

Figure 7.4: The maps (7.3) and (7.4).

Proposition 7.11 *The maps F and T are topologically conjugate on $I = [0, 1]$.*

Proof: Introduce

$$x = h(y) = \sin^2\left(\frac{\pi y}{2}\right).$$

This is a homeomorphism of the interval $[0, 1]$ onto itself. Moreover, for $y \in [0, \frac{1}{2}]$,

$$\begin{aligned} F(h(y)) &= 4\sin^2\left(\frac{\pi y}{2}\right)\left(1 - \sin^2\left(\frac{\pi y}{2}\right)\right) = 4\sin^2\left(\frac{\pi y}{2}\right)\cos^2\left(\frac{\pi y}{2}\right) = \sin^2(\pi y) \\ &= h(T(y)). \end{aligned}$$

By symmetry, $F(h(y)) = h(T(y))$ for all $y \in [\frac{1}{2}, 1]$. Thus,

$$F \circ h = h \circ T. \quad \square$$

Therefore, all topological properties of F and T are the same and we can concentrate on the piecewise-linear tent map T. $\diamond$

Consider the set Σ_2 of all one-sided infinite sequences of two symbols/digits $\{0, 1\}$. This set is a complete metric space with respect to the distance

$$d(\omega, \theta) = \sum_{k=0}^{\infty} \frac{|\omega_k - \theta_k|}{2^k}. \tag{7.5}$$

Introduce two closed intervals:

$$I_0 = \left[0, \tfrac{1}{2}\right], \quad I_1 = \left[\tfrac{1}{2}, 1\right].$$

Lemma 7.12 *For each sequence $\omega \in \Sigma_2$ there is a unique $x \in I$, such that*

$$T^k(x) \in I_{\omega_k}, \quad k = 0, 1, 2, \ldots .$$

Proof: The Markov graph for T associated to $\{I_0, I_1\}$ is

$$\underset{\circlearrowleft}{I_0} \rightleftarrows \underset{\circlearrowright}{I_1}$$

Thus, by Lemma 7.3, for any $n \geq 0$, and any given finite sequence $\omega = \omega_0\omega_1\omega_2 \ldots \omega_n$, there is a closed interval in I:

$$J_{\omega_0\omega_1\cdots\omega_n} = \{x \in I : T^k(x) \in I_{\omega_k} \ \text{ for } \ 0 \leq k \leq n\}.$$

Let $T^{-k}(J)$ denote the *full preimage* of $J \subset I$ under T^k, i.e.

$$T^{-k}(J) = \{x \in I : T^k(x) \in J\}.$$

Then

$$J_{\omega_0\omega_1\cdots\omega_n} = I_{\omega_0} \cap T^{-1}(I_{\omega_1}) \cap T^{-2}(I_{\omega_2}) \cap \cdots \cap T^{-n}(I_{\omega_n}) = J_{\omega_0\omega_1\cdots\omega_{n-1}} \cap T^{-n}(I_{\omega_n}),$$

so that

$$J_{\omega_0\omega_1\cdots\omega_n} \subset J_{\omega_0\omega_1\cdots\omega_{n-1}}.$$

This means that the intervals $J_{\omega_0\omega_1\cdots\omega_n}$ are nested. Furthermore,

$$|J_{\omega_0\omega_1\cdots\omega_n}| = \frac{1}{2}|J_{\omega_0\omega_1\cdots\omega_{n-1}}|,$$

where $|J| = |a - b|$ is the length of $J = [a, b]$.

Thus, there exists a *unique* common point

$$x = \bigcap_{n \geq 0} J_{\omega_0\omega_1\cdots\omega_n}$$

and for this point $T^k(x) \in I_{\omega_k}, \quad k = 0, 1, 2, \ldots$, by the construction. □

Therefore we have a well-defined map

$$\Phi : \Sigma_2 \to I, \quad \omega \mapsto \Phi(\omega) = x \ .$$

Moreover, it can be shown that Φ is continuous, i.e. two points $x = \Phi(\omega)$ and $x' = \Phi(\omega')$ are close if the sequences ω and ω' are close with respect to the distance (7.5). Note that the construction used in the above proof will be repeated several times in what follows.

Definition 7.13 *The* ***shift map*** $\sigma : \Sigma_2 \to \Sigma_2$ *is defined by the formula* $\sigma(\omega) = \theta$, *where*

$$\theta_k = \omega_{k+1}, \quad k = 0, 1, 2, \ldots . \tag{7.6}$$

The shift map $\sigma : \Sigma_2 \to \Sigma_2$ (cf. Example 1.4 in Chapter 1) is a continuous map with respect to the distance (7.5). It is noninvertible in Σ_2.

Theorem 7.14 *The discrete-time noninvertible dynamical system* $\{\mathbb{N}, \Sigma_2, \sigma^k\}$ *has*
(*i*) *a countable dense set of periodic orbits with arbitrarily long periods*;
(*ii*) *an uncountable set of nonperiodic orbits*;
(*iii*) *a dense orbit, i.e. an orbit that passes arbitrarily close to any given sequence.*

Proof:
(*i*) Any periodic sequence with a repeating an block of length n defines an n-periodic orbit. The set of all periodic sequences is countable. To show that the periodic orbits are dense, take an arbitrary sequence and produce a periodic sequence by repeating a sufficiently long initial block of the given sequence.

(*ii*) Suppose that all nonperiodic sequences are listed. Then we can construct a sequence ω by choosing ω_0 different from ω_0 in the first listed sequence, ω_1 different from ω_1 in the second listed one, ω_2 different ω_2 in the third listed one, *etc.* The sequence ω thus constructed is not in the list, since it differs from any listed one. A contradiction.

(*iii*) To produce a dense orbit, list all blocks of length k for all k and glue them together in the order of increasing k. The resulting sequence γ is called the *Morse sequence.* An appropriate shift of this sequence, say $\sigma^k(\gamma)$, agrees with any given sequence ω in the first l positions, where l is as large as we want. Therefore, the distance $d(\sigma^k(\gamma), \omega)$ defined by (7.5) can be made smaller than any given positive number. $\square$

It happens that the shift map (7.6) acts on a one-sided sequence $\omega \in \Sigma_2$ as the tent map (7.4) on $x = \Phi(\omega) \in I$.

Lemma 7.15 $\Phi(\sigma(\omega)) = T(\Phi(\omega))$ *for all* $\omega \in \Sigma_2$.

Proof: Let $x = \Phi(\omega)$ with some $\omega \in \Sigma_2$. Then, according to Lemma 7.12,

$$T^k(x) \in I_{\omega_k}, \quad k = 0, 1, 2, \ldots.$$

Thus

$$T^{l-1}(T(x)) \in I_{\omega_l}, \quad l = 1, 2, 3, \ldots,$$

or, equivalently,

$$T^k(T(x)) \in I_{\omega_{k+1}}, \quad k = 0, 1, 2, \ldots.$$

This implies that $T(x) = \Phi(\theta)$, where $\theta_k = \omega_{k+1}, \quad k = 0, 1, 2, \ldots$. Since $\theta = \sigma(\omega)$, we get

$$\Phi(\sigma(\omega)) = \Phi(\theta) = T(x) = T(\Phi(\omega))$$

for all $\omega \in \Sigma_2$. $\square$

This lemma implies that all assertions in Theorem 7.14 are also valid for the dynamical system $\{\mathbb{N}, I, T^k\}$ generated by the tent map (7.4), and, taking into account Proposition 7.11, for the special logistic map (7.3). In particular, both maps have infinite number of periodic orbits.

Remark:

For each $x \in I$, consider its positive half-orbit

$$x, T(x), T^2(x), T^3(x), \dots$$

and associate to it a *symbolic sequence*

$$\omega = \omega_0\omega_1\omega_2\omega_3\dots,$$

where

$$\omega_k = \begin{cases} 0, & \text{if } T^k(x) \in I_0, \\ 1, & \text{if } T^k(x) \in I_1 \setminus \{\frac{1}{2}\}, \end{cases} \tag{7.7}$$

for $k = 0, 1, 2, \dots$. The sequence ω codes the future fate of the point x and is called its *itinerary*. Since $I_0 \cap I_1 = \{\frac{1}{2}\}$, we have introduced a special convention for assigning $\omega_k = 0$ in case $T^k(x) = \frac{1}{2}$ (which happens for some k whenever $x = p2^{-q}$ for positive integer p and q). We adopted the convention that $\omega_k = 0$ in that case with the consequence that half-orbits that reach $x = 0$ in $k + 2$ steps are coded by the sequence

$$\omega_0\omega_1\dots\omega_{k-1}01000\dots$$

and that sequences of the

$$\omega_0\omega_1\dots\omega_{k-1}11000\dots$$

do not represent any half-orbit.

Next introduce the set

$$\tilde{\Sigma}_2 = \{\omega \in \Sigma_2 : \omega \text{ is not of the form } \omega_0\omega_1\dots\omega_{k-1}11000\dots\}.$$

One easily verifies that $\tilde{\Sigma}_2$ is a closed subset of Σ_2, so it is a complete metric space with respect to the same distance (7.5). Moreover, $\tilde{\Sigma}_2$ is invariant under σ and one can restrict the shift map on $\tilde{\Sigma}_2$.

The formula (7.7) defines a map

$$\Psi : I \to \tilde{\Sigma}_2,\ x \mapsto \Psi(x) = \omega,$$

which is the inverse of the map $\Phi : \tilde{\Sigma}_2 \to I$. One can also verify that this map is continuous. Thus, Φ is a homeomorphism. This implies that the restriction of the tent map T to $\Omega = \Phi(\tilde{\Sigma}_2)$ and the shift map $\sigma : \tilde{\Sigma}_2 \to \tilde{\Sigma}_2$ are topologically conjugate,

$$T|_\Omega = \Phi \circ \sigma \circ \Phi^{-1},$$

and, consequently, have identical topological properties. $\diamond$

Next we introduce the notion of “sensitive dependence on initial conditions”.

Definition 7.16 *Let X be a metric space with distance d. A dynamical system $\{\mathbb{T}, X, \varphi^t\}$ is said to exhibit* ***sensitive dependence on initial conditions*** *if there exists $r > 0$ such that for every point $x \in X$ and for each $\varepsilon > 0$ there is a point $y \in X$ with $d(x, y) < \varepsilon$ and a positive $t \in \mathbb{T}$ such that $d(\varphi^t(x), \varphi^t(y)) \geq r$.*

This phenomenon was observed by E.N. Lorenz in the early 1960's when he numerically integrated a system of three ODEs (See Section 7.6 below). He restarted some computations on the basis of printed intermediate coordinate values and found that the new results were in good agreement with the original outcome for small times, but did differ greatly for large times. He attributted this to the difference in decimal precision between the printed coordinate values and their machine representation.

It is easy to establish sensitive dependence on initial conditions for $\{\mathbb{N}, \Sigma_2, \sigma^k\}$ and $\{\mathbb{N}, \tilde{\Sigma}_2, \sigma^k\}$. Indeed, if ω and θ agree up to symbol k but differ at in the k-th position, then $d(\omega, \theta) \leq 2^{1-k}$. For any $\varepsilon > 0$ we may choose k such that $2^{1-k} < \varepsilon$. Yet, since $\sigma^k(\omega)$ and $\sigma^k(\theta)$ differ in the first position, they are at least at distance one apart. Since sensitive dependence on initial conditions is preserved under topological conjugacy, it follows that also the dynamical systems generated by F and T exhibit sensitive dependence on initial conditions. For the tent map T we see most clearly the underlying mechanism: At every time step small differences are increased by factor two, but the folding prevents that the orbits themself move to infinity. So there is a combination of local stretching and global bending that keeps orbits bounded but expands the distance between nearby points.

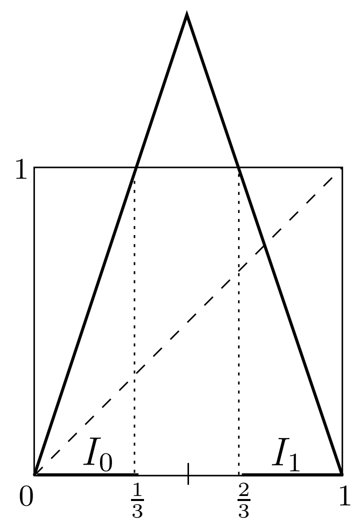

Figure 7.5: 'Big Tent Map' (7.8).

If we make the tent bigger by raising the top, while preserving that both zero and one are mapped to zero, then part of the interval $I = [0, 1]$ is mapped outside that interval (see Figure 7.5). A bigger part is mapped outside in two steps, etc. But something remains, i.e. there are points for which the entire forward half-orbit belongs to I. What is the structure of the set of such points ? Note that a similar behaviour – but in the plane – is demonstrated by the *Smale horseshoe map* to be treated in Section 7.2.1.

For definiteness let the map $\tilde{T} : \mathbb{R} \to \mathbb{R}$ shown in Figure 7.5 be defined by

$$\tilde{T}(x) = \begin{cases} 3y & \text{for } x \leq \frac{1}{2}, \\ 3 - 3y & \text{for } \frac{1}{2} < x, \end{cases} , \tag{7.8}$$

so that once an orbit is outside $I = [0,1]$ it tends to $-\infty$. Note that $\tilde{T}^{-1}(I) = I_0 \cup I_1$ with $I_0 = [0, \frac{1}{3}]$ and $I_1 = [\frac{2}{3}, 1]$. Next note that $\tilde{T}^{-1}(I_0) = I_{00} \cup I_{10}$ and $\tilde{T}^{-1}(I_1) = I_{01} \cup I_{11}$ with $I_{00} = [0, \frac{1}{9}], I_{10} = [\frac{8}{9}, 1], I_{01} = [\frac{2}{9}, \frac{1}{3}]$, and $I_{11} = [\frac{2}{3}, \frac{7}{9}]$. By induction we define for $\omega_i \in \{0,1\}$ the intervals

$$J_{\omega_0\omega_1\ldots\omega_{n-1}} = \{x : \tilde{T}^k(x) \in I_{\omega_k} \ \text{ for } \ 0 \le k \le n-1\}.$$

Let

$$S_n := \{x : \tilde{T}^k(x) \in I \ \text{ for } \ 0 \le k \le n\},$$

then

$$S_n = \bigcap_{k=0}^{n} \tilde{T}^{-k}(I) = \bigcap_{k=0}^{n-1} \tilde{T}^{-k}(I_0 \cup I_1) = \bigcup_{\omega_i \in \{0,1\}} J_{\omega_0\omega_1\ldots\omega_{n-1}} \ ,$$

the union of 2^n closed intervals of length $(\frac{1}{3})^n$. Note that the S_n are nested: $S_{n+1} \subset S_n$. Finally, define

$$\Lambda = \bigcap_{n=0}^{\infty} S_n = \{x : \tilde{T}^k(x) \in I \ \text{ for all } \ k \ge 0\}. \tag{7.9}$$

Our aim is to characterize the structure of Λ and the dynamics generated by $\tilde{T}$ on this set. In order to do so we need more terminology.

Let X be a metric space and let $Y \subset X$. By definition, Y is *nowhere dense* if the interior of the closure of Y is empty. One calles Y *totaly disconnected* if the connected components of Y are single points. A closed subset of $\mathbb{R}$ is nowhere dense if and only if it is totally disconnected (but a smooth curve in the plane is nowhere dense, yet not totally disconnected). Finally, Y is *perfect* if it is closed and every point $p \in Y$ is the limit of points $y_n \in Y$ with $y_n \neq p$.

Definition 7.17 *The set Y is called a **Cantor set** if it is compact, totally disjoint, and perfect.*

The classical example of a Cantor set is obtained by removing the middle open interval of length $\frac{1}{3}$ from the unit interval I, next removing the middle open interval of length $\frac{1}{9}$ from the remaning two closed intervals, and repeating this "surgery" indefinitely. So the following result should not come as a surprise.

Proposition 7.18 *Λ is a Cantor set.*

Proof: Since S_n is closed, so is $\Lambda = \cap_{n=0}^{\infty} S_n$. Consider any $y \in Y$. For arbitrary $j \in \mathbb{N}$ choose $n = n(j)$ such that $3^{-n} < 2^{-j}$. Let $J_{\omega_0\omega_1\ldots\omega_{n-1}}$ be the component of S_n to which y belongs. Note that

$$J_{\omega_0\omega_1\ldots\omega_{n-1}} \cap S_{n+1} = J_{\omega_0\omega_1\ldots\omega_{n-1}0} \cup J_{\omega_0\omega_1\ldots\omega_{n-1}1}$$

and that $J_{\omega_0\omega_1\ldots\omega_{n-1}} \setminus S_{n+1}$ is not empty. For any choice of $p_j \in J_{\omega_0\omega_1\ldots\omega_{n-1}} \setminus S_{n+1}$ we have that $p_j \notin S_{n+1}$ hence $p_j \notin \Lambda$ and

$$|p_j - y| \le 3^{-n} < 2^{-j}.$$

It follows that $p_j \to y$ for $j \to \infty$ and hence that Λ is nowhere dense.

Next choose q_j to be the endpoint of $J_{\omega_0\omega_1\ldots\omega_{n-1}j_n}$. Then $q_j \in \Lambda$ and

$$|q_j - y| \leq 3^{-n} < 2^{-j},$$

so also $q_j \to y$ for $j \to \infty$ but $q_j \neq y$. Thus, Λ is perfect. $\square$

A Cantor set has "holes" everywhere and yet contains no isolated points. The st Λ is *self-similar*: If we restrict to $\Lambda \setminus [0, \frac{1}{3}]$ we recover Λ in the sense that any point in Λ is obtained from that in $\Lambda \cap [0, \frac{1}{3}]$ by multiplication by 3. Likewise we recover Λ if we zoom in on appropriate "smaller" parts of Λ: At every scale Λ contains a copy of itself. There are multiple inequivalent ways in which we can think about the "size" of Λ. As we will show below, Λ is not countable. Since the total length $(\frac{2}{3})^n$ of S_n converges to zero as $n \to \infty$, the Lebesgue measure of Λ is zero (but Cantor sets with positive Lebesgue measure do exist, so this is *not* a consequence of the Cantor structure).

Let's return to dynamics. Clearly, $\tilde{T}(\Lambda) \subset \Lambda$. If $\tilde{T}(x) \in \Lambda$, then $\tilde{T}^k \in I$ for all k, so by the very definition of Λ the point x belong to Λ. Hence $\tilde{T}(\Lambda) = \Lambda$, i.e. Λ is forward-invariant under $\tilde{T}$. We define $h : \Lambda \to \Sigma_2$ by

$$h(x) = \omega \quad \text{with} \quad \omega_k = \begin{cases} 0, & \text{if } \tilde{T}^k(x) \in I_0, \\ 1, & \text{if } \tilde{T}^k(x) \in I_1, \end{cases} \tag{7.10}$$

and call $h(x)$ the *itinerary* of x and h the *itinerary map*. From this definition it follows at once that

$$h(\tilde{T}(x)) = \sigma(h(x)) \tag{7.11}$$

for all $x \in \Lambda$. Hence $\{\mathbb{N}, \Lambda, \tilde{T}^k\}$ and $\{\mathbb{N}, \Sigma_2, \sigma^k\}$ are topologically conjugate if we can show that h is a homeomorphism. And this is exactly what we prove next.

Proposition 7.19 *The map $h : \Lambda \to \Sigma_2$ defined by* (7.10) *is a homeomorphism.*

Proof: Let $\omega \in \Sigma_2$. The intervals $J_{\omega_0\omega_1\ldots\omega_n}$ introduced above are nested as n increases. If $x \in J_{\omega_0\omega_1\ldots\omega_n}$ the $h(x)$ and ω agree up to position n. So for $x_0 \in \cap_{n=0}^{\infty} J_{\omega_0\omega_1\ldots\omega_n}$ we have $h(x_0) = \omega$ and we conclude that h is onto.

Suppose that $h(x) = h(y) = \omega$. Then both x and y belong to $J_{\omega_0\omega_1\ldots\omega_n}$ and hence $|x - y| \leq 3^{-n}$. As this estimate holds for arbitrary n, we conclude that $x = y$ and hence that h is one-to-one.

Let $x \in \Lambda$ and $\omega = h(x)$. For every $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that $2^{-n} < \varepsilon$. Let $\delta = \delta(\varepsilon)$ be small enough to guarantee that from $y = \lambda$ and $|y - z| < \delta$ follows that $y \in J_{\omega_0\omega_1\ldots\omega_n}$. Let $\theta = h(y)$. Then $d(\omega, \theta) \leq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n} < \varepsilon$. So h is continuous.

Likewise it follows that $d(\omega, \theta) < 2^{-n}$ implies that $h^{-1}(\theta) \in J_{\omega_0\omega_1\ldots\omega_n}$ and hence that $|h^{-1}(\omega) - h^{-1}(\theta)| \leq |J_{\omega_0\omega_1\ldots\omega_n}| = 3^{-(n+1)}$. So h^{-1} is also continuous. $\square$

We can now apply Theorem 7.14 to conclude that the restriction of $\tilde{T}$ to the invariant set Λ has a very rich orbit structure. Also note that the diagonalization argument used to prove part (ii) of that theorem establishes that Σ_2 is not countable (in fact any perfect set is uncountable). Among the points in Λ are the boundary points of $J_{\omega_0\omega_1\ldots\omega_n}$. They form a countable subset of Λ and their itinerary ends with an infinite sequence of zeros (since they are mapped to zero in finitely many iterates). So Λ is *not* just the limit of the ever increasing set of boundary points.

The fact that $\tilde{T}$ restricted to Λ has a dense orbit shows that we cannot divide Λ into invariant pieces. To formulate this precisely, we introduce another important notion.

Definition 7.20 *Let X be a metric space and let $f : X \to X$ be continuous. The dynamical system $\{\mathbb{N}, X, f^k\}$ is called **topologically transitive** if for every pair of non-empty open subsets U and V of X there exists $n = n(U, V)$ such that $f^n(U) \cap V \neq \emptyset$.*

If f has a dence orbit and X has no isolated points then $\{\mathbb{N}, X, f^k\}$ is topologically transitive. The *Birkhoff Transitivity Theorem* asserts that, under some extra conditions on X, a topologically transitive dynamical system has a dense orbit.

Definition 7.21 *Let X be a metric space and let $f : X \to X$ be continuous. Let Y be an invariant subset of X, i.e. $f(Y) = Y$. The dynamical system $\{\mathbb{N}, Y, f^k\}$ is called **chaotic** if*
(i) it exhibits sensitive dependence on initial conditions;
(ii) it is topologically transitive.

It should be noted that some authors include the condition that periodic orbits are dense in Y in the definition of chaotic.

We can now summarise our earlier results in a simple statement: *The dynamical system $\{\mathbb{N}, \Lambda, \tilde{T}^k\}$ is chaotic.*

7.1.3 Feigenbaum universality

As we have already mentioned in Example 5.28 in Chapter 5, the *Ricker map*

$$x \mapsto \alpha x \mathrm{e}^{-x}, \quad x \in \mathbb{R}, \tag{7.12}$$

undergoes an infinite cascade of flip bifurcations at parameter values

$$\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots,$$

which form (asymptotically) a geometric progression:

$$\frac{\alpha_k - \alpha_{k-1}}{\alpha_{k+1} - \alpha_k} \to \mu_F,$$

as $k \to \infty$, where $\mu_F = 4.6692\ldots$ This is a system-independent (universal) constant. The sequence $\{\alpha_k\}$ has a limit α_∞. This phenomenon is called *Feigenbaum universality*.

It was first explained for maps, which, for all parameter values, satisfy the following conditions:

(1) $f : [-1, 1] \to [-1, 1]$ is an even smooth function;
(2) $f'(0) = 0$, $x = 0$ is the only maximum, $f(0) = 1$;
(3) $f(1) = -a < 0$;
(4) $b = f(a) > a$;
(5) $f(b) = f^2(a) < a$;

where a and b are positive with $a < 1$ (see Figure 7.6). We denote by $\mathcal{Y}$ the set of maps having

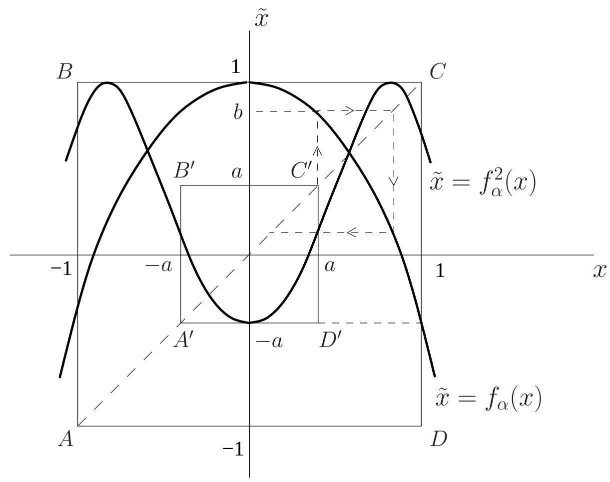

Figure 7.6: The graph of a map satisfying conditions (1) through (5) and the graph of its second iterate.

these properties. The function $f_{(\alpha)}(x) = 1 - \alpha x^2$ is in this class for $1 < \alpha < 2$. Note that it is linearly conjugate to the logistic map (7.2).

Consider the second iterate f^2 of a map satisfying conditions (1) through (5). In the square $A'B'C'D'$ (see Figure 7.6), the graph of f^2, after a coordinate dilatation and a sign change, is similar to the graph of f in the unit square $ABCD$. This observation leads to the introduction of a map defined on functions in $\mathcal{Y}$,

$$(\mathcal{T}(f))(x) = -\frac{1}{a} f(f(ax)), \quad a = -f(1). \tag{7.13}$$

Notice that a depends on f.

Definition 7.22 *The map $\mathcal{T}$ is called the* ***doubling operator****.*

Since f is even, we can also write

$$(\mathcal{T}(f))(x) = -\frac{1}{a} f(f(-ax)), \quad a = -f(1). \tag{7.14}$$

It can be checked that the map (7.13) transforms a function $f \in \mathcal{Y}$ into some function $\mathcal{T}(f) \in \mathcal{Y}$. Therefore, we can consider a *discrete-time dynamical system* $\{\mathbb{N}, \mathcal{Y}, \mathcal{T}^k\}$. This dynamical system has the infinite-dimensional state space $\mathcal{Y}$, which becomes a *metric space* if the following distance is introduced:

$$d(f, g) = \sup_{x \in [-1,1]} |f(x) - g(x)|, \quad f, g \in \mathcal{Y}.$$

Note that the space $\mathcal{Y}$ is neither linear nor complete. However, we can use this distance to define the convergence of sequences in $\mathcal{Y}$. Note also that the doubling operator is not invertible. Thus, we have to consider only *positive* iterates of $\mathcal{T}$.

The following theorems have been originally established with the help of a computer and delicate error estimates.

Theorem 7.23 (Fixed-point existence) *The map $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ defined by (7.13) has a fixed point $\varphi \in \mathcal{Y}$: $\mathcal{T}(\varphi) = \varphi$, where*

$$\varphi(x) = 1 - 1.52763\ldots x^2 + 0.104815\ldots x^4 + 0.0267057\ldots x^6 + \ldots .$$

□

Then we can define the *linearization* $\mathcal{L}$ of $\mathcal{T}$ at $f \in \mathcal{Y}$ by the formula

$$(\mathcal{L}(f))h = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathcal{T}(f + \varepsilon h) - \mathcal{T}(f)], \tag{7.15}$$

where h belongs to a linear space $\mathcal{X}$ of smooth functions on $[-1, 1]$ satisfying $h(0) = 0$, so that $f(0) + \varepsilon h(0) = 1$. Thus map $h \mapsto (\mathcal{L}(f))h$ defines a linear operator $\mathcal{L}(f)$ on $\mathcal{X}$ and one can study its spectrum. In Exercise 7.4.7, you are asked to compute $(\mathcal{L}(\varphi))h$ explicitly.

Theorem 7.24 (Saddle properties of the fixed point) *The linear part $\mathcal{L}(\varphi)$ of the doubling operator $\mathcal{T}$ at its fixed point φ has only one eigenvalue $\mu_F = 4.6692\ldots$ with $|\mu_F| > 1$. The rest of the spectrum of $\mathcal{L}$ is located strictly inside the unit circle.* □

Theorems 7.23 and 7.24 imply that the system $\{\mathbb{N}, \mathcal{Y}, \mathcal{T}^k\}$ has a saddle fixed point. This fixed point φ has a codim 1 invariant stable manifold $W^s(\varphi)$ and a one-dimensional invariant unstable manifold $W^u(\varphi)$. The stable manifold consists of functions $f \in \mathcal{Y}$, which become increasingly similar to φ under iteration of $\mathcal{T}$. The unstable manifold consists of functions for which *all* their preimages under the action of $\mathcal{T}$ remain close to φ. This is a curve in the function space $\mathcal{Y}$ (Figure 7.7 sketches the manifold structure).

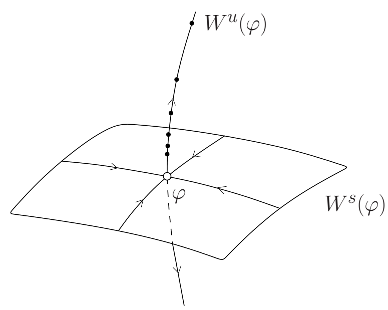

Figure 7.7: Stable and unstable manifolds of the fixed point φ.

Notice that the maps $\mathcal{T}(f)$ and f^2 are topologically equivalent (the relevant homeomorphism is the linear scaling $x \mapsto -ax$; see (7.14)). Hence, if $\mathcal{T}(f)$ has a periodic orbit of period N, f^2 has a periodic orbit of the same period and f therefore has a periodic orbit of period $2N$. Consider all maps from $\mathcal{Y}$ having a fixed point with multiplier $\mu = -1$. Such maps form a codim 1 manifold $\Sigma \subset \mathcal{Y}$. The following result has also been first established with the help of a computer.

Theorem 7.25 (Manifold intersection) *The manifold Σ intersects the unstable manifold $W^u(\varphi)$ transversally.* □

By analogy with a finite-dimensional saddle, it is clear that the preimages $\mathcal{T}^{-k}\Sigma$ will accumulate on $W^s(\varphi)$ as $k \to \infty$ (see Figure 7.8). Taking into account the previous observation, we can conclude that $\mathcal{T}^{-1}\Sigma$ is composed of maps having a cycle of period two with a multiplier -1, that $\mathcal{T}^{-2}\Sigma$ is formed by maps having a cycle of period four with a multiplier -1, and so forth. Any generic one-parameter dynamical system $f_{(\alpha)}$ from the considered class corresponds to a curve Λ in $\mathcal{Y}$. If this curve is sufficiently close to $W^u(\varphi)$, it will intersect *all* the preimages $\mathcal{T}^{-k}\Sigma$. The points of intersection define a sequence of bifurcation parameter values $\alpha_1, \alpha_2, \ldots$ corresponding

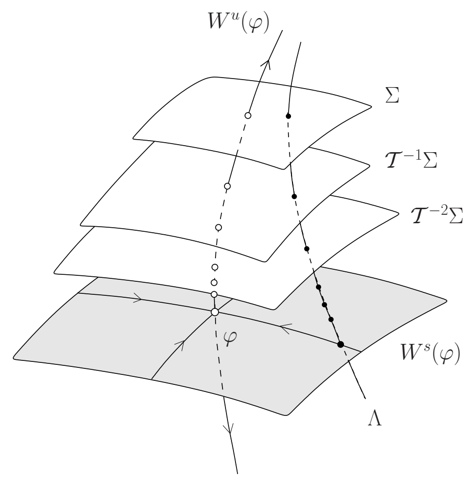

Figure 7.8: Preimages of a surface Σ intersecting the unstable manifold $W^u(\varphi)$.

to a cascade of period doublings. Asymptotic properties of this sequence are clearly determined by the unstable eigenvalue μ_F. Indeed, let ξ be a coordinate along $W^u(\varphi)$, and let ξ_k denote the coordinate of the intersection of $W^u(\varphi)$ with $T^{-k}\Sigma$. The doubling operator *restricted* to the unstable manifold has the form

$$\xi \mapsto \mu_F \xi + O(\xi^2)$$

and is locally invertible, with the inverse given by

$$\xi \mapsto \frac{1}{\mu_F}\xi + O(\xi^2).$$

Since

$$\xi_{k+1} = \frac{1}{\mu_F}\xi_k + O(\xi_k^2),$$

we have

$$\frac{\xi_k - \xi_{k-1}}{\xi_{k+1} - \xi_k} \to \mu_F$$

as $k \to \infty$, as does the sequence of the flip bifurcation parameter values on the curve Λ.